$$3 \cdot \frac{\binom{n}{3} |S_1(\alpha)|}{\binom{n}{3}} \geq n \cdot \binom{n}{2} \frac{|[\alpha]|}{\binom{n}{2}} \Rightarrow$$

$$\Rightarrow 3 \cdot \binom{n}{3} \cdot \|S_1(\alpha)\| \geq n \cdot \binom{n}{2} \cdot \|[\alpha]\|$$

$$\cancel{3} \cdot \frac{n \cdot (n-1)(n-2)}{\cancel{1 \cdot 2 \cdot 3}} \cdot \|S_1(\alpha)\| \geq n \cdot \frac{n \cdot n-1}{2} \cdot \|[\alpha]\|$$

$$\frac{\|S_1(\alpha)\|}{\|[\alpha]\|} \geq \frac{n}{n-2}$$

$$\Downarrow$$

$$\varepsilon_1(\Delta) \geq \frac{n}{n-2} \qquad \blacksquare$$

20/05/14 X - pure s.c. of dim d, $\quad \delta_i : C^i(X, \mathbb{F}_2) \to C^{i+1}(X, \mathbb{F}_2)$

$$\delta_i(f)(\tau) = \sum_{\substack{\sigma \subset \tau \\ \sigma \in X(i)}} f(\sigma)$$

$$B^i = Im(\delta_{i-1})$$

<u>Def</u> Expansion: $\quad \varepsilon_i(X) = \min \left\{ \frac{\|\delta_i(f)\|}{\|[f]\|} \,\middle|\, f \in C^i \setminus B^i \right\}$

$\|\cdot\|$: $\quad \sigma \in X(i)$, $\quad c(\sigma) = \# \{\tau \in X(d) \mid \sigma \subset \tau\}$

$$\omega(\sigma) = \frac{c(\sigma)}{\binom{d+1}{i+1} |X(d)|}$$

<u>Property</u>: $\displaystyle\sum_{\sigma \in X(i)} \omega(\sigma) = 1$. $\quad \blacksquare$

$\alpha \in C^i$, $\quad \|\alpha\| = \displaystyle\sum_{\sigma \in \alpha} \omega(\sigma)$, $\quad [\alpha] = \alpha + B^i$,

$\|[\alpha]\| = \min\{\|\gamma\| \mid \gamma \in [\alpha]\}$

<u>Thm</u> (LM) $X = \Delta_n^{(2)}$ = the complete 2-dim comp on n vertices. Then $\varepsilon_1(X) \geq 1 + o(1)$

<u>Property testing</u>

<u>Def</u> <u>(q, ε)-testability</u>:

Let A be a finite set, $P_n \subseteq A^n$, we say that the membership of $\alpha \in A^n$ in P_n is ~~(q,ε)~~ -testable if $\exists$ $\varepsilon > 0$ and $q \in \mathbb{N}$, and a (randomized) algorithm called tester s.t. the tester reads only q entries of α and answers Yes if $\alpha \in P_n$, and answers NO with probability at least $\varepsilon \cdot \mathrm{dist}(\alpha, P_n)$ if $\alpha \notin P_n$

Thm (B-L-R) Linearity is testable:

Let $V = \mathbb{F}_2^m$, $\underbrace{\{f : V \to \mathbb{F}_2\}}_{\text{all fun}} \simeq \mathbb{F}_2^{(2^m) = n}$

$P^n = \{f : V \to \mathbb{F}_2 \mid f$ is linear$\}$. The alg. checking $x, y \in V$ and answering Yes if $f(x) + f(y) = f(x+y)$ and No otherwise is $(3, \frac{2}{9})$-testable

"Baby-version" $[n] = \{1, \dots, n\}$, $f : [n] \to \mathbb{F}_2$, is f constant.

Assume X is an ε-expander on the set of vertices $[n]$

Test Pick edge e of X and check whether $f(e^+) \stackrel{?}{=} f(e^-)$. If Yes answer YES, if no, NO.

Claim this is $(2, \varepsilon)$- ~~expander~~ tester.

(in fact, This is $(2,\varepsilon)$-tester iff X is ε-expander.)

Proof: exercise.

Cocycle tester

X is a d-dim pure simpl. comp, $\alpha \in C^i(X, \mathbb{F}_2)$

The i-cocycle tester to check if $\alpha \in B^i$

Pick randomly $\tau \in X(i+1)$ with prob $\omega(\tau)$
and if $\delta_i(\alpha)(\tau) = 0$ answer Yes
otherwise, NO

Theorem ~~If ε_i~~ This is an $(i+2, \varepsilon_i(X))$-tester

If $\alpha \in B^i \rightsquigarrow$ Yes 100%

$\text{dist}(\alpha, B^i) = \frac{1}{|X(i)|}$ Hamming dist $(\alpha, B^i) =$

$= \frac{1}{|X(i)|} \min \{ \text{Hamming dis}(\gamma) \mid \gamma \in \alpha + B^i \} =$

$= \text{dist}(\alpha, B^i)$

If $\text{dist}(\alpha, B^i) = d(\alpha)$, then

$\text{Prob}\{ \delta_i(\alpha)(\tau) \neq 0 \} = \frac{\# \{ \tau \mid \delta_i(\alpha)(\tau) \neq 0 \}}{|X(i+1)|} =$

$= \| \delta_i(\alpha) \|$

In our complex $\varepsilon_i(X) \overset{\text{def}}{=} \min_{\text{dist}} \left\{ \frac{\| \delta(\alpha) \|}{\| [\alpha] \|} \mid \alpha \in C^i \setminus B^i \right\}$

$\forall \alpha \notin B^i$, Prob of NO $= \| \delta_i(\alpha) \| \geq \varepsilon_i(X) \cdot \text{dist}(\alpha, B^i)$.

Ex: Let Y be the complete graph on n vertices, i.e. $Y = K_n$. Given $f: V \to \mathbb{F}_2$, denote

$\tilde{f}: E \to \mathbb{F}_2$ the function given by $\tilde{f}(e) = f(e^+) + f(e^-)$

Now Given a function $g: E \to \mathbb{F}_2$, is $g = \tilde{f}$ for some f?

Prop 1 a) $g = \tilde{f}$ for some f iff $\forall x, y, z \in V$:

(*) $g(xy) + g(yz) + g(zx) = 0$

Proof Let $X = \Delta_n^2$, $\{ \tilde{f} \mid f \in \text{Func}(V, \mathbb{F}_2) \} = B^1$

and cond (*) means $g \in Z^1$. $H^1(\Delta_n^2) = 0$,

so $g \in Z^1 \Leftrightarrow g \in B^1$ ☒

Prop 1.b Checking (*) on $g \in \mathrm{Func}(E, \mathbb{F}_2)$ is a $(3,1)$ tester for g being equal to $\tilde{f}$ for some f

Proof by the prev. thm. (Δ^2_n is 1-expander)

Tensor power question

Given $\alpha \in \{\pm 1\}^n$. Let $A = \alpha \otimes \alpha$, i.e. A is $n \times n$ matrix s.t. $A_{ij} = \alpha(i) \cdot \alpha(j)$, so it's ± 1, symmetr, and diag $= 1$.

Q: Given such a A, is $A = \alpha \otimes \alpha$ for some $\alpha \in \{\pm 1\}^n$?

Prop: 1) $A = \alpha \otimes \alpha$ iff $\forall (i,j,k) \in \binom{[n]}{3}$

(*) $A_{ij} A_{jk} A_{ki} = 1$

2) (*) is $(3,1)$ - tester for A to be $\alpha \otimes \alpha$

Pf it's equiv to the previous one. $\{\pm 1\} \sim \{0,1\}$

Pach's thm $P_1, \ldots P_{d+1}$ sets of n pts (in general position), $|P_i| = n$ $\forall i$.

$\exists$ constant $C_d > 0$ s.t.

in $\mathbb{R}^d$, $Q_1, \ldots Q_{d+1}$, $Q_i \subseteq P_i$, $|Q_i| \geq C_d |P_i|$ and a point $O \in \mathbb{R}^d$, such that for every choice $q_i \in Q_i$

$O \in \mathrm{conv}\{q_1, \ldots, q_{d+1}\}$ ($O \in \mathrm{Int\,Conv}(q_1, \ldots, q_{d+1})$)

We use: 1) Colorful Tverberg

2) Fractional Helly

3) (version) of Szemeredi Regularity Lemma

4) Ham sandwich thm

5) sep. of sets