07.01.14 Ron Rosenthal.

Thm (Echmann) $B^j = Z_j^\perp$, $Z^j = B_j^\perp$

$H^j \cong H_j \cong \mathcal{H}_j$

the space of j-harmonic functions

$\Omega^{j-1}(X) \to \Omega^{j-1}(X)$

$\Delta_j^+ = \partial_j \delta_j$ $\quad \Delta_j^- = \delta_{j-1}\partial_{j-1}$

$\Delta_j = \Delta_j^+ + \Delta_j^-$

$\mathcal{H}_j = \ker \Delta_j$

for a d-complex we will be interested in Δ_d^+

Spectral gap: if $f = \delta_{j-1} g$ i.e. $f \in B^{j-1}$ then $\Delta f = 0$.

$$\lambda(X) = \min \operatorname{Spec}\left(\Delta_d^+\big|_{B_{d-1}^\perp}\right) = \min \operatorname{Spec}\left(\Delta_d^+\big|_{Z_{d-1}}\right)$$

~~Conjecture~~

Def for complete skeleton:

$$h(X) = \min_{\substack{\coprod A_i = V \\ A_i \neq \emptyset}} \frac{|V|\,|F(A_0,\dots A_d)|}{\prod_{i=1}^{d} |A_i|}$$

General case

$$h(X) = \min_{\coprod A_i = V} \frac{|V|\,|F(A_0,\dots A_d)|}{|F^\partial(A_0,\dots A_d)|}$$

how many skeletons for a d-cell

Thm (Gundert, Szedlack) Let X be a general d-complex. For a $(d-1)$-cell σ and a partition $A_0,\dots,A_d$

$$\deg_{A_0,\dots A_d}(\sigma) = |\{\tau^\partial \in F^\partial(A_0,\dots,A_d) : \sigma \subseteq \tau^\partial\}|$$

and define

$$C(X) = \max_{\substack{\sqcup A_i = V \\ A_i \neq \emptyset}} \max_{\tau^d \in F^d(A_0,\dots A_d)} \sum_{\substack{\sigma \subset \tau^d \\ \sigma \in X^{d-1}}} \deg^{\partial}_{A_0 \dots A_d}(\sigma)$$

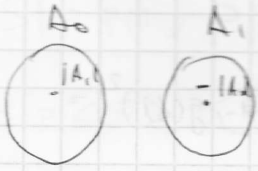
Then

$$\lambda(X) \le \underbrace{\frac{C(X)}{|V|}}_{\le 1} h(X)$$

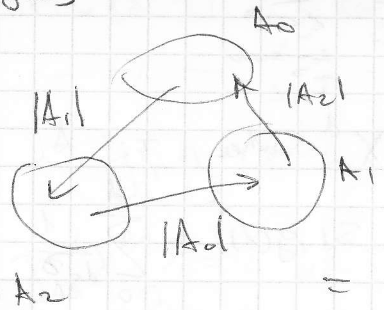
<u>Proof</u> find $f \in Z_{d-1}$ s.t. $\frac{\langle \Delta^+ f, f\rangle}{\langle f, f\rangle} = h(X)$

then $\lambda(X) = \min_{g \in Z_{d-1}} \frac{\langle \Delta^+ g, g\rangle}{\langle g, g\rangle} \le \frac{\langle \Delta^+ f, f\rangle}{\langle f, f\rangle} = h(X)$

idea for the function.



Triangles



$$f([\sigma_0, \dots \sigma_{d-1}]) = \Big\{$$

$$= \begin{cases} \operatorname{sgn}\pi \cdot |A_{\pi(d)}| & \exists\, \pi \in S_n : \sigma_i \in A_{\pi(i)} \\ 0 & \end{cases}$$

<u>Lemma 1</u> If X is any d-complex then

$$\langle \Delta^+ f, f\rangle = |V|^2 \cdot |F(A_0, \dots, A_d)|$$

<u>Lemma 2</u> If X has a complete $(d-1)$-skeleton then $f \in Z_{d-1}$ and $\langle f, f\rangle = |V| \prod_{i=0}^{d} |A_i|$

$$f = \underset{Z_{d-1}}{z} + \underset{B^{d-1}}{b}, \quad \lambda(X) \le \frac{\langle \Delta^+ z, z\rangle}{\langle z, z\rangle} = \frac{\langle \Delta^+ f, f\rangle}{\langle z, z\rangle} \overset{\text{Lemma 1}}{=} \frac{|V|^2 |F(A_0, \dots, A_d)|}{\langle z, z\rangle}$$

$$z = f - \underbrace{\delta_{d-1} g}_{b}$$

<u>Lemma 3</u>: $\forall\, g \in \Omega^{d-2}$

$$\|f - \delta_{d-1} g\|^2 \ge \sum_{\tau^d \in F^d(A_0, \dots A_d)} \sum_{\substack{\sigma \in \tau^d \\ \dim \sigma = d-1}} \frac{1}{\deg^{\partial}_{A_0 \dots A_d}(\sigma)} \cancel{f(\sigma)} \left(f(\sigma) - \delta_{d-1} g(\sigma)\right)^2$$

2) $\forall\, \tau^\partial \in F^\partial(A_0, \dots A_d) \quad \sum \frac{1}{\deg^\partial_{A_0,\dots A_d}(\sigma)} (f(\sigma) - \delta_{d-1} g(\sigma))^2 \geq$

$$\geq \frac{|V|^2}{C(X)}$$

Thus $\quad \forall g \in \Omega^{d-2}$

$$\|f - \delta_{d-1} g\|^2 \geq \frac{|V|}{C(X)} \cdot (V) \, |F^\partial(A_0, \dots A_d)|$$

$$\langle z, z \rangle \geq \min \quad \geq \quad \text{------} 1 \text{------}$$

$\square$ (of the theorem)

Proof of the lemma 3

σ is a $(d-1)$-cell.

basically trivial $\rightarrow$ 1) $\|f - \delta_{d-1} g\|^2 = \sum_{\sigma \in X^{d-1}} (f(\sigma) - \delta_{d-1} g(\sigma))^2 =$

$$= \sum_{\substack{\sigma \not\subset \tau^\partial \\ \text{for all } \tau^\partial}} + \sum_{\substack{\sigma \subset \tau^\partial \\ \text{for some } \tau^\partial}} \geq \sum_{\substack{\sigma \subset \tau^\partial \\ \text{for some } \tau^\partial}}$$

2) $\tau^\partial \in [\tau_0, \dots \tau_d] \in X^d_{\pm}$, when $\tau_i \in A_i$

$$\sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X^{d-1}}} \frac{1}{\deg^\partial_{A_0 \dots A_d}(\sigma)} (f(\sigma) - \delta_{d-1} g(\sigma))^2 = \sum_{i=0}^{d} \frac{1}{\deg^\partial_{A_0 \dots A_d}(\tau \setminus \tau_i)} \cdot$$

$$\cdot \left[(-1)^i |A_i| - \delta_{d-1} g(\tau \setminus \tau_i) \right]^2$$

$$= \sum \frac{1}{\deg^\partial_{A_0,\dots A_d}(\tau \setminus \tau_i)} \cdot \left[|A_i| - (-1)^i \delta_{d-1} g(\tau \setminus \tau_i) \right]^2 = \text{✡}$$

Since $\left(\delta_{d-1} g \in B^{d-1} \Rightarrow \right) \; \delta_d \delta_{d-1} g = 0$

$$0 = \delta_d \delta_{d-1} g = \sum_{i=0}^{d} (-1) \, \delta_{d-1} g(\tau \setminus \tau_i)$$

$$\text{✡} = \sum_{i=0}^{d-1} \frac{1}{\deg^\partial_{A_0 \dots A_d}(\tau \setminus \tau_i)} \Big[|A_i| - \underbrace{(-1)^i \delta_{d-1} g(\tau \setminus \tau_i)}_{x_i} \Big]^2 +$$

$$+ \frac{1}{\deg^\partial_{A_0 \dots A_d}(\tau \setminus \tau_d)} \Big[|A_d| + \underbrace{\sum_{j=0}^{d-1} (-1)^j \delta_{d-1} g(\tau \setminus \tau_j)}_{\sum_{j=0}^{d-1} x_j} \Big]^2$$

As a function of $(x_0, \dots x_{d-1})$ the global minimum is reached at

$$X_i = \frac{\deg^{\partial}_{A_0 \dots A_n}(\tau \setminus \tau_i)}{\sum_{j=0}^{d} \deg^{\partial}_{A_0 \dots A_d}(\tau \setminus \tau_j)} \big| |V| - |A_i| \big|$$

$$\cancel{p+1} + 2\sqrt{q}$$
$$(p+1+2\sqrt{q}) -$$
$$(p-1) - 2\sqrt{p}$$
$$= p+1-q$$

take as a bound

$$\text{✡} \geq \text{✡ with this values} = \frac{|V|}{\sum_{j=0}^{d} \deg^2_{A_0 \dots A_d}(\tau \setminus \tau_j)} \geq \frac{|V|^2}{C(X)}$$

Alex

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$$X^{p,q} = Cay(H, S)$$

$|S| = p+1$, $S \subseteq PGL_2(q)$

$$H = \langle S \rangle \begin{cases} \to PSL_2(q), & \text{if } \left(\frac{p}{q}\right) = 1 \\ \to PGL_2(q), & \text{if } \left(\frac{p}{q}\right) = -1 \end{cases}$$

Thm 1) girth $(X^{p,q}) \geq \frac{2}{3} \log_p(n)$ if $\left(\frac{p}{q}\right) = 1$

2) — " — $\geq \frac{4}{3}$ — " — $\quad -1$

Recall girth $(X^{p,q}) \leq 2 \log_p(n)$

Thm $\chi(X^{p,q}) = 2$, if $\left(\frac{p}{q}\right) = -1$

$\chi(X^{p,q}) \geq \frac{p+1}{2\sqrt{p}}$, ~~now~~, if $\left(\frac{p}{q}\right) = 1$.

so when $p \to \infty$, $q \to \infty$ we get graphs with arbitrarily large girth and arb large chrom. number

Pf 1: Hoffman inequality. X k-reg graph,

then $\chi(X) \geq \frac{k}{-\lambda_{n-1}(x)} + 1$ ← the smallest e.V. of adj.

note that X is bi-part. iff ~~a~~ $\lambda_{n-1}(X) = -k$.

in our case $X^{p,q}$ is not bi-partn and

$$\lambda_{n-1} \in [-2\sqrt{p}, 2\sqrt{p}],$$

so $\chi(X_{p,q}) \geq \frac{p+1}{-2\sqrt{p}} + 1 \geq \frac{p+1}{2\sqrt{p}} + 1$

Pf 2

$i(X) =$ ind number $= \max \{ |A| \mid A \subset V, \text{ no two vertices } A \text{ are adj} \}$

clearly $\chi(X) \geq \frac{|V|}{i(X)}$ since every monochromatic subset is independent