Homework 1 high dimensional expanders

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You may work in groups of 2-3 students. Please write up your homework in \LaTeX{} and submit the pdf by 20.12.2022 to yotamd@weizmann.ac.il. Please feel free to approach Yotam or Irit with any questions.

1 The sparse random complex

Let \( X(0) \) be a set of \( n \) vertices, and let \( d \) be a fixed parameter. A triangle is a set of three distinct vertices in \( X(0) \). For each \( v \in X(0) \) let us choose \( d \) triangles containing \( v \), uniformly and independently. Let \( X(2) \) be the collection of at most \( nd \) triangles chosen in this way. Let \( X(1) \) be all edges that participate in a triangle of \( X(2) \).

1. Show that the probability that the same triangle is chosen more than once is very small. Namely, for every fixed \( d \), \( \lim_{n \to \infty} P[|X(2)| = nd] = 1. \)

2. Show that if \( d \ll n \) then with high probability the link of a vertex \( v \) is highly disconnected. Describe how a typical link looks like. (Conclude that this is not a high-dimensional expander).

2 The paths complex

Let \( G = (V, E) \) be a \( d \)-regular expander for \( d \geq 3 \). We construct a 2 dimensional complex by setting \( X(0) = V \), and constructing \( X(2) \) by placing a triangle on every triple of distinct vertices \( u, v, w \) such that \( uv, vw \in E \) (i.e. where \( uvw \) is a length-2 path). We let \( X(1) \) be the edges of the triangles described above.

1. Show that \( X(1) \supset E \). Who are the edges in \( X(1) - E \) ?

2. Show that \( |X(2)| = O(nd^2) \).

3. Prove that the link of each vertex \( v \in X(0) \) is a connected graph.

4. The \textit{girth} of a graph is the length of its shortest cycle. Assume that the girth of \( G \) is at least 5. Fix \( v \in X(0) \) and let \( X_v \) be its link.
   
   - Give a full description of the link of a vertex \( v \).
   - The edge expansion or Cheeger constant of a (not necessarily regular) graph \( G \) is
     \[
     h(G) = \inf_{\{S:0<\sum_{v\in S}\deg(v)\leq|E|\}} \left\{ \frac{|\{\{v,u\}\in E\mid v\in S,u\notin S\}|}{\sum_{v\in S}\deg(v)} \right\}.
     \]
(Observe that the set $S$ has smaller “volume” than $V \setminus S$, when volume is measured by sum-of-degrees). Show that for every $v \in X(0)$, $h(X_v) \leq \frac{1}{2}$.

**Discussion:** The quotient inside the infimum is the fraction of edges from $S$ to $V \setminus S$ out of all the edges touching $S$. $h(G)$ measures whether there is a set $S$ with few outgoing edges. We have mentioned Cheeger’s inequality for expander graphs, which relates $h(\cdot)$ to the spectral expansion.

In conclusion, this complex has connected links, which is better than the random complex. However, the expansion inside a link cannot get better than some constant (if you are curious, the constant is approximately $\frac{1}{\sqrt{2}}$).

### 3 The swap walk graph

In this question $X$ is a 2-dimensional two-sided $\lambda$-high dimensional expander. You may assume further that it is regular. We will use the trickle down technique to analyze the bipartite graph $G$ whose left vertices are $X(0)$ and whose right vertices are $X(1)$ and we connect $v$ to $\{u, w\}$ if $\{v, u, w\} \in X(2)$. This is called the vertex-to-edge swap walk graph.

Our goal in this exercise is to show that the graph has excellent expansion, namely $\lambda_2(G) \leq 2\lambda$. \(^1\) Consider the following three-partite simplicial complex $Y$ whose parts are $A = B = X(0)$ (two disjoint copies) and $C = X(1)$. A triangle in $Y$ has three vertices $\{u, v, e\} \in Y(2)$ so that $u \in A, v \in B, e \in C, v \in e$ and $\{u\} \cup e \in X(2)$.

#### 3.1 Expansion in bipartite graphs

Let $G = (L, R, E)$ be a bipartite biregular graph. biregular means, there is some $d_1, d_2$ so that every $v \in L$ has $\text{deg}(v) = d_1$, and every $u \in R$ has $\text{deg}(u) = d_2$. Let $\ell_2(L) = \{f : L \to \mathbb{R}\}, \ell_2(R) = \{f : R \to \mathbb{R}\}$, these are inner product spaces where

$$
\langle f, g \rangle_L = \mathbb{E}_{v \in L} [f(v)g(v)] = \frac{1}{|L|} \sum_{v \in L} f(v).
$$

(and the same for the right side of $G$). The **left bipartite adjacency operator** of $G$ is $A : \ell_2(L) \to \ell_2(R)$ defined by

$$A f(u) = \mathbb{E}_{v \sim u} [f(v)] = \frac{1}{d_2} \sum_{v \sim u} f(v).$$

Its conjugate is the **right bipartite adjacency operator** is $A^* : \ell_2(R) \to \ell_2(L)$

$$A^* g(v) = \mathbb{E}_{u \sim v} [g(u)] = \frac{1}{d_1} \sum_{u \sim v} g(u).$$

$A^* A$ is self-adjoint, so it has a basis of eigenvectors with real eigenvalues. The largest eigenvalue is $\lambda_1(A^* A) = 1$ corresponding to the constant eigenvector. The second largest eigenvalue is denoted $\lambda_2(A^* A)$.

It is well known (and a not-too-hard exercise in linear algebra) that the spectral norm of the left/right bipartite adjacency operator is also the square root of the second-largest eigenvalue of the usual (non-bipartite) adjacency operator.

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\(^1\)This is in contrast to the bipartite graph $B_1$ on the same sets of nodes where $u \in X(0)$ is connected to $e \in X(1)$ if $u \in e$ (recall that $\lambda_2(B_1) \geq 1/2$.)
3.2 Analyzing the swap walk

1. Show that \( Y \) is type-regular, namely, that the bipartite graphs \((A, B), (B, C)\) and \((C, A)\) are all bi-regular. Calculate the respective degrees.

   (a) How does the graph between \( A \) and \( B \) relate to the 1-skeleton of \( X \)? Give a bound its one-sided spectral expansion.

   (b) Show that the graph between \((A \text{ and } C)\) is the vertex-to-edge swap walk graph \( G \).

2. Let \( u \in B \). Give a description of the bipartite graph in the link of \( u \), \( Y_u \).

3. Let \( f : A \to \mathbb{R} \), let \( u \in B \) and let \( f_u : Y_u \cap A \to \mathbb{R} \) be the restriction of \( f \) to the link of \( u \). We can decompose \( f_u = f_u^0 + f_u^\perp \) where \( f_u^0 \) is a constant function and \( f_u^\perp \) is orthogonal to the constant functions. Describe the value of \( f_u^0 \) using the left adjacency operator of the graph between \((A, B)\) (note that a similar relation holds for functions \( g : C \to \mathbb{R} \)).

4. Localization. For \( f, g : C \to \mathbb{R} \) we denote by \( \langle f, g \rangle = \mathbb{E}_{e \in C} [f(e)g(e)] \). For \( u \in B \) and \( f', g' : Y_u \cap C \to \mathbb{R} \) we denote by \( \langle f', g' \rangle_u = \mathbb{E}_{e \in Y_u \cap C} [f'(e)g'(e)] \). Show that

\[
\langle f, g \rangle = \mathbb{E}_{u \in B} \left[ \langle f_u, g_u \rangle_u \right]
\]

where \( f_u, g_u \) are the restrictions of \( f, g \) to the link of \( u \).

5. Prove that the vertex-to-edge swap walk has \( \lambda(G) \leq 2\lambda \) (in fact, your proof should give a slightly better bound).

   **hint:** Use the same localization arguments we used to show the trickle-down theorem.

6. **Bonus:** generalize the question above to \( k \)-face vs. \( \ell \)-face swap walks.