Lecture 3: Random walks on HDX

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In this lecture we will study higher order random walks. Given a d-dimensional simplicial complex, one can walk on its i-dimensional faces. We will define what expansion means in this context and relate this definition to the previous definition through links. We will then derive the breakthrough result of [1] showing that the basis-exchange random walk on matroid bases mixes rapidly.

1 Walking in i dimensions

In graphs, the simple random walk walks from vertex to edge to the opposite vertex. The graph is connected if this walk can reach every vertex from every start vertex.

In higher dimensions, we can "walk" on i-dimensional faces. There are several natural walks. In two dimensions, a walk on the edges can go edge-vertex-edge, or edge-triangle-edge. The connectivity of the first walk is the same as the connectivity of the underlying graph (why?). The second walk can be disconnected.

- Upper random walk: start from an i-face, walk up to an $i+1$ face and then down to an i face.
- Lower random walk: start from an i-face, walk down to an $i-1$ face and then up to an i face.

To analyze these walks, let us consider their transition matrices. These are naturally described in terms of the Up and Down operators, which we define next.

Let the space of i-chains on X be

$$C^i(X) = \{f : X(i) \to \mathbb{R}\}.$$

The Down operator takes an i-chain to an $(i-1)$-chain, by averaging over the faces immediately above it:

$$\forall s \in X(i-1), \qquad D_i f(s) = \mathop{\mathbb{E}}_{t>s} f(t),$$

where we use the notation $t > s$ to denote that $t \supset s$ and $\dim(t) = \dim(s) + 1$.

The Up operator takes an i-chain to an $(i+1)$-chain.

$$\forall t \in X(i+1), \qquad U_i f(t) = \mathop{\mathbb{E}}_{s<t} f(s).$$

The upper walk operator is $D_{i+1}U_i : C^i(X) \to C^i(X)$. It averages the values of an i chain at a face s over all faces s' obtained by first going up to an $i+1$ face t and then down to an i face s'.

The lower walk operator is $U_{i-1}D_i$. The corresponding *lower random walk* goes from i-face to $i-1$-face to another i-face.

1.1 Up and Down operators viewed as a bipartite graph

We can think of the bipartite graph $\mathcal{B}$ whose left side is $X(i)$ and right side is $X(i+1)$, and there is an edge between $s \in X(i)$ and $t \in X(i+1)$ iff $s \subset t$. The transition matrix of $\mathcal{B}$ is a block matrix with two zero components and one U and one D component.

$$B = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & U \end{array}\right].$$

This operator is self-adjoint with respect to the "natural" distribution[1]. This is precisely because U, D are duals of each other with respect to the appropriate inner product which we defined as

$$\forall f, f' \in C^i(X), \quad \langle f, f' \rangle = \mathop{\mathbb{E}}_{s \sim X(i)}[f(s)f'(s)].$$

Claim 1.1. The up and down operators are duals in the sense that for every $f \in C^i(X)$ and $g \in C^{i+1}(X)$,

$$\langle D_{i+1}g, f \rangle = \langle g, U_i f \rangle.$$

Proof. We drop subscripts and maintain that $s \in X(i)$ and $t \in X(i+1)$,

$$\begin{aligned} \langle Dg, f \rangle = \mathop{\mathbb{E}}_{s}[f(s)Dg(s)] &= \mathop{\mathbb{E}}_{s}[f(s) \mathop{\mathbb{E}}_{t>s} g(t)] \\ &= \mathop{\mathbb{E}}_{s<t}[f(s)g(t)] = \\ &= \mathop{\mathbb{E}}_{t} g(t) \mathop{\mathbb{E}}_{s<t}[f(s)] = \langle g, Uf \rangle. \end{aligned}$$

□

So far we have considered the second largest eigenvalue λ_2 of random walk operators. For such an operator A it is well known, and can be easily verified, that

$$\lambda_2(A) = \sup_{f \perp \mathbf{1}, \|f\|=1} \|Af\| = \sup_{f \perp \mathbf{1}, \|f\|=1, \|g\|=1} \langle Af, g \rangle.$$

We now need to define a similar notion for an operator that moves from one space to another. Let us define the spectral norm of an operator $T : \ell_2(W_1) \to \ell_2(W_2)$ by

$$\lambda(T) = \sup_{\|f\|=1, f \perp \mathbf{1}} \|Tf\| = \sup_{f \perp \mathbf{1}, g} \langle Tf, g \rangle$$

This is the standard operator norm of T when restricted to the space of functions $f \perp \mathbf{1}$.

Lemma 1.2. *Let $U : C^{i-1}(X) \to C^i(X)$ and let $D : C^i(X) \to C^{i-1}(X)$ be as before.*

1. $\lambda(D) = \lambda(U)$
2. $\lambda(D)^2 = \lambda_2(UD)$ *where λ_2 is the second largest eigenvalue of the self-adjoint operator UD.*
3. $\lambda_2(B) = \lambda(D) = \lambda(U)$ *where B is the transition matrix of the bipartite graph $\mathcal{B}$.*

[1] The distribution on the nodes of the graph that is derived by selecting a random edge in the graph and then a random endpoint. This implies that half of the weight is on the right hand side vertices, and half is on the left.

We claim that $\lambda(D) = \lambda(U)$. Indeed, if $f \perp \mathbf{1}$ then $Df \perp \mathbf{1}$, so

$$\lambda(D) = \sup_{f \perp \mathbf{1}, g} \langle Df, g \rangle = \sup_{f \perp \mathbf{1}, g \perp \mathbf{1}} \langle Df, g \rangle = \sup_{f \perp \mathbf{1}, g \perp \mathbf{1}} \langle f, Ug \rangle = \sup_{f, g \perp \mathbf{1}} \langle f, Ug \rangle = \lambda(U).$$

We further claim that the second largest eigenvalue of UD is equal to $\lambda(D)^2$. Indeed, let $f \in C^i(X)$ be the eigenfunction corresponding to λ_2, with $\|f\| = 1$.

$$\lambda_2 = \lambda_2 \langle f, f \rangle = \langle UDf, f \rangle = \langle Df, Df \rangle = \|Df\|^2 \leqslant \lambda(D)^2.$$

Let now $f \perp \mathbf{1}$ be a function attaining the supremum in the definition of $\lambda(D)$. Then

$$\lambda(D)^2 = \|Df\|^2 = \langle Df, Df \rangle = \langle UDf, f \rangle \leqslant \lambda_2.$$

1.2 Connectivity of $X(i)$

Let us say that X is i-connected if the upper walk on $X(i)$ is connected.

The lower walk on $X(i)$ is connected iff the upper walk on $X(i-1)$ is connected. However, even when the lower walk on $X(i)$ is connected, the upper walk on $X(i)$ could be disconnected.

If, in addition, we know that the links of vertices are all $(i-1)$-connected, then the upper walk in dimension i will be connected.

Lemma 1.3 (Connectivity from local to global)**.** *If X is $(i-1)$-connected, and the link of every vertex $v \in X(0)$ is $(i-1)$-connected, then X is i-connected.*

Proof. The proof of this lemma relies on the fact that we can "extend" a path in the lower walk to a path in the upper walk. Suppose $e = e_1, \ldots, e_n = e'$ is a path in the lower walk, namely every consecutive pair of edges have a common vertex. We can construct a new path $e = f_1, \ldots, f_m = e'$ in the upper walk, namely where every consecutive pair of edges are in a triangle. We do so by replacing the step e_i, e_{i+1} by a path $e_i, e_{i,1}, \ldots, e_{i,t} = e_{i+1}$ in the upper walk as follows. Suppose $e_i = \{v, u\}$ and $e_{i+1} = \{v, u'\}$. Look at the path $u = u_1, \ldots, u_t = u'$ in the link of the vertex v. This path gives a sequence of edges $e_{i,j} = \{v, u_j\}$ where every consecutive pair of edges is contained in a triangle, namely $\{v, u_j, u_{j+1}\}$. The same can be shown in higher dimensions. $\square$

1.3 Random-walk expansion

When would we say that a simplicial complex is an expander from the point of view of random walks? For expander graphs, we used

$$\|A - J\|^2 \leqslant \lambda \qquad \text{(two-sided)}$$

or

$$A - J \leqslant \lambda \cdot Id \qquad \text{(one-sided)}$$

where A is the transition matrix of the random walk on the vertices of the graph, and J is $\frac{1}{n}$ times the all ones matrix. Observe that $U_{-1}D_0 = J$. We can rewrite the above as

$$\|A - U_{-1}D_0\|^2 \leqslant \lambda.$$

or

$$A - U_{-1}D_0 \leqslant \lambda \cdot Id.$$

Interestingly, we can express A itself in terms of the up and down operators:

$$A = 2D_1U_0 - Id.$$

Indeed when we walk up from a vertex to an edge and then back down to a vertex we stay in place with probability 1/2 and move to a neighbor with probability 1/2. The same is true for walk in higher dimensions. We call a walk non-lazy if the probability to stay in place is zero. The up-down walk on i faces has $\frac{1}{i+2}$ laziness probability. We denote M_i^+ the non-lazy upper walk. Note that

$$DU = \frac{i+1}{i+2}M_i^+ + \frac{1}{i+2}Id.$$

This suggests the following definition,

Definition 1.4. A complex is a λ-random-walk expander if for every $i < d$,

$$\|M_i^+ - U_{i-1}D_i\|^2 \leqslant \lambda, \tag{two-sided}$$

or

$$M_i^+ - U_{i-1}D_i \preccurlyeq \lambda \cdot Id, \tag{one-sided}$$

where

$$M_i^+ = \frac{i+2}{i+1}D_{i+1}U_i - \frac{1}{i+1}Id$$

is the non-lazy upper random walk.

This definition seems complicated and technical, so let us say a few words about it.

Expanders are viewed as graphs that are on one hand sparse and on the other hand, at least from random-walk point of view, imitate the complete graph. So they are sparse objects that are *pseudo-dense*. A generalization to higher dimensions is to find simplicial complexes that are sparse, yet imitate the complete complex, and are *pseudo-dense*.

How would we measure if a given s.c. imitates the complete complex? If we fix the same number of vertices n, there are very different numbers of edges and triangles and so on. So the random-walk operators work in spaces whose dimensions are different, and it is unclear how to compare them.

Instead, we use a comparison between upper walks and lower walks of a complex to itself. The complete complex demonstrates identical behavior between upper and lower, so we let this be the model for our definition.

Is this definition any good? The following two theorems give some nice justification for it. We will first prove, in Theorem 2.1, that this definition is very much related to the link-expansion definition we gave in lecture 1. We will then analyze the second largest eigenvalue of i-walks in the complete complex and show that complexes satisfying this definition have nearly the same eigenvalues.

2 Link expansion and Random-walk expansion

Theorem 2.1 (link-expansion and random walk expansion)**.** *Every one-sided (two-sided) γ-link HDX is a one-sided (two-sided) γ-random-walk HDX.*

Moreover, if X is a two-sided γ-random-walk HDX, then it is a $3d\gamma$-two-sided link HDX.

Proof. We prove the first implication. Assume that X is a γ-link expander, one-sided. Let $f \in C^i(X)$, $i < d$. We have

$$\langle M_i^+ f, f\rangle = \mathop{\mathbb{E}}_{s\in X(i)} \mathop{\mathbb{E}}_{t>s} \mathop{\mathbb{E}}_{s'<t, s'\neq s}[f(s)f(s')] = \mathop{\mathbb{E}}_{t\in X(i+1)} \mathop{\mathbb{E}}_{s,s'<t, s\neq s'}[f(s)f(s')].$$

Let $r = t \setminus \{x, y\}$. Since $t \sim X(i+1)$ and $x \neq y \in t$ are chosen at random, we have $r \sim X(i-1)$. Given such an r, the probability to get specific (t, x, y) is exactly like choosing a random edge in X_r so

$$\langle M_i^+ f, f\rangle = \mathop{\mathbb{E}}_{r\sim X(i-1)} \mathop{\mathbb{E}}_{(x,y)\in X_r(1)}[f(r\cup\{x\})f(s\cup\{y\})]. \tag{2.1}$$

In other words, we have shown that

$$\langle M_i^+ f, f\rangle = \mathop{\mathbb{E}}_{r\sim X(i-1)}[\langle A_r f_r, f_r\rangle], \tag{2.2}$$

where $f_r \colon X_r(0) \to \mathbb{R}$ is defined by

$$f_r(x) = f(r \cup \{x\}).$$

We now note that

$$\mathop{\mathbb{E}}_{x\sim w_r}[f(r\cup\{x\})] = (Df)(r).$$

Therefore we have, by the γ-expansion in the links that

$$|\langle M_i^+ f, f\rangle - \langle UDf, f\rangle| = \Big| \mathop{\mathbb{E}}_{r\sim X(i-1)} \mathop{\mathbb{E}}_{(x,y)\sim w_r}[f(r\cup\{x\})f(r\cup\{y\})] - (Df)(r)^2 \Big| \leqslant \mathop{\mathbb{E}}_{r\sim X(i-1)}\Big[\lambda(A_r) \mathop{\mathbb{E}}_{x\sim w_r}[f(r\cup\{x\})^2]\Big].$$

If X is a γ-two-sided link expander then $\lambda(A_r) \leqslant \gamma$ for all s, and so

$$|\langle (M_i^+ - UD)f, f\rangle| \leqslant \gamma \|f\|^2.$$

□

3 Bibliographic notes

Higher order random walks were first studied in [4] where the authors defined a combinatorial "random-walk" type of expansion, and showed that it is implied by expansion of the links, thus proving that in high dimensional link-expanders, higher order random walks are rapidly mixing. This was refined in [3, 5], who gave stronger convergence bounds, showing that random walks on a high-dimensional expander converge at "optimal" rates, namely at nearly the same speed as random walks on the complete complex. (In [3] the focus is on two-sided link expansion and in [5] the one-sided link expansion is analyzed). The work [2] showed a reverse connection in the case of two-sided expansion, where random walk expansion also implies link expansion (with a loss in parameters) and therefore they are qualitatively equivalent. The random walk definition of high dimensional expansion is also generalized beyond simplicial complexes also to ranked posets (partially ordered sets).

References

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