Lecture 2: Trickle down theorem, and random walks on HDX

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In the first part of the lecture we will describe a local-to-global phenomenon in which expansion of "local" links implies expansion in the entire "global" complex. This is called the trickle down theorem because the expansion trickles down from the top links to the bottom links. (The lowest link, of the -1-dimensional face which is the empty set, is the entire complex).

In the second part of the talk we will study random walks on *i*-dimensional faces of a complex. We will show how expansion in the links allows us to upper bound the second largest eigenvalue of these random walks.

1 Notation

First, let us recall some definitions from the previous lecture. A *d*-dimensional simplicial complex consists of vertices X(0), edges X(1), and so on $X(i) \subseteq \binom{X(0)}{i+1}$. It satisfies a downwards closure property: if $s \in X(i)$ and $t \subset s$ then $t \in X$. An element in $X(0) \cup \cdots \cup X(d)$ is called a face. The link of a face $s \in X$ is

$$X_s = \{t \in X \mid s \cup t \in X, s \cap t = \phi\}.$$

We also add the empty face to X for convenience, namely $X(-1) = \{\phi\}$. The link of the empty face is the entire complex.

We said that a graph is a λ one-sided (two-sided) expander if its non-trivial eigenvalues are bounded from above by λ (and from below by $-\lambda$). An equivalent definition is to say that

$$||A - J||^2 \le \lambda \qquad (\text{two-sided}) \tag{1.1}$$

or

 $A - J \leq \lambda Id$ (one-sided)

where $A \leq B$ means that B - A is positive semi-definite; and J is the matrix whose entries are all 1/n, corresponding to the transition matrix of the random walk on the complete graph with self loops.

One interpretation of (1.1) is that A approximates J in operator norm. A random walk on the complete graph is uniformly random, and this allows us to say a random walk on an expander is *pseudo-random*.

Finally, we defined X to be a high-dimensional λ -link expander if for any $s \in X$, where $dim(s) \leq d-2$, X_s is a λ -spectral expander.

1.1 Regularity Assumption

In this lecture we will assume, for simplicity, that the complexes under discussion are *regular*. Namely, there are natural numbers $r_0 > r_1 > \cdots > r_{d-1}$ such that every *i* face is contained in r_i (i + 1)-faces. This is the HD analog of an *r*-regular graph. This assumption is not necessary, but will help us avoid unimportant technicalities. In the last part of the lecture we will discuss how to remove this condition. (Many important constructions of high dimensional expanders are not regular, so this is needed.)

2 Examples of *d*-dimensional simplicial complexes

Example 2.1 (*d*-dimensional complete complex). The *n*-simplex, denoted Δ^n , is the *n*-dimensional simplicial complex that contains all possible subsets of n + 1 elements.

The *d*-dimensional complete complex on *n* vertices is the *d*-skeleton of Δ^{n-1} . Namely, its *d*-faces are all subsets of $\{1, ..., n\}$ of size at most *d*+1. It is an example for a two-sided high dimensional expander. The 1- skeleton of every link is a complete graph, which is a $\left(\frac{1}{n-d}\right)$ -two-sided spectral expander (check!).

Example 2.2 ((d + 1)-partite d-dimensional complete complex). Let $X(0) = A_0 \cup A_1 \cup \cdots A_d$, define $X(d) = \{\{v_0, \ldots, v_d\} \mid v_i \in A_i\}$, and let X be the downward closure of X(d).

This is a multi-partite complex in that there is a (d+1)-coloring of the vertices such that there are no colors inside a color class.

It is a 0-one-sided spectral expander.

Example 2.3 (Spanning trees complex). Let G = (V, E) be a graph, and let m = |E|. Define an (n-2)-dimensional complex X as follows

- -X(0) = E
- $-X(n-2) = \{ s \subset E \mid s \text{ is a spanning tree in } G \}.$
- -X(i) is defined by closing downwards

This complex was studied in [1]. They studied a more general complex whose faces are bases of a general matroid. They proved that the links of this complex expand, and used this for showing that random walks mix rapidly.

Example 2.4 (Subspaces complex, aka the spherical building). Let \mathbb{F}_q be a finite field, d > 1, and let X(0) have a vertex for each non-trivial linear subspace of \mathbb{F}_q^d . For every maximal chain of subspaces of the form $v_1 \subset v_2 \subset \cdots \subset v_{d-1} \subset \mathbb{F}_q^d$, we add the set $\{v_1, \ldots, v_{d-1}\}$ to X(d-2). The remaining faces are defined by downward closure.

Chains of subspaces $v_1 \subset v_2 \subset \cdots \subset v_{d-1}$ are called flags, and this is sometimes called a flag complex.

For d = 3 this is the lines vs. planes graph. For d = 4 this is a two-dimensional complex. Some links are the d = 3 complex and some links are complete bipartite graphs.

Note that this complex is only regular when d = 3. However, it is an HDX for all d > 2 (when appropriately defined).

The examples above are not bounded-degree. Are there bounded-degree high dimensional expanders? For dimension 1, almost all *d*-regular graphs are expanders. For dimensions two and above random complexes were studied by Linial, Meshulam and Wallach [2, 3]. One requires super constant density before the resulting complex is a high dimensional expander.

For simplicial complexes, even 2-dimensional, there are some combinatorial constructions that achieve weak expansion in the links. However, the only known constructions with arbitrarily good link expansion are algebraic. We will see some of these in future lectures.

3 Trickle-Down theorem

A priori, the definition of HDX requires information on all links of the complex. However, the following theorem by Izhar Oppenheim [4], tells us that if the links of the (d-2)-faces are good expanders, then the links of the lower dimension faces are also expanders, as long as they are connected. More precisely:

Theorem 3.1 (Trickling-Down Theorem, two-dimensional). Let X be a 2-dimensional simplicial complex such that the graph (X(0), X(1)) is connected and $\forall v \in X(0) X_v$ is a one-sided λ -expander. Then (X(0), X(1)) is a μ -expander where $\mu = \frac{\lambda}{1-\lambda}$.

Note that the theorem is useless for $\lambda \ge \frac{1}{2}$. By applying the theorem iteratively, we get the following useful corollary:

Corollary 3.2 (Trickling-Down Theorem, *d*-dimensional). Let X be a *d*-dimensional simplicial complex such that the 1-skeleton of every link (including the entire simplicial complex) is connected and $\forall v \in X(d-2) X_v$ is a one-sided λ -expander. Then X is a μ -expander where $\mu = \frac{\lambda}{1-(d-1)\lambda}$.

Proof of Theorem 3.1. Let A be the adjacency operator associated with the 1-skeleton (X(0), X(1)).

Suppose $f : X(0) \to \mathbb{R}$ is an eigenfunction with eigenvalue γ , and assume $f \perp \mathbf{1}$. Also assume ||f|| = 1, namely $\mathbb{E}[f^2] = 1$. We have:

$$\gamma = \langle f, Af \rangle = \mathbb{E}_{\{u,w\} \in X(1)} [f(u)f(w)] = \mathbb{E}_{v \in X(0)} \mathbb{E}_{\{u,w\} \in X_v(1)} [f(u)f(w)]$$
(3.1)

Next, let A_v be the adjacency operator associated with X_v . By assumption X_v is a one-sided λ -spectral expander and so the second largest eigenvalue of A_v satisfies $\lambda_2 \leq \lambda$.

For any function $g: X_v(0) \to \mathbb{R}$ satisfying $g \perp \mathbf{1}$ we have by the spectral decomposition of A that

 $\langle Ag, g \rangle \leq \lambda \|g\|^2.$

Denote the restriction of f to a link X_v by f^v , namely:

$$f^v: X_v \to \mathbb{R}$$
$$f^v(u) = f(u)$$

Define $g^v = f^v - \gamma f(v)\mathbf{1}$. Recall that $Af(v) = \gamma f(v)$ and so we have $g^v \perp \mathbf{1}$ since

$$\mathbb{E}_{\iota \in X_v(0)}[f^v(u)] = Af(v) = \gamma f(v).$$

Now we evaluate

$$\mathbb{E}_{uw \in X_v}[f^v(u)f^v(w)] = \mathbb{E}_v[\langle f^v, A_v f^v \rangle] = \mathbb{E}_v[\langle g^v, A_v g^v \rangle] + \mathbb{E}_v(f(v)\gamma)^2 =$$

$$= \mathbb{E}[\langle g^v, A_v g^v \rangle] + \gamma^2$$

On the other hand, in (3.1) we can switch f to f^v since we only evaluate f on X_v and so we also have

$$\mathbb{E}_{v \in X(0)uw \in X_v} \mathbb{E}_{v}[f^v(u)f^v(w)] = \mathbb{E}_{v \in X(0)uw \in X_v} \mathbb{E}_{v}[f(u)f(w)] = \gamma$$

Therefore,

$$\gamma - \gamma^2 = \mathop{\mathbb{E}}_{v}[\langle g^v, A_v g^v \rangle] \leq \lambda \mathop{\mathbb{E}}_{v}[\|g^v\|^2] = \lambda(1 - \gamma^2)$$

We assumed that G is connected thus $\lambda_1 = 1$ has multiplicity 1, and we have $\gamma < 1$. and so we divide by $1 - \gamma$:

$$\gamma \leqslant \frac{\lambda}{1-\lambda}.$$

Note that when $\lambda < \frac{1}{2}$ we have $\gamma \leq 2\lambda$. This theorem shows us that we can infer global properties of the graph based on local properties given by the links of the (d-2)-faces.

4 Walking in *i* dimensions

In graphs, the simple random walk walks from vertex to edge to the opposite vertex. The graph is connected if this walk can reach every vertex from every start vertex.

In higher dimensions, we can "walk" on *i*-dimensional faces.

There are several natural walks. In two dimensions, a walk on the edges can go edgevertex-edge, or edge-triangle-edge. The connectivity of the first walk is the same as the connectivity of the underlying graph (why?). The second walk can be disconnected.

- Upper random walk: start from an *i*-face, walk up to an i + 1 face and then down to an *i* face.
- Lower random walk: start from an *i*-face, walk down to an i 1 face and then up to an *i* face.

To analyze these walks, let us consider their transition matrices. These are naturally described in terms of the Up and Down operators, which we define next.

Let the space of i-chains on X be

$$C^{i}(X) = \{ f : X(i) \to \mathbb{R} \}.$$

The Down operator takes an *i*-chain to an (i - 1)-chain, by averaging over the faces immediately above it:

$$\forall s \in X(i-1), \qquad D_i f(s) = \mathop{\mathbb{E}}_{t>s} f(t),$$

where we use the notation t > s to denote that $t \supset s$ and $\dim(t) = \dim(s) + 1$.

The Up operator takes an *i*-chain to an (i + 1)-chain.

$$\forall t \in X(i+1), \qquad U_i f(t) = \mathbb{E}_{s < t} f(s).$$

Claim 4.1. The up and down operators are duals in the sense that for every $f \in C^{i}(X)$ and $g \in C^{i+1}(X)$,

$$\langle D_{i+1}g, f \rangle = \langle g, U_i f \rangle.$$

Proof. We drop subscripts and maintain that $s \in X(i)$ and $t \in X(i+1)$,

$$\begin{split} \langle Dg, f \rangle &= \mathop{\mathbb{E}}_{s}[f(s)Dg(s)] = \mathop{\mathbb{E}}_{s}[f(s)\mathop{\mathbb{E}}_{t>s}g(t)] \\ &= \mathop{\mathbb{E}}_{s < t}[f(s)g(t)] = \\ &= \mathop{\mathbb{E}}_{t}g(t)\mathop{\mathbb{E}}_{s < t}[f(s)] = \langle g, Uf \rangle. \end{split}$$

The upper walk operator is $D_{i+1}U_i: C^i(X) \to C^i(X)$. It averages the values of an i chain at a face s over all faces s' obtained by first going up to an i + 1 face t and then down to an i face s'.

The lower walk operator is $U_{i-1}D_i$. The corresponding *lower random walk* goes from *i*-face to i - 1-face to another *i*-face.

4.1 Connectivity of X(i)

Let us say that X is *i*-connected if the upper walk on X(i) is connected.

The lower walk on X(i) is connected iff the upper walk on X(i-1) is connected. However, even when the lower walk on X(i) is connected, the upper walk on X(i) could be disconnected. Still, if we know that the links of vertices are all (i-1)-connected, then the upper walk in dimension i will be connected. (Why?)

Lemma 4.2 (Connectivity from local to global). If X is (i - 1)-connected, and the link of every vertex $v \in X(0)$ is (i - 1)-connected, then X is i-connected.

In expanders, statements about connectivity are replaced by more robust statements.

4.2 Random-walk expansion

When would we say that a simplicial complex is an expander from the point of view of random walks?

For expander graphs, we used (1.1), which, by observing that $U_{-1}D_0 = J$, amounts to $\|A - U_{-1}D_0\|^2 \leq \lambda.$

$$A - U_{-1}D_0 \leq \lambda Id.$$

Interestingly, we can express A itself in terms of the up and down operators:

$$A = 2D_1U_0 - Id.$$

Indeed when we walk up from a vertex to an edge and then back down to a vertex we stay in place with probability 1/2 and move to a neighbor with probability 1/2. The same is true for walk in higher dimensions. We call a walk non-lazy if the probability to stay in place is zero. The up-down walk on *i* faces has $\frac{1}{i+2}$ laziness probability. We denote M_i^+ the non-lazy upper walk. Note that

$$DU = \frac{i+1}{i+2}M_i^+ + \frac{1}{i+2}Id.$$

This suggests the following definition,

Definition 4.3. A complex is a λ -random-walk expander if for every i < d,

$$\|M_i^+ - U_{i-1}D_i\|^2 \le \lambda$$

or

$$M_i^+ - U_{i-1}D_i \leq \lambda \cdot Id,$$

where

$$M_i^+ = \frac{i+2}{i+1}D_{i+1}U_i - \frac{1}{i+1}Id$$

is the non-lazy upper random walk.

We will prove

Theorem 4.4 (link-expansion and random walk expansion). Every one-sided (two-sided) γ -link HDX is a one-sided (two-sided) γ -random-walk HDX.

Moreover, if X is a two-sided γ -random-walk HDX, then it is a $3d\gamma$ -two-sided link HDX.

Proof. We prove the first implication. Assume that X is a γ -link expander, one-sided. We need to show that

$$\|M_i^+ - UD\| \le \gamma,$$

for all i < d, where M_i^+ is the non-lazy upper walk. Let $f \in C^i(X), i < d$. We have

$$\langle M_i^+ f, f \rangle = \mathop{\mathbb{E}}_{t \in X(i+1)} \mathop{\mathbb{E}}_{x \neq y \in X_t(0)} \left[f(t \setminus \{x\}) f(t \setminus \{y\}) \right]$$

Let $s = t \setminus \{x, y\}$. Since $t \sim X(i+1)$ and $x \neq y \in t$ are chosen at random, we have $s \sim X(i-1)$. Given such an s, the probability to get specific (t, x, y) is exactly like choosing a random edge in X_s so

$$\langle M_i^+ f, f \rangle = \mathbb{E}_{s \sim X(i-1)} \mathbb{E}_{(x,y) \in X_s(1)} [f(s \cup \{x\}) f(s \cup \{y\})].$$
(4.1)

In other words, we have shown that

$$\langle M_i^+ f, f \rangle = \mathbb{E}_{s \sim X(i-1)} [\langle A_s f_s, f_s \rangle], \qquad (4.2)$$

where $f_s: X_s(0) \to \text{ is defined by}$

$$f_s(x) = f(s \cup \{x\}).$$

We now note that

$$\mathbb{E}_{x \sim w_s}[f(s \cup \{x\})] = (Df)(s).$$

Therefore we have, by the γ -expansion in the links that

$$\begin{split} |\langle M_i^+f, f\rangle - \langle UDf, f\rangle| &= \Big| \underset{s \sim X(i-1)}{\mathbb{E}} \underset{(x,y) \sim w_s}{\mathbb{E}} [f(s \cup \{x\})f(s \cup \{y\})] - (Df)(s)^2 \Big| \leqslant \\ & \underset{s \sim X(i-1)}{\mathbb{E}} [\lambda(A_s) \underset{x \sim w_s}{\mathbb{E}} [f(s \cup \{x\})^2]]. \end{split}$$

If X is a γ -two-sided link expander then $\lambda(A_s) \leq \gamma$ for all s, and so

$$|\langle (M_i^+ - UD)f, f \rangle| \leq \gamma ||f||^2.$$

5 The non-regular case

Let X be a general d-dimensional complex that is not regular, and let π_d be any probability distribution on X(d). A popular choice will be the uniform distribution on X(d). We can inductively define distributions π_i on X(i) by

$$\boldsymbol{\pi}_i(s) = \sum_{t>s} \boldsymbol{\pi}_{i+1}(t) \cdot \frac{1}{i+2}.$$

To check that this is a valid distribution think of the random process of first choosing an i + 1 face according to π_{i+1} and then removing one its i + 2 vertices at random, thus obtaining an i face. Even if π_d is uniform, when X is not regular π_i will not in general be uniform.

For example, when X is a one-dimensional complex, namely a graph, and π_1 is the uniform distribution over edges, then π_0 gives each vertex probability proportional to its degree. The transition matrix of a weighted graph is defined to be

$$A(u,v) = \mathbb{P}[v|u] = \frac{\pi_1(uv)}{\sum_{v \sim u} \pi_1(v)}.$$

This matrix satisfies $A\mathbf{1} = \mathbf{1}$ and the remaining eigenvalues are between -1 and 1. A weighted graph is a λ -expander if $\lambda_2 \leq \lambda$ just like before.

We define up and down operators to average according to π_i :

$$D_i f(s) = \mathbb{E}_s f(t)$$

where the expectation is over choosing a random face $t \sim \pi_i$ conditioned on t > s. Similarly,

$$U_i f(s) = \mathop{\mathbb{E}}_{t < s} f(t)$$

where the expectation is over choosing a random face $t \sim \pi_i$ conditioned on t < s. Notice that this coincides with the previous definition, because going down the conditional distribution is always uniform.

The link complexes X_s naturally inherit distributions from X by conditioning:

$$\boldsymbol{\pi}_{s,i}(t) = \frac{\boldsymbol{\pi}_i(t \cup s)}{\sum_{t \in X_s(i-|s|-1)} \boldsymbol{\pi}_i(t \cup s)}$$

The proof above goes through when we define inner products on the spaces $C^{i}(X)$ by

$$\langle f,g\rangle_{\boldsymbol{\pi}_i} = \mathop{\mathbb{E}}_{\boldsymbol{s}\sim\boldsymbol{\pi}_i}[f(\boldsymbol{s})g(\boldsymbol{s})], \qquad \|f\|_{\boldsymbol{\pi}_i}^2 = \langle f,f\rangle_{\boldsymbol{\pi}_i}.$$

We will write $\langle f, g \rangle$ instead of $\langle f, g \rangle_{\pi_i}$ when the distribution is clear from the context. Going back to the previous sections with these definitions, the proofs go through almost identically. For example (3.1) should be rewritten as

$$\gamma = \langle f, Af \rangle = \mathop{\mathbb{E}}_{\{u,w\}\sim\pi_1} [f(u)f(w)] = \mathop{\mathbb{E}}_{v\sim\pi_0} \mathop{\mathbb{E}}_{\{u,w\}\sim\pi_{v,1}} [f(u)f(w)]$$
(5.1)

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