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Let X - finite s.c, pure, i.e. all max. faces are of dim d. For $\sigma \in X(i) = i$-cells denote

$$c(\sigma) = |\{\tau \in X(d) \mid \tau \supseteq \sigma\}|$$

$$\omega(\sigma) = \frac{c(\sigma)}{\binom{d+1}{i+1} \cdot |X(d)|}$$

Lemma $\sum_{\sigma \in X(i)} \omega(\sigma) = 1.$

Proof

$$\sum_{\sigma \in X(i)} \omega(\sigma) = \frac{1}{\binom{d+1}{i+1} \cdot |X(d)|} \sum_{\sigma \in X(i)} c(\sigma) = \frac{1}{\binom{d+1}{i+1} \cdot |X(d)|} \cdot \sum_{\substack{(\sigma,\tau) \\ \sigma \in X(i) \\ \tau \in X(d) \\ \sigma \subset \tau}} 1$$

$$= \frac{1}{\binom{d+1}{i+1} |X(d)|} \sum_{\tau \in X(d)} \cdot \binom{d+1}{i+1} = 1$$

number of i-cells in τ ∎

co-boundary

$$\delta_i : C^i(X, \mathbb{F}_2) \to C^{i+1}(X, \mathbb{F}_2)$$

$$C^i(X,\mathbb{F}_2) = \{f : X(i) \to \mathbb{F}_2\}$$

$$\delta_i(f)(\tau) = \sum_{\substack{\sigma \subset \tau \\ \dim \sigma = i}} f(\sigma) \quad , \tau \in X(i+1)$$

Notation

$$\alpha \in C^i(X, \mathbb{F}_2)$$

$$|\alpha| = |\{\sigma \in X(i) \mid \alpha(\sigma) \neq 0\}|$$

$$\|\alpha\| = \sum_{\sigma \in \alpha} \omega(\sigma)$$

Definition (Linial-Meshulam / Gromov) i-Expansion of X is defined for ~~even~~ each $0 \le i \le d-1$ as

$$\min \left\{ \frac{\|\delta_i \alpha\|}{\|[\alpha]\|} \;\middle|\; \alpha \in C^i(X,\mathbb{F}_2) \setminus \underset{\mathrm{Im}(\delta_{i-1})}{B^i(X,\mathbb{F}_2)} \right\}$$

where $[\alpha] = \alpha + B^i(X, \mathbb{F}_2)$ and

$$\|[\alpha]\| = \min\{\|\alpha'\| \mid \alpha' \in [\alpha]\} =$$

$$= \min\{\|\alpha + \delta_{i-1}(\gamma)\| \mid \gamma \in C^{i-1}(X, \mathbb{F}_2)\}$$

Remarks: 1) $\varepsilon_i > 0 \iff H^i(X, \mathbb{F}_2) = 0$

$$\Updownarrow$$

$$Z^i(X, \mathbb{F}^2) = B^i(X, \mathbb{F}^2)$$

$$\Updownarrow$$

$\forall\ \alpha \in C^0 \setminus B^i$

$\delta(\alpha) \neq 0 \Leftrightarrow$

$\Leftarrow \|\delta(\alpha)\| = 0 \Leftrightarrow \varepsilon_i > 0$

2) If X is a k-reg graph, $X = (V, E)$,

$|V| = n$, $|E| = \frac{nk}{2}$

$\forall\ v \in V$, $c(v) = k$, $\omega(v) = \dfrac{k}{\binom{2}{1}|E|} = \dfrac{k}{2 \cdot \frac{nk}{2}} = \dfrac{1}{n}$

$\forall\ e \in E$, $c(e) = 1$, $\omega(v) = \dfrac{1}{\binom{2}{2}|E|} = \dfrac{2}{kn}$

$C^0 = \mathbb{F}_2$-funct on V = subsets of V

$B^0 = \mathrm{Im}(\delta_{-1}) = \{\emptyset, V\}$, since

$\delta_{-1} : C^{-1}(X, \mathbb{F}_2) \to C^0(X, \mathbb{F}_2)$

$X(-1) = \{\emptyset\}$

Hence, $\alpha + B^0 = \{\alpha, \underbrace{\bar{\alpha}}_{\alpha + \mathbb{1}}\} \Rightarrow \|[\alpha]\| = \min\left\{\frac{\|\alpha\|}{n}, \frac{n - \|\alpha\|}{n}\right\}$

$= \min\left\{\dfrac{|\alpha|}{|V|}, \dfrac{|V \setminus \alpha|}{|V|}\right\}$

ε_0: $\delta_0(\alpha)(e) = \alpha(e^+) + \alpha(e^-) \Rightarrow$

$\Rightarrow |\delta_0(\alpha)| = |E(\alpha, \bar{\alpha})|$, therefore

$$\varepsilon_0 = \min_{\alpha \neq \emptyset, V} \left\{ \frac{|E(\alpha, \bar{\alpha})|}{|E|} \cdot \frac{|V|}{\min\{|\alpha|, |\bar{\alpha}|\}} \right\} :$$

$$= \frac{2}{k} \min_{\alpha \neq \emptyset, V} \left\{ \frac{|E(\alpha, \bar{\alpha})|}{\min\{|\alpha|, |\bar{\alpha}|\}} \right\} = \frac{2}{k} h(X),$$

Cheeger constant

Def A family of s.c. (pure) of dim. d is called a family of (co-boundary-) expanders is $\exists\ \varepsilon > 0$ s.t. $\forall X\ \forall\ i = 0, \dots d-1 : \ \mathcal{E}_i(X) \geq \varepsilon$.

Statement $\Delta = \Delta_n^{(2)}$ = complete 2-s.c. is an expander

$V = \{1, \dots n\}$ $\quad \omega(v) = \frac{1}{n}$

$E = \binom{[n]}{2}$ $\quad \omega(e) = \frac{1}{\binom{n}{2}}$

$T = \binom{[n]}{3}$ $\quad \omega(t) = \frac{1}{\binom{n}{3}}$

Proof $\mathcal{E}_0(\Delta) = \min\left\{ \frac{\| \delta_0(\alpha) \|}{\| [\alpha] \|} \,\middle|\, \alpha \in C^0 \setminus B^0 \right\} =$

$$= \min_{0 < |\alpha| \leq \frac{|V|}{2}} \frac{|E(\alpha, \bar{\alpha})| / |E|}{|\alpha| / |V|} = \frac{|V|}{|E|} \min \frac{|\alpha| \cdot (n - |\alpha|)}{|\alpha|} =$$

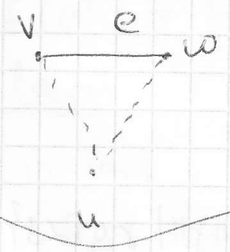
$$= \frac{|V|}{|E|} \cdot \min_{0 < |\alpha| \leq \frac{|V|}{2}} (|V| - |\alpha|) = \frac{n}{\frac{n(n-1)}{2}} \cdot \frac{n}{2} = \frac{n}{n-1} \geq 1$$

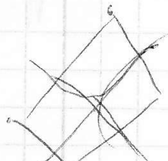
$\mathcal{E}_1(\Delta)$:

$\alpha \in C^1,\ \alpha \subseteq E,\ e \in E,\ \underset{u \notin e}{u \in V} \Rightarrow ue$ = the only triangle defined by them

$u \in V$: $\quad \alpha_u(v) = \begin{cases} \alpha(uv) & v \neq u \\ 0 & , u = v \end{cases}$

(*) $\quad (\alpha + \underbrace{\delta_0(\alpha_u))(e)}_{\parallel} = \begin{cases} \delta_1(\alpha)(ue), & u \notin e \\ 0, & u \in e \end{cases}$





$\delta_0(\alpha_u)(e) = \alpha(uv) + \alpha(uw)$

$= \alpha(vw) + \alpha(uv) + \alpha(uw) = \delta_1(\alpha)(ue)$

$$3\, |\delta_1(\alpha)| = |\{ (\overset{\text{vertex}}{u}, \underset{\text{triangle}}{t}) \mid u \in t,\ t \in \delta_1(\alpha) \}| =$$

(*) $$\geq = |\{ (\overset{\text{vertex}}{u}, \underset{\text{edge}}{e}) \mid u \notin e,\ e \in \alpha + \delta_0 \alpha_u \}| =$$

$$= \sum_{u \in V} |\alpha + \delta_0 \alpha_u| \geq n \cdot \| [\alpha] \| \Rightarrow 3 |\delta_1(\alpha)| \geq n \cdot |[\alpha]|$$

$$3 \cdot \binom{n}{3} \frac{|S_1(\alpha)|}{\binom{n}{3}} \geq n \cdot \binom{n}{2} \frac{|[\alpha]|}{\binom{n}{2}} \Rightarrow$$

$$\Rightarrow 3 \cdot \binom{n}{3} \cdot \| S_1(\alpha) \| \geq n \cdot \binom{n}{2} \cdot \| [\alpha] \|$$

$$\cancel{3} \cdot \frac{n \cdot (n-1)(n-2)}{\cancel{1 \cdot 2 \cdot 3}} \cdot \| S_1(\alpha) \| \geq n \cdot \frac{n \cdot n-1}{2} \cdot \| [\alpha] \|$$

$$\frac{\| S_1(\alpha) \|}{\| [\alpha] \|} \geq \frac{n}{n-2}$$

$$\Downarrow$$

$$\varepsilon_1(\Delta) \geq \frac{n}{n-2} \qquad \blacksquare$$

20/05/14 X - pure s.c. of dim d, $\quad \delta_i : C^i(X, \mathbb{F}_2) \to C^{i+1}(X, \mathbb{F}_2)$

$$\delta_i(f)(\tau) = \sum_{\substack{\sigma \subset \tau \\ \sigma \in X(i)}} f(\sigma)$$

$$B^i = \operatorname{Im}(\delta_{i-1})$$

<u>Def</u> Expansion: $\quad \varepsilon_i(X) = \min \left\{ \frac{\| \delta_i(f) \|}{\| [f] \|} \;\middle|\; f \in C^i \setminus B^i \right\}$

$\|\cdot\|$: $\quad \sigma \in X(i)$, $\quad c(\sigma) = \# \{ \tau \in X(d) \mid \sigma \subset \tau \}$

$$\omega(\sigma) = \frac{c(\sigma)}{\binom{d+1}{i+1} |X(d)|}$$

Property: $\sum_{\sigma \in X(i)} \omega(\sigma) = 1$. $\quad \blacksquare$

$\alpha \in C^i$, $\quad \|\alpha\| = \sum_{\sigma \in \alpha} \omega(\sigma)$, $\quad [\alpha] = \alpha + B^i$,

$\| [\alpha] \| = \min \{ \|\gamma\| \mid \gamma \in [\alpha] \}$

<u>Thm</u> (LM) $X = \Delta_n^{(2)}$ = the complete 2-dim comp on n vertices. Then $\varepsilon_1(X) \geq 1 + o(1)$

<u>Property testing</u>

<u>Def</u> <u>(q, ε)-testability</u>: