$\Leftarrow$ morphisms weakly order-preserving maps

$f : \{0,1,\dots,i\} \to \{0,1,\dots,j\}$

$Z_d(X)$: $(Z_d(X))_0 = Z^d(X)$

$(Z_d(X))_1$ (edges) : $\{$ triples :

$(a_1^d, a_2^d, a_{12}^{d-1}) \in Z^d(X) \times Z^d(X) \times C^{d-1}(X)$

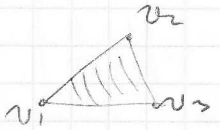
$\delta a_{12}^{d-1} = a_1^d + a_2^d$ $\}$

$(Z_d(X))_2$ (triangles) : $= \{$ tuples : $(a_1^d, a_2^d, a_3^d, a_{12}^{d-1}, a_{13}^{d-1}, a_{23}^{d-1}, a_{123}^{d-2})$:

$\delta a_{ij}^{d-1} = a_i^d + a_j^d$

$\delta a_{123}^{d-2} = a_{12}^{d-1} + a_{23}^{d-1} + a_{13}^{d-1}$

Lemma $F : X \to \mathbb{R}^d$, choose a fine triangulation B of a bounding box for $f(X)$. Extend to $S = \mathrm{sh}\,d(v_\infty * B)$: If τ k-simplex of S $v_1, \dots, v_{k+1}$



$v_i \mapsto a_i^d = F^\#(v_i)$

$v_i v_j \mapsto a_{ij}^{d-1} = F^\#(v_i v_j)$

$F^\#$ $C_*(S) \to C_*(Z_d(X))$.

Magic lemma is: If ω is the generator of $Z_d(S)$ the $F^\#(\omega) \in Z_d(Z_d(X))$ is non trivial (not null-homologous).

18/02/14

Let X be a set of points in $\mathbb{R}^d$, $|X| = n$. There are $\binom{n}{d+1}$ d-simplices spanned by $(d+1)$-tuples of the points.

Thm (Barany) $\exists$ a constant c_d s.t. for every such X $\exists$ $y \in \mathbb{R}^d$ s.t. that is contained in $c_d \cdot \binom{n}{d+1}$ such simplicies

Thm (Boros - Furedi) $c_2 = \frac{2}{9}$.

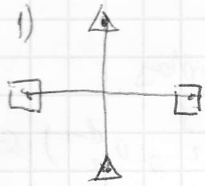
I Radon's theorem

Thm (Radon, 1920) Given a set $X = \{x_1, \dots, x_n\}$ of n points in $\mathbb{R}^d$, $n \geq d+2$, $\exists$ partition $S \sqcup T = [n] = \{1, 2, \dots, n\}$ s.t.

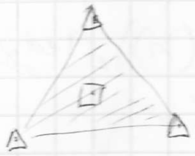
$$\operatorname{conv}(x_i,\ i\in S)\cap \operatorname{conv}(x_i,\ i\in T)$$

$d=1$

$d=2$ 1)



2)



Proof Since $n > d+1$, the points in X are affinely dependent $\Rightarrow \exists\ \alpha_i\in\mathbb{R}$:

$$\sum_{i=1}^{n}\alpha_i x_i = 0,\ \&\ \sum_{i=1}^{n}\alpha_i = 0$$

now $S := \{i : \alpha_i \geq 0\}$, $T := \{i : \alpha_i < 0\}$

$$\begin{cases}\sum_{i\in S}\alpha_i x_i = \sum_{j\in T}(-\alpha_j)x_j\\ \sum_{i\in S}\alpha_i = \sum_{j\in T}(-\alpha_j) =: \lambda\end{cases} \Rightarrow \sum_{i\in S}\frac{\alpha_i}{\lambda}x_i = \sum_{j\in T}\frac{-\alpha_j}{\lambda}x_j =: y$$

$y\in$ convex hulls. $\square$

Order type

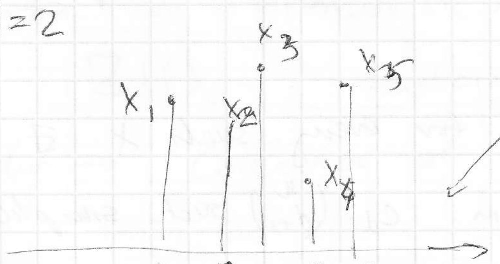
$X=\{x_1, \dots x_n\}\subset\mathbb{R}^d$ & $Y=\{y_1,\dots,y_n\}\subset\mathbb{R}^d$ have the same order type if for every $i_1, i_2, \dots i_{d+1}$

$$\operatorname{sgn}\left(\det\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ | & | & & & |\\ x_{i_1} & x_{i_2} & & & x_{i_{d+1}}\\ | & | & & & |\end{pmatrix}\right) = \operatorname{sg}\left(\det\begin{pmatrix}1 & \dots & 1\\ | & & |\\ y_{i_1} & & y_{i_{d+1}}\\ | & & |\end{pmatrix}\right)$$

~~$\operatorname{sgn}\in\{\pm$~~ $\operatorname{sgn}\in\{+,0,-\}$

Order type is a map $X\subset\mathbb{R}^d$ to $\{+,0,-\}$

$d=2$

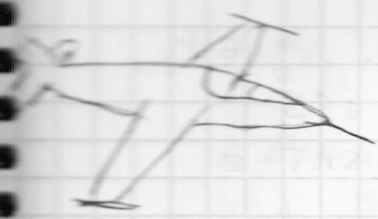


Start rotating the line until you reverse it, so what you get is

$$(id)(\pi_1)(\pi_2)\dots(\pi_m) = (n, n-1, \dots 1)$$

$$\pi_i = (k, k+1)$$

$$m = \binom{n}{2}$$



open problem: $\overset{\text{even}}{n}$ pts in $\mathbb{R}^2$

wants to find a halving line.

$f_k(n) = \max\limits_{\substack{\text{all conf of } n \text{ pts} \\ \text{in } \mathbb{R}^2}}$ # of ways to partition them in k and $n-k$ points

$$n \cdot e^{\sqrt{\log n}} \le f_{\frac{n}{2}}(n) \le n^{4/3}$$

conj: $f_{\frac{n}{2}}(n) \le C_\varepsilon \cdot n^{1+\varepsilon}$ for every $\varepsilon > 0$

If $\pi_1, \dots, \pi_{\binom{n}{2}}$ is reduced decomposition in S_n

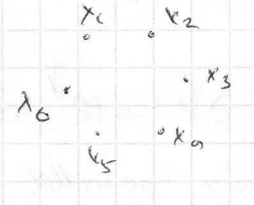
then $(j, j+1)$ appears at most $O(n^{1+\varepsilon})$ $\forall \varepsilon > 0$

a slightly more general conj

<u>Important order type</u>

$$\begin{pmatrix} 1 & \cdots & 1 \\ | & | & | \\ x_1 & x_2 \cdots & x_n \\ | & | & | \end{pmatrix}$$ all signs are $+$, then points are in <u>cyclic</u> position.

<u>Example</u> $d=2$



<u>d general</u> $x(t) = (t, t^2, \dots t^d)$

$x_i := x(t_i)$ $\quad t_1 < t_2 < \dots < t_n$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ \vdots & & & \\ t_1^d & \cdots & & t_n^d \end{pmatrix}$$

$\frac{\lambda k}{\lambda - k} =$

$= \frac{\lambda - k}{\lambda k} = \frac{1}{k} - \frac{1}{\lambda}$

II Helly's thm

Given n convex sets in $\mathbb{R}^d$, $K_1, K_2, \dots K_n$, $n \ge d+1$, if every $(d+1)$ of the sets $K_{i_1}, K_{i_2}, \dots K_{i_{d+1}}$ have a point in common then all of them have a point in common

<u>Proof</u> induction on n. Base $n = d+1$, trivial

Assume every $(n-1)$ points have a point in common

denote $y_m \in K_1 \cap K_2 \cap \dots \cap \widehat{K_m} \cap \dots \cap K_n$, $m = 1, \dots n$

$y_1, \dots y_n$, by Radon's theorem, $\exists\ S \sqcup T = [n]$:

$\operatorname{conv}\{y_i, i \in S\} \cap \operatorname{conv}\{y_i, i \in T\} \ne \emptyset$ denote $z \in$

$\partial X \geq k \cdot |X| \leq \frac{\lambda - k}{k}$

$\frac{n}{X}$

$\partial X \geq \alpha \cdot |X| \geq \alpha \cdot \frac{n}{X}$

$\frac{X_\alpha}{\alpha} \geq \frac{x_k}{k}$

$X \leq \frac{k}{\alpha} \lambda$

$z \in \operatorname{conv} \{ y_i, i \in S \}$, for each y_i, $y_i \in K_j$, $j \notin S$

$\Rightarrow z \in K_j$, $j \notin S$

Analogously, $z \in K_\ell$, $\forall \ell \notin T$ $\Rightarrow$ since $S \cap T = \emptyset$ $\Rightarrow z \in K_j$, all j ☒

III Caratheodory's thm

If $S \subset \mathbb{R}^d$, $|S| > d+1$, $x \in \operatorname{conv} S$, then $\exists R \subset S$

$i_\lambda \leq n \cdot \frac{k}{\lambda \alpha}$ s.t. $|R| \leq d+1$, $x \in \operatorname{conv}(R)$

Pf $S = \{x_1, \dots x_n\}$

$i_\alpha \leq \frac{\partial}{\alpha} \leq \frac{nk}{\alpha}$

$x = \sum_{i=1}^{m} \alpha_i x_{j_i}$, $\alpha_i > 0$, $\sum \alpha_i = 1$

If $m > d+1$, then $\sum p_i x_{j_i} = 0$, $\sum p_i = 0$, not all 0 & one can add this to this and kill one of the coefficients.., ie we can assume $m = d+1$. ☒

IV Borsuk Ulam thm

$f : S^n \to \mathbb{R}^n$ cont., $S^n = \{x \in \mathbb{R}^{n+1}, \|x\| = 1\}$, $\exists y \in S^n$, $f(y) = f(-y)$

K smooth strictly convex body in $\mathbb{R}^{d+1}$, $f : \partial K \to \mathbb{R}^d$ cont.

$\exists$ $x, y \in \partial K$, x, y belong to two parallel support hyperplanes; $f(x) = f(y)$

P - polytope in $\mathbb{R}^{d+1}$, f: vert. of $P \to \mathbb{R}^d$

$\exists$ 2 parallel supp. hyperplanes H_1, H_2 : $x \in H_1 \cap P$, $y \in H_2 \cap P$, $f(x) = f(y)$

If P = simplex we get Radon's theorem.

If $K_1, \dots K_n$ are compact sets in $\mathbb{R}^d$, $n \geq d+1$, if each K_i is contractible, if each non-empty intersection of K_i's is contractible then Helly's theorem applies

$\}$

nerve theorems

$Y_1, \dots Y_n$ sets

Nerve N is an abstract s.c. on $\{1, \dots n\} = [n]$ s.t. $S \subset [n]$ iff $\bigcap_{i \in S} Y_i \neq \emptyset$

Nerve thm If $Y_1, \dots Y_n$ is a good cover of X, then N is homotopically equiv to X.

V ~~Colorful Caratheodory thm~~
(Barany)

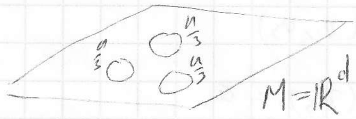
If $x \in \mathrm{conv}(Z_1) \cap \mathrm{conv}(Z_2) \dots \cap \mathrm{conv}(Z_{d+1})$ in $\mathbb{R}^d$ then $\exists\, z_i \in Z_i$, $i = 1, \dots d+1$, s.t.

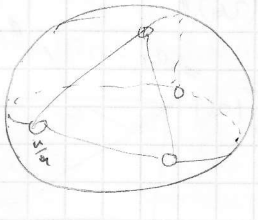
$x \in \mathrm{conv}(z_1, \dots, z_{d+1})$

Uli Wagner

18/02/14

$F : X \to \mathbb{R}^d$ (or a manifold M)





$\left(\frac{n}{4}\right)^3 = c' \cdot \binom{n}{3}$

the worst possible constant

$\left(\frac{n}{3}\right)^3 = \frac{n^3}{27} = c \binom{n}{3}$

Choose a fine triangulation of M

$$\begin{array}{ccc} C_k(M) & \xrightarrow{F^\#} & C^{d-k}(X) \\ & \searrow \quad \nearrow & \\ & C^{d-k}(M) & \end{array}$$

Poincaré duality

pullback by F

intersection number + pullback by F

$F^\#(\partial c) = \delta F^\#(c)$

$$\begin{array}{ccccccccc} 0 & \xleftarrow{\partial} & C_0(M) & \xleftarrow{\partial} & C_1(M) & \xleftarrow{\partial} \dots & \xleftarrow{\partial} & C_d(M) & \leftarrow 0 \\ & & \downarrow F^\# & & \downarrow F^\# & & & \downarrow F^\# & \\ 0 & \xleftarrow{\delta} & C^d(X) & \xleftarrow{\delta} & C^{d-1}(X) & \xleftarrow{\partial} \dots & & C^0(X) & \leftarrow 0 \end{array}$$

$F^\#$ is a chain map

Recall: Suppose that G is another chain map $C_*(M) \to C^{d-*}(X)$

Claim $F^\#$ is not chain homotopic to any chain map G : ~~G~~ $C_*(M) \to C^{d-*}(X)$ s.t. $G : C_d(M) \to C^0(X)$ maps the fund. cycle of M, $[M]$, to 0.