$$X_i = \frac{\deg^{\partial}_{A_0 \dots A_n}(\tau \setminus \tau_i)}{\sum_{j=0}^{\cancel{d},i} \deg^{\partial}_{A_0 \dots A_d}(\tau \setminus \tau_j)} \big(|V| - |A_i|\big)$$

take as a bound

$\geq$ with my values $= \dfrac{|V|}{\sum_{j=0}^{d} \deg^2_{A_0 \dots A_d}(\tau \setminus \tau_j)} \geq \dfrac{|V|^2}{C(X)}$

$\cancel{p}+1 + 2\sqrt{p}$

$(p+1+2\sqrt{q}) -$

$(p-1) - 2\sqrt{p}$

$= p+1-4$

Alex

14.01.14

$X^{p,q} = Cay(H, S)$

$|S| = p+1$, $S \subseteq PGL_2(q)$

$H = \langle S \rangle$ → $PSL_2(q)$, if $\left(\frac{p}{q}\right) = 1$

→ $PGL_2(q)$, if $\left(\frac{p}{q}\right) = -1$

Thm 1) $\text{girth}(X^{p,q}) \geq \frac{2}{3} \log_p(n)$ if $\left(\frac{p}{q}\right) = 1$

2) — / — $\geq \frac{4}{3}$ — / — -1

Recall $\text{girth}(X^{p,q}) \leq 2 \log_p(n)$

Thm $\chi(X^{p,q}) = 2$, if $\left(\frac{p}{q}\right) = -1$

$\chi(X^{p,q}) \geq \frac{p+1}{2\sqrt{p}}$, ~~now~~, if $\left(\frac{p}{q}\right) = 1$.

so when $p \to \infty$, $q \to \infty$ we get graphs with arbitrarily large girth and arb large chrom. number

Pf 1: Hoffman inequality. X k-reg graph,

then $\chi(X) \geq \dfrac{k}{-\lambda_{n-1}(x)} + 1$ ← the smallest e.V. of adj.

note that X is bi-part. iff $\lambda_{n-1}(X) = -k$.

in our case $X^{p,q}$ is not bi-partn and

$\lambda_{n-1} \in [-2\sqrt{p}, 2\sqrt{p}]$,

so $\chi(X_{p,q}) \geq \dfrac{p+1}{-2\sqrt{p}} + 1 \geq \dfrac{p+1}{2\sqrt{p}} + 1$

Pf 2

$i(X)$ = ind number = $\max\{|A| \mid A \subset V$, no two vertices A are adj$\}$

clearly $\chi(X) \geq \dfrac{|V|}{i(X)}$ since every monochromatic subset is independent

Prop If X is k-reg. Ramanujan, not-bi-partite then $i(X) \le \frac{2\sqrt{k-1}}{k} n$

Pf let B be an ind. size τ

let $f(x) = \begin{cases} 1 & \text{on } B \\ -c & \text{on } \overline{B} \end{cases}$ $\quad \tau + (n-\tau)\cdot c = 0 \quad c = -\frac{\tau}{n-\tau}$ $\quad f \in L_2^0(X)$

If $A = Adj(X)$

$$\|Af\|_2^2 \le (2\sqrt{k-1})^2 \|f\| \quad \text{since } f \in L_2^0$$

$$(Af)(x) = -ck \quad \text{if } x \in B \Rightarrow$$

$$\Rightarrow \|Af\|_2^2 \ge c^2 \cdot k^2 \cdot \tau$$

$$\|f\| = \tau + c^2 \cdot (n-\tau) = \tau + \frac{\tau^2}{n-\tau} = \frac{n\tau - \tau^2 + \tau^2}{n-\tau} = \frac{n\tau}{n-\tau}$$

$$\Downarrow$$

$$\frac{\tau^2}{(n-\tau)^2} k^2 = c^2 k^2 \tau \le 4(k-1)\cdot \frac{n\tau}{n-\tau}$$

$$\tau^2 \cdot k^2 \le 4(k-1)\cdot n \cdot (n-\tau)$$

$$k^2 \cdot \tau^2 + n \cdot 4(k-1)\cdot \tau - 4(k-1)\cdot n^2 \le 0$$

$$\cancel{D/4 = 4n^2(k-1)^2 + 4k^2\cdot(k-1)n^2 = 4n^2(k-1)(k-1+k^2)}$$

$$\cancel{\tau \le \frac{-2n(k-1) \pm 2n\sqrt{(k-1)\cdot(k^2+k-1)}}{k^2}}$$

$$c^2 k^2 \le 4(k-1)\left(1 + c^2\cdot\left(\frac{1}{\tau} - 1\right)\right)$$

$$\frac{\nu^2}{(1-\nu)^2} k^2 \le 4(k-1)\left(1 + \frac{\nu^2}{(1-\nu)^2} - \frac{1-\nu}{\nu}\right)$$

$$\nu^2 k^2 \le 4(k-1)(1-\nu)$$

$$\nu^2 k^2 \le 4(k-1) \Rightarrow \nu^2 \le \frac{4(k-1)}{k^2} \quad \nu \le \frac{2\sqrt{k-1}}{k}$$

$$PGL_2(F) \rightsquigarrow B_2(F) = {}^{PGL_2(F)}/_{K} = T_{p+1} \rightsquigarrow {}^{T_{p+1}}/_{\Gamma(q)}$$

$\Gamma(q)$ cong. subgp. is a "nice" arith. gp acting simply transitive on $V(B_2(F))$

$$\frac{T_{p+1}}{\Gamma(q)} = Cay\left(\Gamma/\Gamma(q); S\right)$$ these graphs are remarkable Ramanujan exp

$PGL_d(F) \curvearrowright B_d(F) = \frac{PGL_d(F)}{K = PGL_d(\mathcal{O})}$ = building $\rightsquigarrow$

$\rightsquigarrow \frac{B_d(F)}{\Gamma(f)}$

$\Gamma(f)$ is a cong subg of a remarkable arith lattice Γ_0 = Cartwright-Steger lattice where Γ_0 acts simply transitively on the vertices of $B_d(F)$

$\frac{B_d(F)}{\Gamma(f)}$ = Cayley Complex $(\Gamma_0/\Gamma(f); \text{"}S\text{"})$

$B_d(F)$ comes with d type function ("coloring") of the vertices:

$$\tau : V \to \mathbb{Z}/d\mathbb{Z}$$

Define $A_1, \dots, A_{d-1}$ colored adj oper. on $L^2(B_d(F))$

$$A_i(f)(x) = \sum_{\substack{y \sim x \\ \tau(y)-\tau(x)=i}} f(y)$$

$$Adj = \sum_{i=1}^{d-1} A_i$$

$A_1, \dots, A_{d-1}$ are normal operators but not self-adjoint and they all commute with each other.

and can be diagonilized simultaneously.

$Spec(A_1, \dots, A_{d-1}) \subseteq \mathbb{C}^{d-1}$

Thm Let $\Sigma_d = \{(z_1, \dots, z_d) \in \mathbb{C}^d \mid \prod z_i = 1, |z_i| = 1\}$

$\delta : (z_1, \dots, z_d) \to (\lambda_1, \dots, \lambda_{d-1})$

$$\lambda_k = q^{\frac{k(d-k)}{2}} \sigma_k(z_1, \dots, z_d)$$

where $\sigma_k = \sum z_{i_1} \cdots z_{i_k}$ — the k^{th} elem. symm. function

Then $Spec(A_1, \dots, A_{d-1}) = \delta(\Sigma_d)$

$k_{\mathcal{F}} \cancel{\leq} \frac{\lambda_{max} - k}{\lambda} \cdot k$

$k - \frac{\lambda_{max} - k}{\lambda} \cdot k =$

$= k \cdot$

$k_{\mathcal{F}}$

k_c

$\frac{k_{\mathcal{F}} \cdot k_1 \cdot k_0^2}{k_{\mathcal{F}} \cdot k_c \cdot k_0} \leq \lambda_{max}$

$k_c \geq \frac{k_1 k_0}{\lambda_{max}}$

$\alpha \geq \frac{k_1 k_0}{\lambda_{max}^0}$

$\frac{\lambda_{max}^0}{\lambda_{max}^0 - \frac{k_1 k_0}{\lambda_{max}^0}}$

$\sqrt{k_0} + \sqrt{k_0}$

$q^2 + 6q$

$q^2 + 6q - \frac{3(q+1)q^2}{3(q^2+6q)}$

<u>$d=2$</u> $\quad \Sigma_q = \{(z_1, z_2) \mid |z_i| = 1, |z_i| = 1\}$

$$\sigma(z_1, z_2) \longrightarrow \lambda$$

$$\lambda = q^{\frac{1(2-1)}{2}} (z_1 + z_2) = \sqrt{q} \cdot (z_1 + z_1^{-1}) \subset$$

$$\subset \sqrt{q} \cdot [-2; 2]$$

$\Gamma \leq PGL_d(F)$ preserving the colors

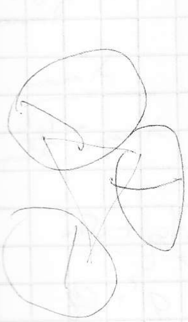
$X = {}_{\Gamma}\backslash^{B_d(F)}$ has coloring and A_i are well defined on $L^2(X)$

Def X is Ramanujan iff every non-trivial "e.v" is inside $\sigma(\Sigma_d)$

- trivial "e.v"'s
- Alon - Boppana" $\quad$ "$X_j = \frac{B_d(F)}{\Gamma_j}$" $\to \infty$

$$\mathrm{Spec}(X_j) \subseteq \mathbb{C}^{d-1},$$

$$\overline{\bigcup_{j=1}^{\infty} \mathrm{Spec}(X_j)} \supseteq \sigma(B_d)$$



The trivial "e.v's of $(A_1, \dots A_{d-1})$ are obtained in the following way

let $\xi \in \mu_d = \{z \in \mathbb{C} \mid z^d = 1\}$

$$\lambda(\xi) = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_q \xi^1, \begin{bmatrix} d \\ 2 \end{bmatrix}_q \xi^2, \dots, \begin{bmatrix} d \\ d-1 \end{bmatrix}_q \xi^{d-1} \right)$$

$\begin{bmatrix} d \\ i \end{bmatrix}_q = \#$ of subspaces of $\mathbb{F}_q^d$ of dim i

$d=2$, $\quad \xi = \pm 1$

$$\lambda(1) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \cdot 1 \right) = q+1$$

$$\lambda(-1) = -(q+1)$$

$d=3$, $\quad \xi = 1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = q^2+q+1 ; q^3+q$$

(q^2+q+1, q^2+q+1)

$\pm(q^2+q+1) e^{\frac{2\pi i}{3}}, (q^2+q+1) \cdot e^{-\frac{2\pi i}{3}})$

$$|\lambda_n| \le \binom{d}{k} q^{\frac{k(d-k)}{2}}$$ – Ramanujan bound

Uli Wagner 10/02/14

Cohomological expansion, topological overlap.

X – finite s.c.

<u>Def</u> we say that X is ε-expanding in dim i if the following holds

$$\alpha \in C^{i-1}(X, \mathbb{Z}_2) \qquad \|\delta\alpha\| \ge \varepsilon \cdot \|[\alpha]\|$$

$\gamma \in C^i(X, \mathbb{Z}_2)$, $|\gamma| = |supp(\gamma)|$, $\|\gamma\| = \frac{|\gamma|}{\#\, i\text{-simplices of } X}$ ← "density norm"

$$\underbrace{\|[\gamma]\|}_{\gamma \in B^i(X)} = \min_{\beta \in B^i(X)} \|\gamma + \beta\|$$

<u>Theorem (Gromov)</u> If X is ε_i-expanding in each dim i, $1 \le i \le d$ (w.r.t. density norm). Then X has the topological overlap property for maps $X \to \mathbb{R}^d$: $\exists\ c = c(\varepsilon_1, \dots, \varepsilon_d) > 0$ s.t. for every cont. map $F: X \to \mathbb{R}^d$ $\exists\ p \in \mathbb{R}^d$ s.t. $F^{-1}(p)$ intersects at least a c-fraction of d-simplices of X.

<u>Def</u> X satisfies an ε-coisoperimetric inequality in dim i (w.r.t. some norm $\|\cdot\|$) if $\forall\, \beta \in B^i(X)$ $\exists\ \gamma \in C^{i-1}(X)$; $\delta\gamma = \beta$ s.t. $\|\gamma\| \le \frac{1}{\varepsilon} \cdot \|\beta\|$.

<u>Remark</u> X is ε-expanding in dim i $\Leftrightarrow$ X has ε-coisoper. ineq. and $H^{i-1}(X) = 0$:

"$\Leftarrow$" $\alpha \in C^{i-1} \rightsquigarrow \beta := \delta\alpha \Rightarrow \exists\ \gamma \in C^{i-1}$. $\|\gamma\| \le \frac{1}{\varepsilon}\|\beta\|$, $\delta\gamma = \beta \Rightarrow$
$\Rightarrow \alpha - \gamma \in Z^{i-1} = B^{i-1}$. If γ minimizes $\|\gamma\|$ over all co-fillings of $\beta = \delta\alpha$ then $\|\gamma\| = \|[\alpha]\|$.

$\hookrightarrow$