Lecture 1: Expanders and high dimensional expanders

Irit Dinur

November 13, 2022

In this first lecture we begin with an introduction to expander graphs and a couple of applications. We will meet these applications in "high dimensional" form later on in this course. We then give a definition of high dimensional expanders (Definition 3.5) through the notion of links and simplicial complexes.

1 Expander Graphs

Recall, $G = (V, E)$ is a graph with set of vertices V and set of edges E. Throughout the course, we assume the edges are undirected, but could be weighted. We now give several definitions for expanders.

Assume that $G = (V, E)$ is a d-regular graph. The probability of a vertex set $S \subseteq V$ is

$$\mathbb{P}[S] = \frac{|S|}{|V|}.$$

The probability of a set of edges $F \subseteq E$ is defined in a similar manner:

$$\mathbb{P}[F] = \frac{|F|}{|E|}.$$

Expander graphs are graphs with no "bottlenecks". More precisely, for $\emptyset \neq S \subset V$ with $\mathbb{P}(S) \leqslant \frac{1}{2}$ we ask what is the probability to choose a random edge from a random vertex in S and land inside S. If this probability is much higher than the probability of S then the set S is "non expanding". Define the leaving probability of S to be

$$h(S) = \mathbb{P}_{u \sim v}[u \notin S \mid v \in S] = \frac{\mathbb{P}(E(S, \bar{S}))}{\mathbb{P}(S)} = \frac{|E(S, \bar{S})|}{d|S|}$$

where $E(S, \bar{S})$ is the set of edges between S and its complement $\bar{S} = V \setminus S$.

Definition 1.1. The edge expansion (aka Cheeger constant) of a graph is

$$h(G) = \min_{S \subset V, |S| \leqslant |V|/2} h(S).$$

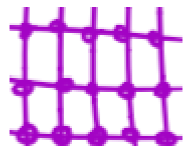
We say that G is an ε - edge expander if for all non-empty $S \subsetneq V$, $\phi(S) \geqslant \varepsilon$. In words, geometrically, this definition says that for every subset of vertices S the ratio of its boundary and its area is bounded from below. Namely, it has a large perimeter.

Example 1.2 (Clique, Cycle, Grid, Plane-vs.-Plane)**.**

- The complete graph K_n on n vertices, where every vertex is connected to every other vertex, including to itself. Here $h(S) = \frac{|\bar{S}||S|}{n|S|} = 1 - \mathbb{P}[S]$ for every S, so taking the minimum over all S such that $|S| \leqslant n/2$ we get $h(G) = \frac{1}{2}$.

– The cycle graph has $h(G) = O(1/n)$.

– The grid. Consider the following family $\{G_n\}$ of subgraphs of $\mathbb{Z}^2$. The vertices of G_n are $\{(i,j) : 0 \leqslant i,j \leqslant n\}$, i.e. vertices in a square of side length n. The edges connecting adjacent vertices as in the figure.



This family is not a family of expanders for any constant $\varepsilon > 0$. Indeed, for any n let S_n be the set of vertices in the form of a rectangle with side lengths n and $[\frac{n}{2}]$. Then S_n contains half the vertices of G_n, namely $\Omega(n^2)$ vertices whereas the boundary of S_n contains $O(n)$ edges. Thus, $h(G) = O(\frac{1}{\sqrt{|V|}})$ and tends to zero as $n \to \infty$.

– The Grassmann graph. Let $Gr(\mathbb{F}_q^m, \ell)$ be the graph whose vertices are all ℓ-dimensional subspaces of $\mathbb{F}_q^m$. Connect two vertices if they intersect on an ℓ-dimensional subspace. One can see that after ℓ steps there is good probability that we reach a completely uniform subspace, so there are no significant bottlenecks and the graph is an expander. However, the degree is not bounded.

It is not trivial to construct a sequence of expander graphs where the degree of all vertices remains uniformly bounded as the number of vertices increases. However, there are known constructions for such objects, both random and deterministic.

1.1 Random walks

The random experiment we just described is a first step in a possibly longer random walk on the graph. At every step, assume we are on v, we choose a random edge containing v, and them "move" to the other endpoint of the edge.

Question 1.3. What is the probability distribution describing the t-th vertex in the random walk?

Let $p_t : V \to \mathbb{R}$ be the distribution at time t. After one step of the random walk, the probability to by at point u is

$$p_{t+1}(u) = \sum_{v \in V} A(u,v) p_t(v)$$

where $A(u,v)$ is the probability to move to v and is equal to $1/d$ if $v \sim u$ and to 0 otherwise.

The matrix A is called the transition probability matrix, and we can write the above in vector notation as

$$p_{t+1} = Ap_t.$$

Lemma 1.4. *Let G be a d-regular graph and let A be the transition probability matrix of the simple random walk on G.*

1. $A\mathbf{1} = \mathbf{1}$

2. *$A(u,v) = A(v,u)$, so A is self adjoint.*

3. *A is diagonalizable with real eigenvalues $1 = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and a corresponding basis of eigenvectors $\mathbf{1} = f_1, \ldots, f_n$.*

4. *$\lambda_2 < 1$ iff G is connected*

Let us write p in the basis of the eigenvectors, and apply the matrix A for several steps,

$$p = \sum_i \alpha_i f_i, \qquad A^t p = \sum_i \alpha_i (\lambda_i)^t f_i.$$

We see that if $\lambda_2 < 1$ then as $t \to \infty$, $A^t p \to \alpha_1 \mathbf{1}$.

1.2 Spectral expansion

We are now ready to give the spectral definition of expansion:

Definition 1.5 (Spectral Expander)**.** A graph G is a λ - one-sided spectral expander if, let $\{\lambda_i\}$ be eigenvalues of the random walk matrix of G,

$$\lambda_n \leqslant \cdots \leqslant \lambda_2 \leqslant \lambda.$$

It is a λ - two-sided spectral expander if

$$-\lambda \leqslant \lambda_n \leqslant \cdots \leqslant \lambda_2 \leqslant \lambda.$$

The notion of a one-sided spectral expander is strictly weaker than the two-sided expander. For example, the complete bipartite graph is a one-sided expander for $\lambda = 0$, but not a two-sided expander, since its most negative eigenvalue is $\lambda_n = -1$.

The definition above is "morally" equivalent to having no high staying probability. Indeed Cheeger's inequality says that

$$(1 - \lambda_2(G))/2 \leqslant h(G) \leqslant \sqrt{2(1 - \lambda_2(G))}.$$

In the literature, it is common to refer to the second largest eigenvalue of the normalized Laplacian of the graph, which is $L = I - A$ in this case, so $\lambda_2(L) = 1 - \lambda_2$. A nice exposition is found in Trevisan's blog, [1].

One example for a theorem that shows a connection between the combinatorial properties of a graph and its spectral expansion is the following,

Lemma 1.6 (Alon-Chung)**.** *Let $G = (V, E)$ be a d-regular λ-one-sided expander. Let $T \subseteq V$ be such that the graph induced on T, denoted $G(T)$, has average degree at least δd. Then $|T| \geqslant (\delta - \lambda) \cdot |V|$, and the number of edges in $G(T)$ is at least $(\delta - \lambda)\delta \cdot |E|$.*

We define an inner product on the space of functions $f : V \to \mathbb{R}$, by $\langle f, g \rangle = \mathbb{E}_{v \in V} f(v)g(v) = \frac{1}{|V|} \sum_{v \in V} f(v)g(v)$. When the graph is not regular, the expectation is defined not with respect to the uniform measure but rather with respect to the stationary distribution on the vertices.

Proof. Denote $p = \mathbb{P}[T]$ and $f = \mathbf{1}_T$. We can write $f = p\mathbf{1} + h$ with $\langle h, \mathbf{1} \rangle = 0$. We get

$$\delta \cdot p \leqslant \langle f, Mf \rangle = \langle p\mathbf{1} + h, M(p\mathbf{1} + h) \rangle \leqslant p^2 + \lambda \langle h, h \rangle \leqslant p^2 + \lambda p.$$

where the last inequality is because $\langle h, h \rangle \leqslant \langle f, f \rangle = p$. When rearranging, this gives the lemma. ∎

2 Applications of expanders

2.1 Rapid mixing

Suppose we have a deck of cards, and are interested in getting the cards in a random order. Namely, one of 52! permutations. We can think of the graph whose vertices are all possible permutations, and put an edge between σ and τ if there is a card-shuffle move that brings you from one to the other. If we manage to bound the second largest eigenvalue of this graph by $1-\varepsilon$ then we can write,

$$p_0 = \sum_i \alpha_i \psi_i$$

for p_0 the initial distribution and $\{\psi_i\}$ the eigenvectors of the graph. After t steps,

$$p_t = \sum_i \alpha_i A^t \psi_i = \sum_i \alpha_i \lambda_i^t \psi_i$$

For all $i > 1$, $\lambda_i^t \to 0$ as t increases. For $i = 1$, $\lambda_i^t = 1^t = 1$. So the distribution becomes more and more similar to the uniform distribution.

2.2 Error Correcting Codes

A linear error correcting code is a linear subspace $C \subseteq \mathbb{F}_2^n$. The rate of the code is $\dim C$ and the relative distance of the code is the minimum Hamming distance between a pair of distinct codewords,

$$\mathrm{dist}(C) = \min_{x \neq y \in C} [\mathbb{P}_i[x_i \neq y_i] \geqslant \delta.$$

A family of codes $C_n \subseteq \mathbb{F}_2^n$ is sometimes called "good" if it has constant rate and constant relative distance (i.e lower bounded by a parameter ε for all n). A random subspace is "good" with high probability. However, there is a major drawback. Given a word w that is close to the code, there is no efficient algorithm for finding the nearest codeword (this is called decoding).

Over the years many explicit good families error correcting codes have been discovered. One beautiful example is that of expander codes, discovered by Sipser and Spielman.

Definition 2.1 (Expander Codes)**.** Let $G = (V, E)$ be a d-regular graph. Let $C_0 \subseteq \mathbb{F}_2^d$ be a code on d-bits with relative distance δ. The code $C(G, C_0)$ is given by

$$C(G, C_0) = \left\{ w \in \mathbb{F}_2^E \,\middle|\, \forall v \in V, \quad w|_{E_v} \in C_0 \right\}.$$

It turns out that when G is a λ-one-sided expander, the code $C(G, C_0)$ has relative distance at least $\delta(\delta - \lambda)$.

3 High Dimensional Hypergraphs

We have spent most of the lecture discussing expansion and giving examples of applications. We are now in shape to generalize expanders to higher dimensions. The first object that comes to mind is a hypergraph,

Definition 3.1. A *hypergraph* is a pair (V, E) with V a set of vertices and E a set of subsets of V.

In the first part of the course we will focus on special types of hypergraphs called simplicial complexes.

Definition 3.2. A *simplicial complex* (abbreviated s.c.) is a hypergraph that is downwards closed under containment. Namely, if $S \in E$ and $S' \subset S$ then $S' \in E$. $S \in E$ is called a face.

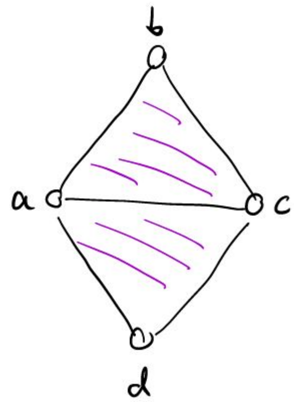
We usually partition a simplicial complex X to

$$X = X(0) \cup X(1) \cup \cdots \cup X(d)$$

where $X(i)$ is the set of faces of size $(i+1)$, or dimension i. In particular, $X(0)$ are identified with the vertices of the simplicial complex.

We say a simplicial complex is of *d-dimensional* if the maximal face size is $d+1$.

Example:. Consider the hypergraph below:



Then

$$\begin{aligned} X(2) &= \{\{a,b,c\},\{a,c,d\}\} \\ X(1) &= \{\{a,b\},\{b,c\},\{a,c\},\{b,c\},\{c,d\}\} \\ X(0) &= \{\{a\},\{b\},\{c\},\{d\}\} \end{aligned}$$

An important definition is that of a link. This is like a neighborhood in a graph, but is a richer structure than just a collection of vertices,

Definition 3.3 (Link)**.** Let X be a d-dimensional simplicial complex and $s \in X(i)$. The link of s is a $(d-i-1)$-dimensional simplicial complex defined by:

$$X_s = \{t \in X \mid \ s \cup t \in X, \ s \cap t = \phi\}.$$

For example, in the figure above, the link of a is the graph $X_a(0) = \{b, c, d\}$ and $X_a(1) = \{\{b,c\},\{c,d\}\}$.

Definition 3.4. Let X be a simplicial complex and $k < d$ some non-negative integer. The k-skeleton of X is the subspace of X that is the union of faces of dimension $\leqslant k$.

We are ready to define high dimensional expansion:

Definition 3.5 (λ-high Dimensional Expander)**.** Let $\lambda < 1$. A d-dimensional pure simplicial complex X is a λ-two-sided (resp. one-sided) high dimensional expander if:

1. The 1-skeleton of X is a λ-spectral two-sided (resp. one-sided) expander, and
2. For any $i \leqslant d-2$ and all $s \in X(i)$ the 1-skeleton of X_s is a λ-spectral two-sided (resp. one-sided) expander.

Example 3.6. The d-dimensional complete complex on n vertices, which consists of all subsets of $\{1, ..., n\}$ of size $\leqslant d+1$, is an example for a two-sided high dimensional expander. The 1- skeleton of every link is a complete graph, which is a $\left(\frac{1}{n-d}\right)$-two-sided spectral expander (check!).

Constructing bounded degree two-sided HDXs is challenging. For graphs, there are well known algebraic constructions, random constructions and combinatorial constructions (see next week's lecture), where the degree of every vertex is uniformly bounded (when the number of vertices goes to infinity).

For simplicial complexes, even 2-dimensional, there are some combinatorial constructions that achieve weak expansion in the links. However, the only known constructions with arbitrarily good link expansion are algebraic. We will see some of these in future lectures.

References

[1] Luca Trevisan. *Lecture 3: Cheeger's inequality*, 2011. `https://lucatrevisan.wordpress.com/2011/01/19/cs359g-lecture-3-cheegers-inequality/`. 3