$\Gamma \hookrightarrow SO(3) \times PGL_2(\mathbb{Q}_p)$

Exercise $\Gamma \hookrightarrow PGL_2(\mathbb{Q}_p)$ is discrete since $SO(3)$ is compact.

Next time: this is a lattice.

24.12 Alex

$R$ – commut ring with Id

$H(R) = \{ x_0 + x_1 i + x_2 j + x_3 k \mid x_i \in R,\ i^2 = j^2 = k^2 = -1,\ ij = k = -ji \}$

If $K$ is a field with $\varepsilon = \sqrt{-1} \in K$, then

$H(K) \simeq M_2(K)$

$$\alpha = x_0 + x_1 i + x_2 j + x_3 k \overset{\eta}{\longmapsto} \begin{pmatrix} x_0 + \varepsilon x_1 & x_2 + \varepsilon x_3 \\ -x_2 + \varepsilon x_3 & x_0 - \varepsilon x_1 \end{pmatrix}$$

Note $\bar{\alpha} = x_0 - x_1 i - x_2 j - x_3 k$,

$$\alpha \bar{\alpha} = \|\alpha\| = \sum_{i=0}^{3} x_i^2$$

Note $\det(\eta(\alpha)) = \|\alpha\|$

Say $H$ splits over $K$ if $H(K) \simeq M_2(F)$

So if $\sqrt{-1} \in K$ then $H$ splits but not iff

In fact for every $p \neq 2$, $H$ splits over $\mathbb{Q}_p$ (but only if $p \equiv 1\ (4)$, $\sqrt{-1} \in \mathbb{Z}_p \subset \mathbb{Q}_p$)

$H$ does not split over $\mathbb{R}$, so $H(\mathbb{R})$ is a division algebra, $H^*(\mathbb{R}) = H(\mathbb{R}) \setminus \{0\}$,

$$H(\mathbb{R})^*/Z \simeq SO(3, \mathbb{R}), \text{ ie. compact}$$

$Z = \mathbb{R}^*$

Let $p$ be odd (we will work with $p \equiv 1\ (4)$)

$$H(\mathbb{Z}[\tfrac{1}{p}]) \underset{\text{discrete}}{\hookrightarrow} H(\mathbb{R}) \times H(\mathbb{Q}_p)^*/Z$$

$$H(\mathbb{Z}[\tfrac{1}{p}])^*/Z \hookrightarrow \underbrace{H(\mathbb{R})^*/Z}_{\text{comp.}} \times \underbrace{H(\mathbb{Q}_p)^*/Z}_{PGL_2(\mathbb{Q}_p)} \sim$$

$\rightarrow \quad \Gamma \longrightarrow PGL_2(\mathbb{Q}_p)$ <u>discrete</u>

<u>Recall Jacobi thm</u>

$$r_4(n) = \#\{(x_0, x_1, x_2, x_4) \in \mathbb{Z}^4 \mid \sum_{i=0}^{3} x_i^2 = n\} =$$

$$= 8 \cdot \sum_{\substack{d \mid n \\ 4 \nmid d}} d$$

In particular, for p - prime:

$$r_4(p) = 8(p+1)$$

<u>Note</u> $r_4(n) = \#\{\alpha \in H(\mathbb{Z}) \mid \|\alpha\| = n\}$,

<u>so</u> $\#\{\alpha \in H(\mathbb{Z}) \mid \|\alpha\| = p\} = 8(p+1)$

<u>Assume</u> $p \equiv 1 \pmod 4$, $\alpha = x_0 + x_1 i + x_2 j + x_3 k$, $\|\alpha\| = p$

then one of the $x_i$'s is odd and 3 even

(since $\forall$ x odd, $x^2 \equiv 1 \pmod 4$)

$$S = \{\alpha \in H(\mathbb{Z}) \subseteq H[\mathbb{Z}[\tfrac{1}{p}]] \mid \begin{array}{l} \|\alpha\| = p \\ x_0 \text{ is odd}, x_0 > 0 \end{array}\}$$

Note $H(R)^* = \{\alpha \in H(R) \mid \|\alpha\| \in R^*\}$,

so $H(\mathbb{Z})^* = \{\alpha \in H(\mathbb{Z}) \mid \|\alpha\| = 1\} = \{\pm 1, \pm i, \pm j, \pm k\}$

$|S| = p+1$.

$\forall \alpha \in S$, $\eta(\alpha) \in M_2(\mathbb{Z}_p)$, $\det(\eta(\alpha)) = \|\alpha\| = p \neq 0$

So we will think of S as a subset of $GL_2(\mathbb{Q}_p)$ or $PGL_2(\mathbb{Q}_p)$

$H(\mathbb{Z}[\tfrac{1}{p}])/Z$ therefore ~~this gro~~ every element is represented by an integer quaternion.

$Z = \{\pm p^j \mid j \in \mathbb{Z}\}$

**Exer** $\mathcal{Z}_0 = \mathbb{Z}_p e_1 + \mathbb{Z}_p e_2$, $[\mathcal{Z}_0]$ - standard vertex.

$$\mathrm{Stab}(\Gamma = H(\mathbb{Z}[\tfrac{1}{p}]^*/\mathbb{Z}), [\mathcal{Z}_0]) = \{\pm 1, \pm i, \pm j, \pm k\}$$

**Prop** $S = \{\alpha \in H(\mathbb{Z}) \mid \|\alpha\| = p,\ x_0 \text{ odd and } > 0\}$

$|S| = p+1$

$S \cdot [\mathcal{Z}_0]$ = the $(p+1)$ neighbors of $[\mathcal{Z}_0]$

"the standard vertex"

**Proof** Each $\alpha \longrightarrow \alpha \in M_2(\mathbb{Z}_p)$, so

$\alpha(\mathcal{Z}_0) \subseteq \mathcal{Z}_0$, $[\mathcal{Z}_0 : \alpha(\mathcal{Z}_0)] = \det(\alpha)$

so $\alpha.(\mathcal{Z}_0)$ is a sublattice of index

$p$, i.e. a neighbor.

Now we show that if $\alpha \neq \beta \in S$, then

$\alpha(\mathcal{Z}_0) \not\sim \beta(\mathcal{Z}_0)$, both of index $p$ in $\mathcal{Z}_0 =$

$= \mathbb{Z}_p e_1 + \mathbb{Z}_p e_2$

Note $\alpha^{-1} = \dfrac{\bar{\alpha}}{\|\alpha\|}$, so if $\alpha(\mathcal{Z}_0) \sim \beta(\mathcal{Z}_0)$

then $\alpha(\mathcal{Z}_0) = \beta(\mathcal{Z}_0)$ (because of the index)

$\Rightarrow \alpha^{-1}\beta(\mathcal{Z}_0) = \mathcal{Z}_0$, so $\alpha^{-1}\beta$ is a unit

$$\|\alpha^{-1}\beta\| = \left\|\frac{\bar{\alpha}}{\|\alpha\|}\beta\right\| = \frac{1}{\|\alpha\|^2}\cdot\|\bar{\alpha}\|\,\|\beta\| = \frac{1}{p^2}\cdot p \cdot p = 1$$

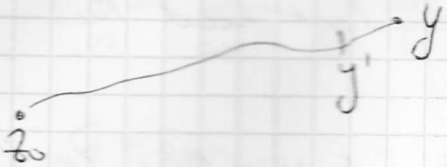
both in $S \Rightarrow \alpha = \beta$ ☒

$$\eta(H(\mathbb{Z}[\tfrac{1}{p}])^*/\mathbb{Z}) = \Gamma \leq PGL_2(\mathbb{Q}_p)$$

$$\Lambda = \langle S \rangle \leq \Gamma$$

**Claim** $\Lambda$ acts transitively on the tree $T_{p+1}$

**Pf** Assume not. ~~So~~ "Pick" $y \in T_{p+1}$, s.t. $y$ is the closest possible to $[\mathcal{Z}_0]$ and $y$ is not in the orbit ~~of~~ : $y \notin \Lambda \cdot [\mathcal{Z}_0]$




$\mathrm{dist}(y, z_0) \geq 2$

Let $y'$ be a vertex with $\mathrm{dist}(y', y) = 1$.
and $\mathrm{dist}([z_0], y') = \mathrm{dist}(y[z_0]) - 1$.

$\exists\, \gamma \in \Lambda$ s.t. $\gamma[z_0] = y'$, Hence

$\gamma \cdot S \cdot [z_0]$ is a sphere of radius $1$ around $y'$, i.e. contains $y$!

In other words.

$$\mathrm{dist}(\underbrace{\gamma^{-1}(y')}_{= z_0}, \gamma^{-1}(y)) = \mathrm{dist}(y', y) = 1$$

$\gamma^{-1}(y) = \alpha([z_0])$ for some $\alpha \in S$

$y = \gamma\alpha[z_0]$ ☒

<u>Remark</u> If $\alpha \in S \Rightarrow \bar{\alpha} \in S$, and $\alpha \cdot \bar{\alpha} = \|\alpha\| \cdot 1 \sim 1$
& $S = S^{-1}$ as a subset of $\Lambda$.

<u>Claim</u> $\Lambda$ is a free group on $\frac{p+1}{2}$ elements

$\Lambda \in H(\mathbb{Z}[\frac{1}{p}])$
$\Lambda \in \mathrm{Ker}\left(H(\mathbb{Z}[\frac{1}{p}]) \to H(\mathbb{Z}[\frac{1}{p}]/2\mathbb{Z}[\frac{1}{p}])\right)$

$w(\alpha_1, \ldots, \alpha_{p+1})$ a reduced word of length $\ell$, then

$\mathrm{dist}(w(\alpha_1, \ldots, \alpha_{p+1})([z_0]), [z_0]) = \ell$ ☒

~~$w = w' \alpha$, $\alpha \in S$~~

<u>Corollary</u> The $B_2(F) = T_{p+1}$ can be identified with $\mathrm{Cay}(\Lambda, S)$

Let $m = 2q$, $q \in \mathbb{Z}$, $(q,p) = 1$, and

$\Lambda(m) = \Lambda(2q) = \mathrm{Ker}\left(\Gamma = H(\mathbb{Z}[\frac{1}{p}])^*/\mathbb{Z} \to H(\underbrace{\mathbb{Z}[\frac{1}{p}]/m\mathbb{Z}[\frac{1}{p}]}_{\text{finite ring}})^*/\mathbb{Z}\right)$

$\cancel{\Lambda(2q) \leq \Lambda \leq \Lambda(2)}$ ~~don't need this~~

$\Lambda$ acts transit. on $PGL_2(\mathbb{Q}_p)/_K \Rightarrow \Lambda \cdot K = PGL_2(\mathbb{Q}_p)$

$$\Lambda(2q) \backslash T_{p+1} = \frac{Cay(\Lambda, S)}{\Lambda(2q)} = Cay\left(\frac{\Lambda}{\Lambda(2q)}, S\right)$$

Believe that $\Lambda = \Lambda(2)$. Assume $(q,2)=1$

$$\Lambda(2)/_{\Lambda(2q)} \cong$$

$$\Gamma = SL_d(\mathbb{Z}), \quad \Gamma/_{\Gamma(2q)} = SL_d(\mathbb{Z}/_{2q}) = SL_d(\mathbb{Z}/_2) \times SL_d(\mathbb{Z}/_q)$$

Strong approximation theorem

$$\cong H(\mathbb{Z}[\tfrac{1}{p}]/_q)^*/_Z = H(\mathbb{F}_q)^*/_Z$$

Now assume $q \equiv 1 \ (4)$, $\varepsilon = \sqrt{-1} \in \mathbb{F}_q$

$$H(\mathbb{F}_q) \simeq M_2(\mathbb{F}_q) \Rightarrow H(\mathbb{F}_q)^*/_Z = PGL_2(\mathbb{F}_q)$$

31.12
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## Arithmetic groups

Let $k$ be a global field (ie $[k:\mathbb{Q}]<\infty$, or $[k:\mathbb{F}_p(t)]<\infty$)

Let $S$ be a finite set of valuations of $k$, including all $S_\infty$ = the arch. val.

subring $\mathcal{O}_S = \{x \in k \mid v(x) \geq 0, \forall v \notin S\}$ of integer

Ex
1) $k=\mathbb{Q}$, $S=\{p,\infty\}$, $\mathcal{O}_S = \mathbb{Z}[\frac{1}{p}]$
2) $k=\mathbb{F}_q(t)$, $\mathbb{F}_q[t, \frac{1}{t+1}]$
0) $\mathbb{Z}$
0') $\mathbb{F}_q[t]$