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$\mathcal{K}$ - a family of sets $\mathcal{K} = \{K_1, \dots, K_n\}$

The nerve of $\mathcal{K}$ is a simp. complex:

$$N(\mathcal{K}) = \{ S \subset [n] : \bigcap_{i \in S} K_i \neq \emptyset$$

Simplicial complexes, homology, related topology

Simplex is the convex hull of aff. ind. points, in a linear space.

(vertices of the simplex)

If K is a convex ~~body~~ set, then

$\dim K = \dim$ affine span of K.

$\dim$ Simplex $= |\text{vertices}| - 1$.

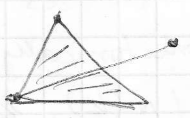
$\emptyset$

(-1)-dim simplex

S.C. K is a collection of simplices:

1) if $T \in K$ and S is a face of T, then $S \in K$.

2) If $T_1, T_2 \in K$, then $T_1 \cap T_2$ is the simplex spanned by $V(T_1) \cap V(T_2)$, i.e. e.g. this is not allowed:



The total space of K, denoted: $|K|$ is the union of all simplices. $|K|$ is a top. space, K is called a triangulation of $|K|$.

Abstract simp. complex K is a collection of subset of a vertex set V with the property that $S \in K$, $R \subset S \Rightarrow R \in K$. $(\emptyset \in K)$

ASC $\to$ GSC

$V = \{v_1, \dots v_n\}$, $\mathbb{R}^n$, $\{e_1, \dots e_n\}$ standard

$s \in K$, $s = \{v_{i_0}, \dots v_{i_k}\} \Rightarrow$ & conv $\{e_{i_0}, \dots e_{i_k}\}$

If dim K = max $\{$dim F : F face of $K\} = d$

choose $x_1, \dots x_n$ pts in general position in $\mathbb{R}^{2d+1}$

Comments: For d-dim manifolds you can replace $2d+1$ by $2d$ (Whitney trick)

K is 1-dimensional $\Rightarrow$ K is a graph

but $K \hookrightarrow \mathbb{R}^2$ is very rare.

(Lipton, Tarjan) For $V(G) = n$ planar G $\exists U \subset V(G)$: $|U| = \sqrt{n}$, and when we remove U, every connected component is of size $\le \frac{2}{3} \cdot n$.

i.e expanders are not planar

Question Is it the case that 2-dim expanders cannot be embedded into $\mathbb{R}^4$?

K sim complex $F \in K$,

$\{T \in K : T \supset F\}$, $\{T \setminus F : T \in F, T \supset F\}$

this is the link of F in K, denoted by link (F, K).

Star of F in K is $\{T \in K : T \supset F\}$

Anti-star of F in K $\{T \in K : T \not\supset F\}$

Star $(F, K) = F *$ link (F, K).

$\langle \delta f^+, \chi_{V^-} \rangle =$

$= \sum_{e \in \delta} f(v)$

$\langle \Delta f^-, \chi_{V^-} \rangle = \sum_{e \in \delta} f(v)$

$\langle \Delta f, f\chi_{V^-} \rangle =$

$= \lambda \cdot \langle f, \chi_v \rangle = \lambda \sum_{u \in V} f(u)$

$\langle \delta f^+, \delta f^+ \rangle -$

$- 2 \langle \delta f^+, \delta f^- \rangle$

Homology

K - sc $V(K) = [n] = \{1, 2, \dots n\}$

$R = \mathbb{Z}/2\mathbb{Z}$, chain complex ass. to K: $\bigoplus_{i}^{\infty} C_i(K)$

$C_i(K)$ = free R module spanned by i-faces of K.

traditionally $i = 0$

sometimes $i = -1$

Boundary operator

$\partial_i : C_i(K) \to C_{i-1}(K)$

$\partial T = \sum \{S : S \subset T,\ \dim S = \dim T - 1 = i-1\}$

$\partial_{i-1} \partial_i = 0$: sketch:

$$\begin{tikzcd} & T \arrow[dl, no head] \arrow[dr, no head] & \\ R\cup\{a\} \arrow[dr, no head] & & R\cup\{b\} \arrow[dl, no head] \\ & R & \end{tikzcd}$$

$\dim T = i$

$\dim R = i-2$ $\qquad R \cup \{a,b\} = T$

cycles $Z_i(K) = \{x \in C_i(K) : \partial_i x = 0\}$

boundaries $B_i(K) = \{x \in C_i(K) : x = \partial_{i+1} y,\ y \in C_{i+1}(K)\}$

$Z_i \subset B_i$

homology $\to$ $H_i(K) = Z_i / B_i$

For a general ring R there are two ways to define:

- ordering of vertices
- orientation

We'll choose the ordering way:

$T = \{j_0 < \dots < j_r\}$ — r-face

$\partial T := \sum_{l=0}^{r} (-1)^l \{j_0 \dots \hat{j_l} \dots j_r\}$

$\partial \partial T = 0$:

$$\begin{tikzcd} & j_0 \cdots j_r \arrow[dl, "(-1)^k"'] \arrow[dr, "(-1)^l"] & \\ j_0 \dots \hat{j_k} \dots j_r \arrow[dr, "(-1)^k \cdot (-1)^{l-1}"'] & & j_0 \dots \hat{j_l} \dots j_r \arrow[dl, "(-1)^k \cdot (-1)^{l+m}"] \\ & j_0\ \hat{j_k}\ \hat{j_l} \dots j_r & \end{tikzcd}$$

$k < l$

their sum is zero.

Given K, $H_i(K)$ is a topological invariant of K.

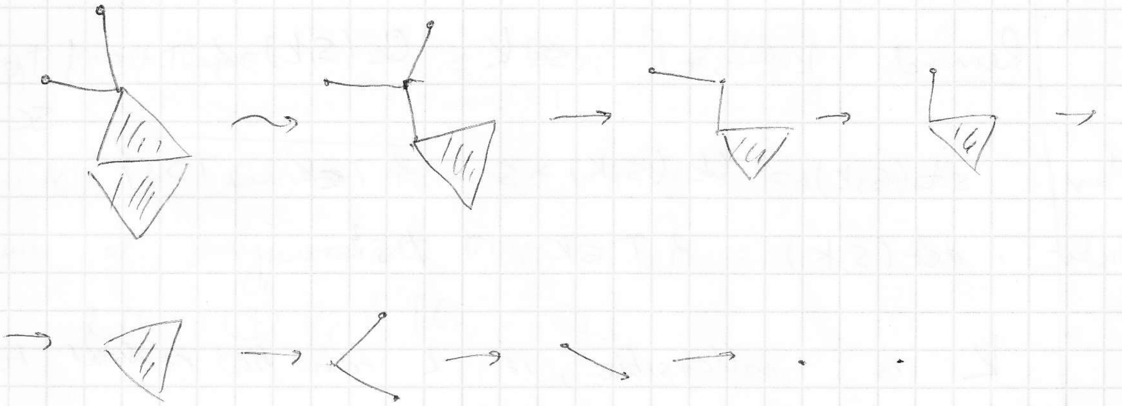
$f : X \to Y$ continuous induces $f_* : H_i(X) \to H_i(Y)$

X contractible : $H_i(X) = H_i(\cdot)$

Collapsability

A face F in s.c. K is free if it's contained in a unique max. face

Elementary collapse of K is the deletion of a free face of dim i and a max face containing it of dim i+1



K is called collapsible if it can be reduced to a point by a series of elem. collapses.

Graph is collapsible $\Leftrightarrow$ it is a tree

An example of a space which is contractible but not collapsible