$\rightsquigarrow$ no more d-faces

$$\sum_{i=1}^{m}(|T_i|-d) = f_d(N) \geq \alpha \cdot \binom{n}{d+1}$$

$$\frac{d! \, (n-d)!}{(d+1)!(n-d-1)!}$$

$$\max(|T_i|-d) \geq \frac{1}{m}\cdot \alpha \cdot \binom{n}{d+1} \geq \alpha \cdot \frac{\binom{n}{d+1}}{\binom{n}{d}} = \alpha \cdot \frac{n-d}{d+1}$$

$$\max |T_i| \geq \frac{\alpha}{d+1}\cdot n$$

25.03
2014

- Pach's theorem
- Cohomology / HD Laplacian

$\mathcal{K} = \{K_1, \dots, K_n\}$ – convex sets in $\mathbb{R}^d$

$N(\mathcal{K})$ (– the nerve) $= \{S \subset [n] : \bigcap_{i \in S} K_i \neq \emptyset\}$

Thm $N(K)$ is d-collapsible

Lemma A s.c. is d-collapsible if it can be reduced into $\text{skel}_{d-1}(K)$ by a sequence of elementary collapse steps of type (d, m), where $m > d$.

Corollary $N(K)$ satisfies: $H_i(K'; \mathbb{R}) = 0$ for every K' which is a link in $N(K)$ for every $i \geq d$.

This property is called „d-Leray property"

Helly's thm implies: if $f_d(K) = \#\{d \text{ faces of } K\} \geq$
$\geq \alpha \cdot \binom{n}{d+1} \Rightarrow \exists\, T \in K : |T| \geq n \cdot \beta, \; \beta = \frac{\alpha}{d+1}$

This is also true for d-Leray complexes.
(requires com. algeb. machinery)

Problem: Find an easy proof showing that $\beta(\alpha)$ exists for d-Leray complexes.

- d-Leray is equiv. to : $\forall i \geq d \; \forall L$-induced subcomplex of K: $H_i(L, \mathbb{R}) = 0$

Thm: Let K be a ~~family of~~ d-collapsible s.c. with n vertices. If $f_{d+r}(K) = 0$, then

$$f_d(K) \le \binom{n}{d+1} - \binom{n-r}{d+1}$$

$$p = 1 - (1-\alpha)^{\frac{1}{d+1}}$$

Thm: Let $A_1, \dots A_\ell$, $B_1, \dots B_\ell$ be collections of sets

1) $|A_i| = a$, $|B_i| = b$ $\forall i$
2) $A_i \cap B_i = \emptyset$ $\forall i$
3) $A_i \cap B_j \neq \emptyset$ $\forall i < j$

Then $\ell \le \binom{a+b}{a}$

(Equality: $|S| = a+b$, A_i all subsets of S, $B_i = S \setminus A_i$.)

Easier version: $|A_i| = a$, $|B_i| = b$

$A_i \cap B_i = \emptyset$ $\forall i$

$A_i \cap B_j \neq \emptyset$ $\forall i \neq j$

then $\ell \le \binom{a+b}{a}$

Proof suppose $\bigcup A_i \cup B_i = X$, $|X| = n$

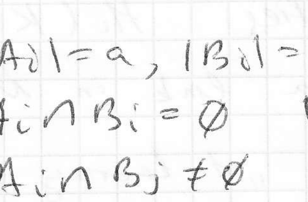
assume $X = [n]$

Count ~~permutations~~ pairs: (π, i)

$\pi \in S_n$, all element of A_i come before all elements of B_i in the ordering

claim for each π $\exists$ at most one i.

proof: suppose not



some element $A_j \cap B_i \neq \emptyset$

$B_j \cap A_i \neq \emptyset$ $\times$

$\#$ pairs $\le n!$

$\forall i$: how many (π, i) are there ?

$$\binom{n}{a+b}\cdot \underbrace{a!}_{\text{order } A_i}\cdot \underbrace{b!}_{\text{order } B_i}\cdot \underbrace{(n-(a+b))!}_{\text{order the rest}}$$

$$\Rightarrow \ell\cdot\binom{n}{a+b}\, a!\, b!\,(n-(a+b))! \le n! \qquad \ell \le \frac{n!}{\frac{n!}{(a+b)!\,\cancel{(n-(a+b))}!}}\cdot\frac{1}{a!\,b!} =$$

$$= \binom{a+b}{a}\cdot\cancel{(n-(a+b))}!$$

<u>Thm</u>: Let $V_1, \dots V_\ell$, $U_1, \dots U_\ell$ be subspaces of $\mathbb{R}^n$

1) $\dim V_i = a$, $\dim U_i = b$

2) $V_i \cap U_i = \{0\}$

3) $V_i \cap U_j \ne \{0\}$ $\quad i \ne j$

Then $\ell \le \binom{a+b}{a}$

If $n > a+b$ we can project to a generic hyperplane so we can assume $n = a+b$.

$W = \mathbb{R}^{a+b}$

ΛW — exterior algebra of W ; $\quad e_1, \dots e_n$ — basis
$S = \{i_1, \dots i_k\}$ $\quad i_1 < \dots < i_k$
$e_S = e_{i_1}\wedge \dots \wedge e_{i_k}$
$e_i \wedge e_j = -e_j \wedge e_i$

If $U \subset W$ and $\dim U = r$ and $u_1, \dots u_r$ – basis for U

$u_1 \wedge \dots \wedge u_r \in \Lambda W$

associate to V_i e_{V_i} and to U_i e_{U_i}

<u>Claim</u> $e_{U_1} \dots e_{U_\ell}$ are lin. ind. (This implies $\ell \le \dim \Lambda^b W = \binom{a+b}{a}$)

<u>Proof</u>: Assume not:

$$\sum \lambda_i e_{U_i} = 0$$

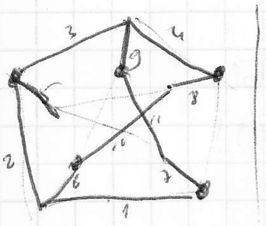
consider $\left(\sum \lambda_i e_{U_i}\right)\wedge e_{V_j} = \lambda_j (e_{U_j}\wedge e_{V_j}) = 0$ ↯

but $e_{V_i}\wedge e_{U_i} \ne 0$
$e_{V_i}\wedge e_{U_j} \ne 0$

$\boxed{\Sigma}$

Similar proof works for the stronger version of the thm for sets.

Now we are ready to prove the THM on collaps.

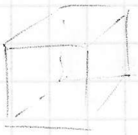


$\frac{11}{5}$

$|E| = 5+5+5 = 15$

$(-1)^4 \; 1^5 \; 3^1$

$0^1 \; 2^5 \; 4^4$



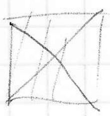
K - d-Leray over $\mathbb{Q}$.

$H_i(K') = 0 \quad i \geq d$, where K' - link in k

and $\dim K = d+r-1$

$$f_{d+r-1} = \binom{n-r}{d}$$

(Analog of trees)



$\langle \Delta(f - \chi_+), (f+\chi) \rangle =$

$= \langle \Delta f, f \rangle + \langle \Delta \chi, \chi_+ \rangle$

$+ \langle \Delta f, \chi_+ \rangle - \langle \Delta \chi_+, f \rangle$

$2 \cdot \lambda \cdot \sum f(v)$

$(f(v) - f(u))^2 = \left(\frac{1}{a}\right)^2$

$-4 \cdot \sum (f(v) - f(u)) + \sum_E \frac{1}{a}$

1

α