Thm Let K be a d-collapsible s.c. with 01/04/19

$f_{d+r}(K) = 0$, then $f_{d+i}(K) \le f_{d+i}(\mathcal{N}(\mathcal{F}))$, where

$\mathcal{F}$ is the following family of convex sets on $\mathbb{R}^d$: r copies of $\mathbb{R}^d$ and $(n-r)$ hyperplanes in general position.

Thm Same conclusion holds for d-Leray complexes (over any field)

Reminder: K is d-Leray if for every $s \in K$ (including $s = \emptyset$), $H_i(\mathrm{lk}(s,K), F) = 0$ for $i \geq d$

($\Leftrightarrow$ $H_i(\hat{K}, F) = 0$ for every induced subcomplex and $i \geq d$)

Conseq. of proof: If equality holds for some $i \geq 0$ then it holds for all $i \geq 0$ and $f_{d-1}(K) = \binom{n}{d}$

If $r = 1$, $d = 1$

1-dim s.c. K which ~~at~~ is 1-collapsible and $f_1(K) = n-1$ is a tree, so

we have a 2-parameter family of s.c. which generalizes trees.

If d arb. and $r = 1$ K - s.c. of dim d, $H_d(K) = 0$

$f_d(K) = \binom{n-1}{d}$

Thm $\displaystyle\sum_{\substack{d\text{-dim s.c. } K \\ \text{on } n \text{ vertices} \\ \text{s.t. } f_d(K) = \binom{n-1}{d} \\ H_d(K, \mathbb{Z}) = 0}} |H_{d-1}(K)|^2 = n^{\binom{n-2}{d}}$

$\hookrightarrow$

For 2-parameter family, $d_m = d e r - 1$, we want to count (d,r)-trees we "root" every (d,r)-tree with $(d-1, r)$ subtree

$\sqrt{\frac{k \cdot \lambda}{2}}$

Statement # trees on n labelled vertices is n^{n-2}

$\sqrt{\frac{3\cdot 5}{2}} \vee \frac{11}{5}$

Pf $T_n(d_1,\dots,d_n)$ = # trees where vertex i has degree d_i

$\frac{3\cdot5}{2} \vee \frac{11^2}{25}$

$135 \vee \frac{2}{3} \cdot 11$

$\sum d_i = 2n-2$, assume $d_k = 1$

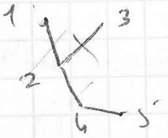
$$T_n(d_1,\dots,d_n) = T_{n-1}(d_1-1, d_2,\dots,\hat{d_k},\dots,d_n) + \dots$$
$$\dots + T_{n-1}(d_1,\dots,\hat{d_k},\dots,d_n-1)$$

Then $T_n(d_1,\dots,d_n) = \binom{n-2}{d_1-1,\dots,d_n-1}$

$$\sum_{\substack{d_1,\dots,d_n \\ \sum d_i = 2n-2 \\ d_i \geq 1}} \binom{n-2}{d_1-1,\dots,d_n-1} = (1+\dots+1)^{n-2}$$

∎

Pf Start with a tree, associate to it a sequence of length $n-2$ of integers:



delete a leaf with the smallest label and write the label of it's neighbor

$(2,2,4,5)$ ∎

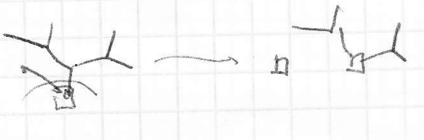
Pf T_n = # rooted trees

F_n = # rooted forests

$T(x) = \sum \frac{T_n x^n}{n!}$, $F(x) = \sum \frac{F_n x^n}{n!}$ =

$F(x) = e^{T(x)}$, since $F(x) = T(x) + \frac{T(x)^2}{2} + \frac{T(x)^3}{3!} + \dots$ ∎

rooted tree ⟷ rooted forest

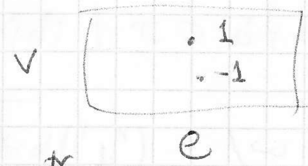


$\Rightarrow$ $x \times F(x) = T(x)$

$$T(x) = 1 + x e^{T(x)}$$

...

Pf G - graph, $I(G)$ is a $v \times e$ matrix representation ∂:



$L(G) = I(G) \cdot I(G)^{tr}$

$\tilde{I}(G)$ incidence matrix where row corresponding 'n' is deleted

$\tilde{L}(G) = \tilde{I}(G) \cdot \tilde{I}(G)^{tr}$.

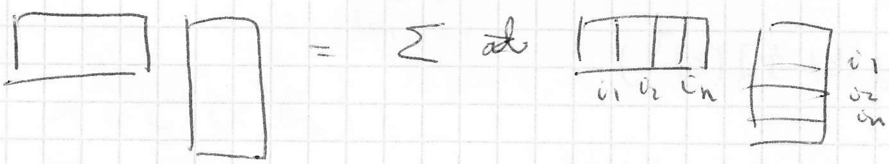
Matrix - tree theorem: $\det \tilde{L}(G) = \#$ spanning trees of G

__Cauchy - Binet thm:__

Let $A \in M_{n \times m}$, $B \in M_{m \times n}$

$$\det(AB) = \sum \det(C)\det(D)$$

C is $n \times n$
D $n \times n$
cut with the same indices



Apply this thm to $\tilde{L}(G) = \tilde{I}(G)\, \tilde{I}(G)^{tr}$.

$$\tilde{I}(K_n) = {\scriptstyle n-1}\left\{\overbrace{\begin{pmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{pmatrix}}^{n-1}\right.$$

complete graph on n vertex

$$\det(\tilde{L}(K_n)) = 1 \cdot n^{n-2} = \#\text{ trees}$$

K - s.c. d-dim on n vertices

$I_d(K)$

$$\begin{array}{c c} & t \quad |t| = n \\ S & \boxed{\begin{matrix} +1 \\ -1 \end{matrix}} \end{array}$$

$|S| = t$

$F \subset K$ subcomplex w/ $\binom{n-1}{d}$ faces

a) $\det \tilde{I}_d(F) = 0$ if $H_d(F) \neq 0$

b) $\det \tilde{I}_d(F) = |H_{d-1}(K)|$ if $H_d(F) = 0$

Basic fact (Euler-Poincare formula)

$$\sum (-1)^i \dim H_i(K, F) = \sum (-1)^i f_i(K)$$

$$H_{d-1}(F) = \mathbb{Z}^{\binom{n}{d}} \Big/ \tilde{I}_d(F)\, \mathbb{Z}^{\binom{n-1}{d}} \rightarrow$$

$$\Rightarrow \det\left(\tilde{I}_d(K)\, \tilde{I}_d(K)^T\right) = \sum_{\substack{F \subset K \\ f_d(F) = \binom{n-1}{d} \\ H_d(F) = 0}} |H_{d-1}(F)|^2$$

Assume $K = K_n^d$

$\det\left(\tilde{I}_d(K)\, \tilde{I}_d(K)^T\right)$