$H(v_1 v_2 \dots v_k) := a^{d-k}_{012\dots k}$

$\forall v_i$ $\quad G(v_i) = G(v_0) = F^{\#}(v_0)$ $\quad G : C_0(M) \to C^d(X)$ constant

$G : C_k(M) \to C^{d-k}(X)$ zero, $k>0$

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$I_d \le \frac{n}{d}\frac{k}{X}$

$\frac{n}{X} \le I$

$I_d \le \frac{k}{d} I$

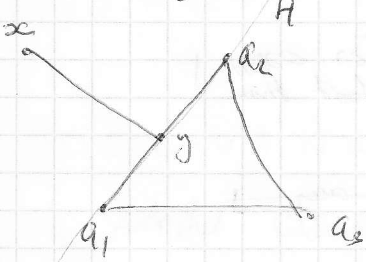
Colored Caratheodory thm:
(Barany)

Let $x \in \text{conv}\, A_1 \cap \text{conv}\, A_2 \cap \dots \cap \text{conv}\, A_{d+1}$

$A_1, \dots, A_{d+1} \subset \mathbb{R}^d$ (finite), then there are $a_1 \in A_1, a_2 \in A_2, \dots, a_{d+1} \in A_{d+1}$

s.t. $x \in \text{conv}(a_1, a_2, \dots, a_{d+1})$.

Proof Let's take $y \in \text{conv}(a_1, \dots, a_{d+1})$ s.t. $a_i \in A_i$, with $\text{dist}(x,y)$ minimal. Take H - hyperplane perpendicular to xy.



Let H^+ be the closed half-space containing x.
H^- be the other one.

Claim 1 H^- contains $\text{conv}(a_1, a_2, \dots a_{d+1})$.

Pf: Otherwise, if $z \in \text{conv}(a_1, \dots a_{d+1}) \cap \text{int}\, H^+$, then $[z,y] \subset \text{conv}(a_1, \dots a_{d+1}) \cap \text{int}\, H^+ \Rightarrow \exists$ point $u \in [z,y]$ closer to x than y. ⊠

Claim 2 $\quad y \in \text{conv}\{a_i : a_i \in H, i = 1, \dots d+1\}$

Pf If $y = \sum_{i=1}^{d+1} \lambda_i a_i$ and $a_j \in \text{int}\, H^-$ then if $\lambda_j \neq 0$ then implies that $y \in \text{int}\, H^-$ therefore $\lambda_j = 0$

$\lambda_j \geq 0, \sum \lambda_j = 1$ ⊠

$y \in \text{conv}\{a_i, i \in H\}$, by Caratheodory thm (for $d-1$))

$y \in \text{conv}(a_1, a_2, \dots \hat{a_j}, \dots a_{d+1})$.

Claim $\quad A_j \not\subset H^-$.

Pf $\quad x \in \text{conv}\, A_j \cap \text{int}\, H^+$ ⊠

Therefore $\exists\ a_j' \in A_j \cap \text{int}\, H^+$. We look at

$\text{conv}(a_1, a_2, \dots a_j', \dots, a_{d+1})$. Both a_j' and y belong to it. $\Rightarrow$ $[y, a_j'] \subset \text{conv}(a_1, a_2, \dots a_j', \dots a_{d+1}) \Rightarrow$

$\Rightarrow$ $\exists$ a point on $[y, a_j]$ which closer to x than y. $\blacksquare$

Colored Helly's thm.

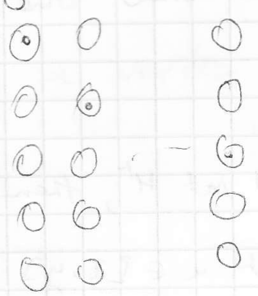
$\mathcal{K}_1, \mathcal{K}_2, \ldots \mathcal{K}_{d+1}$ — $(d+1)$ families of convex sets in $\mathbb{R}^d$, if $\forall\, k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2, \ldots k_{d+1} \in \mathcal{K}_{d+1}$, there is a point in common for them, then for some j $\exists$ a point in common for all sets in $\mathcal{K}_j$.

Colored $\Rightarrow$ regular Helly's thm if $\mathcal{K}_1 = \mathcal{K}_2 = \ldots = \mathcal{K}_{d+1}$.

Nerve thm implies a top. version of colored Helly's thm when $|\mathcal{K}_i| = 2$ for all i.

Straightening colored Helly thm (for case foot thm)

Colored Helly' thm



~~For any j~~ ~~Anyone~~

$\exists\, j$, $\exists\, k_i \in \mathcal{K}_i$, $\forall i \neq j$:

$$\bigcap (k : k \in \mathcal{K}_j) \cap \bigcap_{i \neq j} k_i \neq \emptyset.$$

Thus topological colored Helly's thm is correct.

Tverberg's thm if $x_1, \ldots, x_m \in \mathbb{R}^d$, $m \geq (d+1)(r-1)+1$ then $\exists$ a partition $S_1, S_2, \ldots S_r$ of $[m] = \{1, \ldots, m\}$ s.t. $\mathrm{conv}\{a_i, i \in S_1\} \cap \mathrm{conv}\{a_i, i \in S_2\} \cap \ldots \cap \mathrm{conv}\{a_i, i \in S_r\} \neq \emptyset$.

Pf $v_1, \ldots v_m$ pts in $\mathbb{R}^d$, $m = (d+1)(r-1)+1$, we want find a partition $S_1 \sqcup \ldots \sqcup S_r = \{1, 2, \ldots, m\}$:

$$\bigcap_{j=1}^{r} \mathrm{conv}(v_i, i \in S_j) \neq \emptyset$$

Regard $\mathbb{R}^d$ as a hyperplane H in $\mathbb{R}^{d+1}$

$$H = \{(y_0, \ldots y_d) \in \mathbb{R}^{d+1} : \textstyle\sum y_i = 1\}$$

Take space W ($\dim W = r-1$) with r vectors $w_1, \ldots, w_r$ s.t. $W = \mathrm{span}(w_1, \ldots, w_r)$ and $w_1 + \ldots + w_r = 0$

Now we take tensor product $V \otimes W$

$\dim V \otimes W = (d+1)(r-1)$.

Consider the following sets

$$A_1 = \{ v_1 \otimes w_1, v_1 \otimes w_2, \dots, v_1 \otimes w_r \}$$

$$A_2 = \{ v_2 \otimes w_1, v_2 \otimes w_2, \dots, v_2 \otimes w_r \}$$

$$\vdots$$

$$A_m = \{ \dots \qquad \dots \}$$

m sets in $V \otimes W$

for all $1 \leq j \leq m$: $\frac{1}{r} (v_i \otimes w_1 + \dots + v_i \otimes w_r) =$

$$= v_i \otimes \left(\frac{w_1 + \dots + w_r}{r} \right) = 0.$$

$0 \in \operatorname{conv} A_1 \cap \operatorname{conv} A_2 \cap \dots \cap \operatorname{conv} A_m$.

by colorful Carath. thm $\exists$

$$v_1 \otimes w_{i_1}, v_2 \otimes w_{i_2}, \dots, v_m \otimes w_{i_m}$$

$$0 \in \operatorname{conv} (v_1 \otimes w_{i_1}, \dots, v_m \otimes w_{i_m})$$

$0 = \sum \lambda_j \cdot v_j \otimes w_{i_j} \qquad \sum \lambda_j = 1, \lambda_j \geq 0$

$S_\ell = \{ j : i_j = \ell \}$

$$0 = \sum_{\ell = 1, \dots, r} \underbrace{\left(\sum_{j \in S_\ell} \lambda_j \cdot v_j \right)}_{y_\ell \in \mathbb{R}^{d+1}} \otimes w_\ell =$$

$= y_1 \otimes w_1 + y_2 \otimes w_2 + \dots + y_r \otimes w_r = 0$

look at k-th coordin.

$$y_1^k w_1 + y_2^k w_2 + \dots + y_r^k w_r = 0 \Rightarrow$$

$\Rightarrow y_1^k = y_2^k = \dots = y_r^k \quad \forall k$.

$\sum_\ell \left(\sum_{j \in S_\ell} \lambda_j \right) \cdot w_\ell = 0$, since the sum of the coord. of v_j is zero for all j.

$\Rightarrow \sum_{j \in S_j} \lambda_j = \frac{1}{r}$ | $y_j = \sum \lambda_j v_j$, $y_1 = y_2 = \dots = y_r$

Consequences.

n pts in $\mathbb{R}^d$. Tverberg says that you can partition them into $\frac{n}{d+1}$ parts $X_1, \dots, X_r$ s.t. $\exists$ a point z in the convex hull of each of them

$d=2$, $n=10^6$ $\Rightarrow$ $\frac{n}{3}$ intersecting triangles

Choose three of them.
(d+1)

by color Carath. $z \in$ conv $d+1$ pts ; one from each part.

$\exists$ point z that belongs to at least $\binom{\frac{n}{d+1}}{d+1}$ simp. deter. by $(d+1)$ tuples,

$$\binom{\frac{n}{d+1}}{d+1} \approx \frac{1}{(d+1)^{d+1}}\binom{n}{d+1}$$ i.e. we've proved an overlap thm

4.3.03 We proved Tverberg's thm. ; We derive:

There is a constant $C_d > 0$ s.t. given n pts in $\mathbb{R}^d$ $\exists$ a point $x \in \mathbb{R}^d$ which belongs to at least $C_d \cdot \binom{n}{d+1}$ of the simplices spanned by $(d+1)$-tuples of points.

I.e. the complete d-dim simplicial complex on n vertices has the "geometric overlap property"

Proof: By Tverberg's thm we can divide them to $\frac{n}{d+1}$ sets $A_1, \dots A_n$ of size $(d+1)$ s.t. all the convex hull of these sets will contain a point z. For each $(d+1)$ of these sets we can, by the colored Caratheodory thm, find one point from each set so that z is in their convex hull