18/03/14

K is a s.c. and an (a,b)-collapse step is the deletion from K of a free face S, included in a unique maximal face T, where $|S| = a$, $|T| = b$

If $a < b$, then this gives a contraction and can be replaced by a sequence of ordinary elem. collapse steps.

<u>Lemma</u> K - s.c. $S \in K$ $lk(S,K) = \{T \setminus S \mid T \in K, S \subset T\}$

star $st(S,K) = lk(S,K) * S = \{T \in K : T \supset S\}$

antistar $ast(S,K) = \{T \in K : T \not\supset S\}$

K is collapsible, if K can be reduced to $\emptyset$ by $(i,i-1)$ elementary collapses

K is d-collapsible if K can be reduced to $\emptyset$ by steps of type (a,b) for $a \le d$

K is shellable if K can be reduced to $\emptyset$ by collapse steps of type (a,d)

<u>Lemma</u> K is d-collapsible iff it can be reduced to it's $(d-1)$-skeleton by collapse ~~set~~ steps of type (d,a) when $a \geq d$

(in particular, it is homotopically equivalent to a subcomplex of $skel_{d-1}(K)$ and therefore $H_i(K) = 0$ $\forall i \geq d$)

<u>Lemma</u> If K is d-collapsible then $lk(S,K)$ is d-collapsible for every $S \in K$

Consequence: If K is d-collapsible then $H_i(K') = 0$ $\forall i \geq d$ and every K' which is the link of a face in K.

If K is shellable $S \in K$, $lk(S, K)$ is shellable

Face ring: K - s.c. on vertices $\{1, \dots, n\}$

$R[x_1, \dots x_n]$, I is an Ideal spanned by non-face $I = \langle \prod_{i \in T} x_i \;\; T \notin K \rangle$

$R(K)$ is Cohen-Macaulay if it's a direct sum of polynomial rings

$$R(K) = \bigoplus_{i=1}^{t} \eta_i R[\theta_1, \dots \theta_d]$$

Theorem (Wagner) Let $K_1, \dots, K_n$ be convex sets in $\mathbb{R}^d$, Let $N(F)$ be the nerve of F.

$$N(F) = \{ S \subset [n] : \bigcap_{i \in S} K_i \neq \emptyset \}$$

$N(F)$ is d-collapsible.

Pf We can assume, all K_i's are polytopes.

Denote $K[S] = \bigcap_{i \in S} K_i$

H is a hyperplane in generic position, i.e.

1) Supporting translate of H to any non-empty $K[S]$ support as a unique point

2) the same translate does not support two different $K[S]$'s in two different points.

Take translates of H. We look at first time H passes $K[S]$ for $|S| \leq d$.

We show that S is a free face of $N(F)$

k_+ : [illegible]

$-2 \leq \mu_{m}$

$f(v) \cdot \sum_{u \sim v} f(u) \geq$

$f(v) \cdot \sum_{u \in V} f(u)$

$\frac{a_1}{b_1} + \dots + \frac{a_n}{b_n} \geq$

$\geq \frac{a_1 + \dots + a_n}{b_1 + \dots + b_n}$

$\frac{a_1}{b_1} + \frac{a_2}{b_2} \geq$

$\geq \frac{a_1 + a_2}{b_1 + b_2}$

$a_1 b_2 (b_1 + b_2) + a_2 b_1 (b_1 + b_2) \geq$

$\geq a_1 b_1 b_2 + a_2 b_1 b_2$

$a_1 b_2^2 + a_2 b_2^2 \geq$

$k - \mu_{min}$

$\lambda_m \leq 2k - \rho$

$\beta - k \geq \mu_m$

$T := \{ i \text{ s.t. } K_i \cap K(S) \neq \emptyset \}$

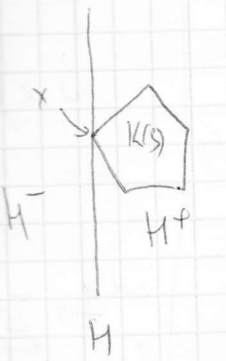
$T \overset{?}{\in} N$

<u>Claim</u> every K_i, $i \in T$ contains $x = H \cap K[S]$

$x_i = \langle \Delta x_A, x_B \rangle$

Suppose that $j \in T$, $j \notin S$, $x \notin K_j$, i.e.

$$H^- \cap \bigcap_{i \in S} K_i \cap K_j = \emptyset$$



By Helly's thm there are $d+1$ sets among $d+2$ with empty intersection

$\exists$ $R = S \setminus \{i\} \cup \{j\}$ s.t. $K(R) \cap H = \emptyset$

contradiction!

...

2.E

<u>Thm</u> If $K_1, \ldots K_n$ sets in $\mathbb{R}^2$ and from every five $\exists$ four with non-empty intersection.
Then $\exists$ two points x and y s.t. $\forall$ K_i contains or x or y.

<u>Pf</u> ...

<u>Fractional Helly Thm</u>

Let N be the nerve of convex sets $K_1, \ldots K_n$ in $\mathbb{R}^d$, $f_i(N) = \#$ i-faces of N,
if $f_d(N) \geq \alpha \cdot \binom{n}{d+1}$ then there is $T \in N$

$$|T| \geq n \frac{\alpha}{d+1}$$

<u>Proof</u>: $|F_i| = d$, $m \leq \binom{n}{d}$

$F_1, F_2, \ldots F_m$ collapse process $\sim$
$T_1 \quad T_2 \quad T_m$

$\rightsquigarrow$ no more d-faces

$$\sum_{i=1}^{m} (|T_i| - d) = f_d(N) \geq \alpha \cdot \binom{n}{d+1}$$

$$\frac{d!\,(n-d)!}{(d+1)!\,(n-d-1)!}$$

$$\max(|T_i| - d) \geq \frac{1}{m} \cdot \alpha \cdot \binom{n}{d+1} \geq \alpha \cdot \frac{\binom{n}{d+1}}{\binom{n}{d}} = \alpha \cdot \frac{n-d}{d+1}$$

$$\max |T_i| \geq \frac{\alpha}{\alpha+1} \cdot n$$

25.03.2014

- Pach's theorem
- Cohomology / HD Laplacian

$\mathcal{K} = \{K_1, \dots, K_n\}$ – convex sets in $\mathbb{R}^d$

$N(\mathcal{K})$ (– the nerve) $= \{S \subset [n] : \bigcap_{i \in S} K_i \neq \emptyset\}$

<u>Thm</u> $N(K)$ is d-collapsible

<u>Lemma</u> A s.c. is d-collapsible if it can be reduced into $skel_{d-1}(K)$ by a sequence of elementary collapse steps of type (d, m), where $m > d$.

<u>Corollary</u> $N(K)$ satisfies: $H_i(K', \mathbb{R}) = 0$ for every K' which is a link in $N(K)$ for every $i \geq d$.

This property is called „d-Leray property"

Helly's thm implies: if $f_d(K) = \#\{d \text{ faces of } K\} \geq$

$\geq \alpha \cdot \binom{n}{d+1} \Rightarrow \exists\, T \in K : |T| \geq n \cdot \beta, \ \beta = \frac{\alpha}{d+1}$

This is also true for d-Leray complexes.
(requires com. algeb. machinery)

Problem: Find an easy proof showing that $\beta(\alpha)$ exists for d-Leray complexes.

- d-Leray is equiv. to: $\forall\, i \geq d\ \forall\, L$-induced subcomplex of K: $H_i(L, \mathbb{R}) = 0$