Lecture 4: Cohomological expansion

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In this lecture we will study topological and co-homological perspectives on expansion. This presentation follows a very nice lecture by Uli Wagner, [4].

1 Basic Definitions

We begin with the definition of the boundary and co-boundary operators.

1.1 The Boundary Operator

The boundary of an edge $e = \{u, v\}$ is its two endpoints, u and v. The boundary of a triangle $\{u, v, w\}$ is its three edges, $\{u, v\}$, $\{u, w\}$ and $\{v, w\}$. The boundary operator is a linear algebraic version of this notion. We don't only consider faces but also formal linear combination of faces, namely chains. Let the set of i-chains be

$$C_i(X, \mathbb{Z}_2) = \mathbb{Z}_2^{X(i)} = \{f : X(i) \to \mathbb{Z}_2\}.$$

We view this set as a group, where addition is done element wise. Let $\mathbf{1}_s$ be the i-chain that is 1 on s and zero elsewhere. The chains $\mathbf{1}_s$ for $s \in X(i)$ span the set of all i-chains. We define the boundary operator by setting

$$\partial_i \mathbf{1}_s = \sum_{t<s} \mathbf{1}_t$$

where $t < s$ refers to all $(i-1)$-faces that are contained in s. This definition is equivalent to

$$\partial_i : C_i \to C_{i-1}, \quad \forall t \in X(i-1),\ \partial f(t) = \sum_{s>t} f(s) \mod 2.$$

Note that in this definition we are summing over all i-faces above t. It looks very similar to the down operator we defined in previous lectures, since it uses the same incidence structure. However, we are now summing modulo 2 and not over the real numbers. When clear from the context we omit the subscript and write ∂ for ∂_i.

Example 1.1. Suppose $f \in \mathbb{Z}_2^{X(1)}$ is an indicator of a path from v to v'. The boundary of f is the two end vertices, namely $\partial f = \mathbf{1}_v + \mathbf{1}_{v'} \in C_0$. If f is an indicator of a cycle, then its boundary is zero.

Who are all of the 1-chains whose boundary is zero? This is $Ker\partial_1 = \left\{f \in C_1 \,\middle|\, \forall v \in X(0),\ \sum_{e>v} f(e) = 0 \mod 2\right\}$. Namely, all of the subgraphs with even degrees.

Suppose $f = \mathbf{1}_{\{u,v,w\}} \in C_2$. Then $g = \partial f = \mathbf{1}_{\{u,v\}} + \mathbf{1}_{\{v,w\}} + \mathbf{1}_{\{u,w\}}$. We can further ask about the boundary of g, which is $\partial g = \mathbf{1}_u + \mathbf{1}_v + \mathbf{1}_v + \mathbf{1}_w + \mathbf{1}_u + \mathbf{1}_w = 0$. In fact, for every $f \in C_2$, since it is a linear combination of functions $\mathbf{1}_{\{u,v,w\}}$, we deduce that $\partial\partial f = 0$. Namely, a boundary has no boundary! This holds in general for any i,

$$\partial_{i-1} \circ \partial_i = 0$$

(check this!). We look at the chain of maps

$$C_i \overset{\partial_i}{\to} C_{i-1} \overset{\partial_{i-1}}{\to} C_{i-2} \overset{\partial_{i-2}}{\to} \cdots \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0$$

as a single “high order” linear-algebraic object, which is called a *chain complex*. This is a sequence of linear maps that satisfies that every consecutive pair is zero, namely the image of one arrow is in the Kernel of the next arrow.

We define the boundaries B_i and the cycles Z_i by

$$B_i = Im(\partial_{i+1}), \qquad Z_i = Ker(\partial_i).$$

By the fact that $\partial_i \circ \partial_{i+1} = 0$,

$$B_i \subseteq Z_i \subseteq C_i,$$

so every boundary is a cycle, but not always vice versa. We define the i-th homology to be the quotient space

$$H_i = Z_i / B_i.$$

The dimension of this space measures how large the gap between cycles and boundaries. When the homology is zero it means that $B_i = Z_i$, so this space is defined both "explicitly" as the images of $i+1$-chains, and implicitly as elements f satisfying linear constraints specified by the requirement $\partial_i f = 0$.

1.2 The Co-boundary Operator

Next, we move to the “adjoint” operators, namely the co-boundary operators,

$$\delta_i : C_i \to C_{i+1}$$

which give us the chain of coboundary maps

$$C_{-1} \overset{\delta_{-1}}{\to} C_0 \overset{\delta_0}{\to} C_1 \overset{\delta_1}{\to} \cdots \overset{\delta_{i-1}}{\to} C_i \overset{\delta_i}{\to} C_{i+1} \to \cdots$$

where δ_i is given by

$$\forall f \in C_i, \forall s \in X(i+1), \quad \delta_i f(s) = \sum_{t<s} f(t) \mod 2.$$

The set of coboundaries is the set $B^i = Im(\delta_{i-1})$ and the set of cocycles is $Z^i = Ker(\delta_i)$. Like before,

$$\delta_i \circ \delta_{i-1} = 0$$

and so

$$B^i \subseteq Z^i \subseteq C^i$$

and we let the i-th cohomology be $H^i = Z^i / B^i$.

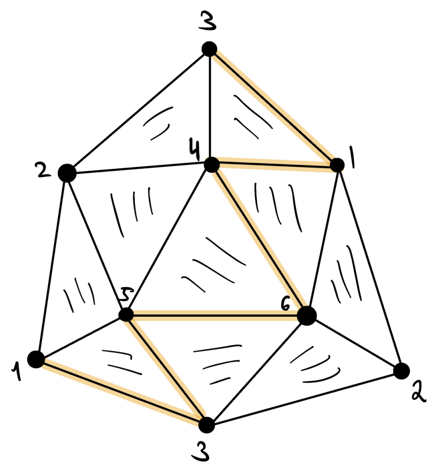
We use superscript to denote the coboundaries, cocycles, and cohomology; and subscripts to denote the boundaries, cycles, and homology.

Let us explore a few example calculating B^1, B^0, Z^0 and H^0.

Example 1.2 (B^1)**.** Fix any $f \in C_0$. By definition, $f = \mathbf{1}_S$ for some set of vertices $S \subseteq X(0)$. What is δf? this is an indicator of all edges crossing between S and $X(0) \setminus S$. So $B^1 = \left\{ \mathbf{1}_{E(S,\bar{S})} \;\middle|\; S \subset X(0) \right\}$.

Example 1.3 (B^0)**.** Fix any $f \in C_{-1}$. Since $X(-1) = \{\phi\}$, $C_{-1} = \mathbb{Z}_2^{X(-1)} = \mathbb{Z}_2$. If $f = 0$, then $\delta f \in C_0$ is the all 0 function. If $f = 1$, then δf is the all-1 function. So $B^0 = \{\bar{0}, \bar{1}\}$.

Example 1.4 (Z^0 and H^0)**.** Fix any $f \in Z^0$. Since $\delta f = 0$ we know that for every $\{uv\} \in X(1)$, $\delta f(uv) = f(u) + f(v) = 0$. In other words $f(u) = f(v)$ whenever u, v have an edge between them. By transitivity, f is constant on every connected component of the graph $(X(0), X(1))$. We have $Z^0 = B^0$ (and $H^0 = 0$) iff the graph is connected. If the graph is disconnected, H^0 is spanned by functions that are constant on the different connected components.



In the figure we see an example of a 2-dimensional complex where the first cohomology is non trivial. This complex has six vertices $X(0) = \{1, 2, 3, 4, 5, 6\}$, and 15 edges, and 10 triangles. The highlighted 1-chain, consisting of the five edges $13, 35, 56, 64, 41$, is a cocycle, since every triangle touches an even number of edges. Why is it not a coboundary? Suppose it were, then there is some $f \in C_0$ such that δf is the highleted edges. Assume wlog that $f(1) = 0$. Then necessarily $f(3) = 1$, so $f(5) = 0$, so $f(6) = 1$, so $f(4) = 0$, so $f(1) = 1$, and we have reached a contradiction!

2 Coboundary Expansion

At this point we already have enough terminology to restate the combinatorial definition of graph expansion in cohomological terms. Recall that the edge expansion of a graph G is

$$h(G) = \min_{S \subset V, |S| \leqslant |V|/2} h(S).$$

In cohomological terms, it is

Claim 2.1. For any connected graph G,

$$h(G) = \min_{f \in C_0 \setminus B^0} \frac{wt(\delta f)}{\text{dist}(f, B^0)}.$$

Proof. Observe that $f \in C_0 \setminus B^0$ is an indicator of some set $\phi \neq S \subsetneq V$. The numerator is equal to $|E(S,\bar{S})|/|E|$. The denominator is equal to the minimum between $|S|/|V|$ and $|\bar{S}|/|V|$. ◻

This now begs to be generalized to $i \geqslant 0$,

Definition 2.2 (Coboundary expansion)**.** The coboundary expansion of a d-dimensional simplicial complex X, at level $i < d$, is defined to be

$$h^i(X,\mathbb{Z}_2) = \min_{f \in C_i \setminus B^i} \frac{wt(\delta_i f)}{\mathrm{dist}(f,B^i)}.$$

Here is a possible interpretation for h^i. Assume a d-dimensional complex X such that for some $i < d$, $B^i = Z^i$, namely $H^i = 0$. A chain in B^i is of the form δg for some $g \in C_{i-1}$. Given a chain $f \in C_i$, we can easily check if it is in $Z^i = B^i$ by computing δf and checking that it is zero. This can be done very efficiently, for every $s \in X(i+1)$, to see if $\delta f(s) = 0$ we simply read f at the $i+2$ faces $t < s$ and check that $\sum_{t<s} f(t) = 0$ mod 2. One immediately sees that it is easy to randomly estimate $wt(\delta f)$ by sampling several $s \in X(i+1)$ and performing this calculation.

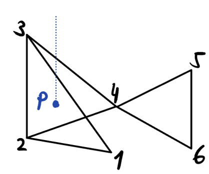
In contrast, given a chain $f \in C_i$, estimating its distance from B^i requires, in principle, going over an exponentially large set (indeed, in some cases this is NP-hard). However, when h^i is a constant, these two measures are related. In particular, the easy-to-calculate $wt(\delta f)$ is an upper bound on $\mathrm{dist}(f,B^i) \cdot h^i$.

The value of h^i can be zero when $B^i \neq Z^i$, or even when $Z^i = B^i$ it can be very small, on the order of $1/|X(i)|$. Nevertheless, it can be large, as we will see.

3 Connection to Topology

In combinatorial topology people are often interested in embedding a simplicial complex into $\mathbb{R}^d$, and studying some topological properties of this so-called "geometric realization" of the complex. This is different from our point of view in most of this course, which is looking at an abstract simplicial complex, and usually not embedding it into $\mathbb{R}^d$.

Let $T : X \to \mathbb{R}^d$ be a linear map. This means that we first map $X(0)$ to points in $\mathbb{R}^d$, and then continue the map linearly. It turns out that $\mathbb{Z}_2$-cohomology is interesting for such maps. In the figure below we see one such embedding of the fish complex in $\mathbb{R}^2$.



Fix some point $p \in Im(T)$ inside the image of one of the triangles of the complex (not on any vertex or edge). Let $f_p \in C_2$ be the indicator of the set of triangles that contain p: $f_p(s) = 1$ iff $T(s) \ni p$.

Claim 3.1. $f_p \in B^2$

Proof. Shoot a ray from p to infinity in an arbitrary direction. We can check that $f_p = \delta g_p$ where $g_p \in C_1$ indicates all edges crossed by this ray. Indeed to exit any triangle that contains p, the ray must cross its boundary once. Every other traiangle is crossed either zero times or twice. ◻

3.1 Topological Overlap Property

In the 1980's Boros and Furedi, who were high school students at the time, asked the following question. Suppose we put n points in $\mathbb{R}^2$, and connect all pairs with edges. We have $\binom{n}{3}$ triangles. Is there a point $p \in \mathbb{R}^d$ that pierces a large fraction of triangles? They proved there is always a point that pierces $2/9$ of the triangles. This was generalized by Barany to d dimensions, but the constant $2/9$ is replaced with some value that is exponentially small in d. This was viewed as a question in discrete geometry, and not related at all to topology.

Gromov, in [2], gave a very strong and surprising generalization, where he showed that the same $2/9$ holds even when the straight lines are replaced with any continuous curves. Moreover, he showed, that the theorem is really about the embedding of the complete complex into $\mathbb{R}^2$ (or $\mathbb{R}^d$ more generally), and follows from the *coboundary expansion* of the complex that is being embedded.

Let's see what happens for $d = 1$. We are embedding the vertices of a graph into the real line, and are asking whether there must be a point on the line that pierces a constant fraction of the edges. This is implied if the graph is an expander. Indeed, take a point that splits the vertices into two equal parts, it must pierce a linear number of edges...

Gromov defined the topological overlap property (TOP): a complex X has TOP if for any embedding $T : X \to \mathbb{R}^d$ there is a point that pierces a constant fraction of $X(d)$. He proved that it is implied by coboundary expansion (which we define shortly next) and then asked if there is a bounded-degree family of d-dimensional complexes with this property. This question energized the area of HDX and in a sequence of works [3, 1] it was shown that certain high dimensional expanders are cosystolic expanders, which is also sufficient for TOP.

<need to complete refs here>

References

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[2] Mikhail Gromov. Singularities, expanders and topology of maps. part 2: from combinatorics to topology via algebraic isoperimetry. *Geometric and Functional Analysis*, 20(2):416–526, 2010. 5

[3] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. Ramanujan complexes and bounded degree topological expanders. In *Proc. 55th IEEE Symp. on Foundations of Comp. Science (FOCS)*, pages 484–493, 2014. 5

[4] Uli Wagner. *Expansion from a Cohomological Viewpoint*, 2019. https://simons.berkeley.edu/talks/cohomological-viewpoint-expansion. 1