Homework 1 high dimensional expanders

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You may work in groups of 2-3 students. Please write up your homework in $LaTeX$ and submit the pdf by 20.12.2022 to *yotamd@weizmann.ac.il*. Please feel free to approach Yotam or Irit with any questions.

1 The sparse random complex

Let $X(0)$ be a set of n vertices, and let d be a fixed parameter. A triangle is a set of three distinct vertices in $X(0)$. For each $v \in X(0)$ let us choose d triangles containing v, uniformly and independently. Let $X(2)$ be the collection of at most nd triangles chosen in this way. Let $X(1)$ be all edges that participate in a triangle of $X(2)$.

1. Show that the probability that the same triangle is chosen more than once is very small. Namely, for every fixed d, $\lim_{n\to\infty} \mathbb{P}\left[|X(2)| = nd\right] = 1$.
2. Show that if $d \ll n$ then with high probability the link of a vertex v is highly disconnected. Describe how a typical link looks like. (Conclude that this is not a high-dimensional expander).

2 The paths complex

Let $G = (V, E)$ be a d-regular expander for $d \geqslant 3$. We construct a 2 dimensional complex by setting $X(0) = V$, and constructing $X(2)$ by placing a triangle on every triple of distinct vertices u, v, w such that $uv, vw \in E$ (i.e. where uvw is a length-2 path). We let $X(1)$ be the edges of the triangles described above.

1. Show that $X(1) \supset E$. Who are the edges in $X(1) - E$?
2. Show that $|X(2)| = O(nd^2)$.
3. Prove that the link of each vertex $v \in X(0)$ is a connected graph.
4. The *girth* of a graph is the length of its shortest cycle. Assume that the girth of G is at least 5. Fix $v \in X(0)$ and let X_v be its link.
   - Give a full description of the link of a vertex v.
   - The edge expansion or Cheeger constant of a (not necessarily regular) graph G is

$$h(G) = \inf_{\{S : 0 < \sum_{v\in S} deg(v) \leqslant |E|\}} \left\{ \frac{|\{\{v,u\} \in E \mid \ v \in S, u \notin S\}|}{\sum_{v \in S} deg(v)} \right\}.$$

(Observe that the set S has smaller "volume" than $V \setminus S$, when volume is measured by sum-of-degrees). Show that for every $v \in X(0)$, $h(X_v) \leqslant \frac{1}{4}$.

Discussion: The quotient inside the infimum is the fraction of edges from S to $V \setminus S$ out of all the edges touching S. $h(G)$ measures whether there is a set S with few outgoing edges. We have mentioned Cheeger's inequality for expander graphs, which relates $h(\cdot)$ to the spectral expansion.

In conclusion, this complex has connected links, which is better than the random complex. However, the expansion inside a link cannot get better than some constant (if you are curious, the constant is approximately $\frac{1}{\sqrt{2}}$).

3 The swap walk graph

In this question X is a 2-dimensional two-sided λ-high dimensional expander. You may assume further that it is regular. We will use the trickle down technique to analyze the bipartite graph G whose left vertices are $X(0)$ and whose right vertices are $X(1)$ and we connect v to $\{u, w\}$ if $\{v, u, w\} \in X(2)$. This is called the vertex-to-edge swap walk graph.

Our goal in this exercise is to show that the graph has excellent expansion, namely $\lambda_2(G) \leqslant 2\lambda$. [1] Consider the following three-partite simplicial complex Y whose parts are $A = B = X(0)$ (two disjoint copies) and $C = X(1)$. A triangle in Y has three vertices $\{u, v, e\} \in Y(2)$ so that $u \in A, v \in B, e \in C$, $v \in e$ and $\{u\} \cup e \in X(2)$.

3.1 Expansion in bipartite graphs

Let $G = (L, R, E)$ be a bipartite biregular graph. biregular means, there is some d_1, d_2 so that every $v \in L$ has $deg(v) = d_1$, and every $u \in R$ has $deg(u) = d_2$. Let $\ell_2(L) = \{f : L \to \mathbb{R}\}, \ell_2(R) = \{f : R \to \mathbb{R}\}$. these are inner product spaces where

$$\langle f, g \rangle_L = \mathop{\mathbb{E}}_{v \in L}[f(v)g(v)] = \frac{1}{|L|}\sum_{v \in L} f(v).$$

(and the same for the right side of G). The *left bipartite adjacency operator* of G is $A : \ell_2(L) \to \ell_2(R)$ defined by

$$Af(u) = \mathop{\mathbb{E}}_{v \sim u}[f(v)] = \frac{1}{d_2}\sum_{v \sim u} f(v).$$

Its conjugate is the *right bipartite adjacency operator* is $A^* : \ell_2(R) \to \ell_2(L)$

$$A^* g(v) = \mathop{\mathbb{E}}_{u \sim v}[g(u)] = \frac{1}{d_1}\sum_{u \sim v} g(u).$$

A^*A is self-adjoint, so it has a basis of eigenvectors with real eigenvalues. The largest eigenvalue is $\lambda_1(A^*A) = 1$ corresponding to the constant eigenvector. The second largest eigenvalue is denoted $\lambda_2(A^*A)$.

It is well known (and a not-too-hard exercise in linear algebra) that the spectral norm of the *left/right bipartite adjacency operator* is also the square root of the second-largest eigenvalue of the usual (non-bipartite) adjacency operator.

[1] This is in contrast to the bipartite graph B_1 on the same sets of nodes where $u \in X(0)$ is connected to $e \in X(1)$ if $u \in e$ (recall that $\lambda_2(B_1) \geqslant 1/2$.

3.2 Analyzing the swap walk

1. Show that Y is type-regular, namely, that the bipartite graphs $(A, B), (B, C)$ and (C, A) are all bi-regular. Calculate the respective degrees.

   (a) How does the graph between A and B relate to the 1-skeleton of X? Give a bound its one-sided spectral expansion.

   (b) Show that the graph between $(A$ and C is the vertex-to-edge swap walk graph G.

2. Let $u \in B$. Give a description of the bipartite graph in the link of u, Y_u.

3. Let $f : A \to \mathbb{R}$, let $u \in B$ and let $f_u : Y_u \cap A \to \mathbb{R}$ be the restriction of f to the link of u. We can decompose $f_u = f_u^0 + f_u^\perp$ where f_u^0 is a constant function and $f_u^\perp$ is orthogonal to the constant funcions. Describe the value of f_u^0 using the left adjacency operator of the graph between (A, B) (note that a similar relation holds for functions $g : C \to \mathbb{R}$).

4. *Localization.* For $f, g : C \to \mathbb{R}$ we denote by $\langle f, g \rangle = \mathbb{E}_{e \in C}[f(e)g(e)]$. For $u \in B$ and $f', g' : Y_u \cap C \to \mathbb{R}$ we denote by $\langle f', g' \rangle_u = \mathbb{E}_{e \in Y_u \cap C}[f'(e)g'(e)]$. Show that

   $$\langle f, g \rangle = \mathop{\mathbb{E}}_{u \in B}[\langle f_u, g_u \rangle_u]$$

   where f_u, g_u are the restrictions of f, g to the link of u.

5. Prove that the vertex-to-edge swap walk has $\lambda(G) \leqslant 2\lambda$ (in fact, your proof should give a slightly better bound).

   hint: Use the same localization arguments we used to show the trickle-down theorem.

6. *Bonus:* generalize the question above to k-face vs. ℓ-face swap walks.