High Dimensional Expanders - Homework 2

Due: December 31, 2018

Instructions: You are welcome to work and submit your solutions in pairs. We prefer that you please type your solutions using LaTex. Please email your solution to `yotamd@weizmann.ac.il`.

Remark Unless stated otherwise, all simplicial complexes in the exercise are pure, and equipped with a distribution on the top-level faces.

1 From One-Sided HDXs to Two-Sided HDXs

In class we talked about the fact that constructing bounded-degree families of two-sided HDXs is a difficult task. In this exercise we show how to obtain a two-sided HDX from a one-sided HDX. More specifically, In this exercise we show we can easily construct a two-sided HDX from a one-sided HDX, by moving to the k-skeleton. Let X be a d-dimensional simplicial complex, and let $k < d$. Recall that the k-skeleton of X is the simplicial complex obtained by removing all faces of dimension $\geq k+1$, that is:

$$X^{(k)} = \bigcup_{j=-1}^{k} X(j).$$

The distributions $(\pi_k, ..., \pi_0)$ on the different levels on the complex $X^{(k)}$ are the same as in the original simplicial complex X. Note that even when π_d was uniform, π_k is *not necessarily uniform.*

We shall prove that if X is a λ-one-sided link expander, then its k-skeleton is a λ'-two-sided link expander, where $\lambda' = \max\{\lambda, \frac{1}{d-k+1}\}$.

1. Recall that in class, we proved that for every 2-dimensional simplicial complex X and $\lambda < 1$. If

   (a) Its 1-skeleton is connected.

   (b) For every $v \in X(0)$, it holds that X_v is a λ-one-sided spectral expander. In other words, if the eigenvalues of the adjacency operator A_v are

   $$\lambda_1^v \geq \lambda_2^v \geq ... \geq \lambda_{k_v}^v,$$

   then $\lambda_1^v \leq \lambda$.

   Then the 1-skeleton of X is a $\lambda' = \frac{\lambda}{1-\lambda}$ one-sided spectral expander.

   We now prove a similar bound but from the opposite direction. Let X be a 2-dimensional simplcial complex, and assume that for every $v \in X(0)$, all the eigenvalues of the adjacency operator of X_v are bounded by λ from below. I.e. if the eigenvalues of the adjacency operator A_v are

$$\lambda_1^v \geq \lambda_2^v \geq ... \geq \lambda_{k_v}^v$$

then for any $1 \leq i \leq k_v$, $\lambda_i^v \geq \lambda$. Show that all the eigenvalues of the one-skeleton are bounded from below by $\frac{\lambda}{1-\lambda}$.

Remark Recall that any graph without selfloops has a negative eigenvalue. For $\lambda < 0$, $\lambda < \frac{\lambda}{1-\lambda}$, so this means that the lower bound actually improves compared to the top-level.

2. Let $k < d$ be positive integers. Recall that a k-skeleton of a d-dimensional simplicial complex X is $X' = \bigcup_{i=-1}^{k} X(i)$. Deduce from the previous item, that for for any d-dimensional simplicial complex X, the eigenvalues of the links of the k-skeleton are lower bounded by $-\frac{1}{d-k+1}$.

   Note that in this item we didn't need any expansion properties for this item!

   Remark This bound is tight. For example, consider the *complete d-partite* complex - the clique-complex obtained from the d-partite graph, with uniform probability on $X(d)$. The underlying graph of the link of a face $\sigma \in X(k)$ is the complete $(d-k)$-partite graph. This graph has a negative eigenvalue of $\frac{-1}{d-k+1}$.

2 Upper and Lower Walks in the Complete Complex

This question analyzes the expansion of the upper walk in the complete complex. In part A we give a lower bound on the second eigenvalue of the upper walk, and in part B we show that this lower bound is tight in the complete complex, and find a combinatorial description to the eigenspaces of DU.

Recall that when we consider the upper walk on the edges, we actually consider a graph $G = (V = X(1), E = E_{X(2)})$ where we connect two elements $e_1, e_2 \in X(1)$ if their union is contained a triangle in $X(2)$.

Part A:

1. Let X be the 2-dimensional complete complex. Show that the second eigenvalue of $D_{\searrow 1} U_{\nearrow 2}$ is at least $\frac{1}{2} - o_n(1)$. Find an explicit vector $f \in \ell_2(X(1))$ s.t.

$$\frac{\langle D_{\searrow 1} U_{\nearrow 2} f, f \rangle}{\langle f, f \rangle} \geq \frac{1}{2} - o(1)$$

   Hint: Consider the indicator for star around a vertex $v \in X(0)$, i.e. the set

$$E_v = \{e \in X(2) : v \in e\}.$$

   What is the probability of choosing an $e' \notin E_v$ when taking a step in the upper walk, given that you begin with some edge $e \in E_v$?

   How does this connect the following expression?

$$\frac{\langle D_{\searrow 1} U_{\nearrow 2} \mathbf{1}_{E_v}, \mathbf{1}_{E_v} \rangle}{\langle \mathbf{1}_{E_v}, \mathbf{1}_{E_v} \rangle}$$

2. Generalize this to the d-dimensional complete complex and $D_{\searrow d-1}U_{\nearrow d}$. That is, Show that the second eigenvalue of $D_{\searrow d-1}U_{\nearrow d}$ is at least $\left(1-\frac{1}{d+1}\right)-o_n(1)$. Find an explicit vector $f\in\ell_2(X(d-1))$ s.t.

$$\frac{\langle D_{\searrow d-1}U_{\nearrow d}f,f\rangle}{\langle f,f\rangle}\geq\left(1-\frac{1}{d+1}\right)-o(1)$$

3. (bonus) is this bound true for an arbitrary simplicial complex on n vertices?

Part B:

1. Let X be the 2-dimensional complete complex on n vertices, and let $f:X(1)\to\mathbb{R}$ be some real valued function on the edges. Show that we can decompose f to three parts:

$$f(x)=U_{\nearrow 0}U_{\nearrow 1}f^{=-1}+U_{\nearrow 1}f^{=0}+f^{=1},$$

where

- $U_{\nearrow 0}U_{\nearrow 1}f^{=-1}$ is constant.
- $D_{\searrow -1}f^{=0}=0$.
- $D_{\searrow 0}f^{=1}=0$.

Show this decomposition is orthogonal.

Hint: Use the well-known fact from linear algebra that if U is adjoint to D ($U=D^*$) then $(kerD)^{\perp}=ImU$.

2. Recall that

$$D_{\searrow 1}U_{\nearrow 2}=\frac{1}{3}I+\frac{2}{3}M_1^+,$$

where M_1^+ is the non-lazy upper walk. Show that in the complete complex the following identity is also true:

$$U_{\nearrow 1}D_{\searrow 0}=\frac{1}{n-1}I+\frac{n-2}{n-1}M_1^+,$$

where M_1^+ is the same non-lazy operator. Conclude that

$$D_{\searrow 1}U_{\nearrow 2}=\left(\frac{1}{3}-\frac{3}{4(n-2)}\right)I+\left(\frac{2(n-1)}{3(n-2)}\right)U_{\nearrow 1}D_{\searrow 0}.$$

3. Use the previous item to show that $U_{\nearrow 0}U_{\nearrow 1}f^{=-1}, U_{\nearrow 1}f^{=0}$ and $f^{=1}$ are eigenvectors of $D_{\searrow 1}U_{\nearrow 2}$ (when they are different from 0). Calculate their respective eigenvalues. In this item you may assume that n is large, and calculate the eigenvalues asymptotically.