Lecture 8: Cosystolic expansion: local to global

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January 8, 2023

In this lecture we will prove cosystolic expansion for bounded-degree complexes, via a local to global connection. We will show that if a simplicial complex has links that are coboundary expanders, together with spectral expansion, this implies global cosystolic expansion. This theorem was proven by Kaufman Kazhdan and Lubotzky [3] for dimension 1, and by Evra and Kaufman [1] for higher dimensions.

Cosystolic expansion, as we have mentioned before, is related to

- Property testing (we have seen, for example, that coboundary expansion of the complete complex is the same as bi-clique testability)
- Locally testable codes and PCPs
- Topological overlap property (TOP)
- Stability of covers

1 Definitions

Recall from lecture 4 that the i-coboundaries are $B^i = Im(\delta_{i-1})$ and the i-cocycles are $Z^i = Ker(\delta_i)$. We noted that $B^i \subseteq Z^i \subseteq C^i$ and defined the i-th cohomology to be $H^i = Z^i/B^i$

A simplicial complex X is a β-*cosystolic expander* in dimension i if for every i-chain $f \in C^i(X, \mathbb{Z}_2)$,

$$wt(\delta_i f) \geqslant \beta \cdot \mathrm{dist}(f, B^i).$$

If, furthermore, $H^i = 0$, we say that X is a β-*coboundary expander* in dimension i. We also denoted by h^i the largest β for which X is a β-cosystolic expander:

$$h^i(X, \mathbb{Z}_2) = \min_{f \in C_i \setminus B^i} \frac{wt(\delta_i f)}{\mathrm{dist}(f, B^i)}.$$

Is there a "Cheeger's inequality" for high dimensional expanders? One could hope that every spectral HDX is a cosystolic expander, and possibly also vice versa. However, this is not true. Gundert and Wagner [2] give a random construction of a complex which is a very good spexctral excpander yet the cosystilic constant is $o(1)$.

2 Local to Global: cosystolic expansion comes from expansion in the links

Theorem 2.1. *Let X be a 3 dimensional simplicial complex. Suppose that*

– *X is a γ-two-sided link expander.*

– *For every $v \in X(0)$, X_v is a β_0-coboundary expander in dimension 1.*

Then, X is a β cosystolic expander in dimension 1 for some constant β.

Observe that the statement makes sense only if the dimension of the link is at least 2 (otherwise we cannot speak of coboundary expansion of 1-chains), so the complex needs to be 3 dimensional for such a statement to hold. Are there 2-dimensional HDX that are not 2-skeletons of 3 dimensional HDX and yet are cosystolic expanders? This is not known.

The theorem can be generalized to expansion in any dimension i, as long as the complex is $i+2$ dimensional [1].

Proof. Let $f \in C^1(X, \mathbb{Z}_2)$, and let $\varepsilon = wt(\delta f)$. We will show that there is some $\tilde{f} \in Z^1$ such that

$$\beta' \cdot \mathrm{dist}(f, \tilde{f}) \leqslant wt(\delta f).$$

We use the following local correction algorithm:

Algorithm. If there is a vertex such that changing f on all edges touching v can reduce $wt(\delta f)$, change it. Repeat

Observe that the algorithm must halt because every iteration reduces the size of δf by at least one. Let f be the initial chain, and let $\tilde{f}$ be the chain after the algorithm halts.

Claim 2.2. There is some constant d_0 such that $\mathrm{dist}(f, \tilde{f}) < \varepsilon \cdot d_0$.

Proof. The number of iterations of the algorithm is at most the support of δf, which we denote by $|\delta f| = \varepsilon |X(2)|$. So the number of edges that are modified during the course of the algorithm is at most $|\delta f| = \varepsilon |X(2)| = \varepsilon |X(1)| \cdot d_0$, where $d_0 = \frac{|X(2)|}{|X(1)|}$. □

Our main lemma is the following

Lemma 2.3. *If $\delta \tilde{f} \neq 0$ then $wt(\delta \tilde{f}) > \tau$.*

This implies the theorem with $\beta = \min(\tau, \frac{1}{d_0})$ because if $\delta \tilde{f} = 0$ then the above claim gives the required inequality, and if $\delta \tilde{f} \neq 0$ then $wt(\delta f) > \tau \cdot 1 \geqslant \tau\, \mathrm{dist}(f, \tilde{f})$. □

2.1 Proof of Lemma 2.3

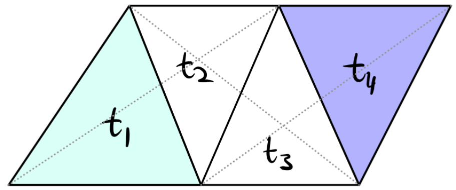
In this section let f denote the chain after the algorithm ended. Let

$$T^* = \{t \in X(2) \mid \ \delta f(t) \neq 0\}$$

The heart of the matter is to look at redundancies between the triangle constraints, and use them to propagate errors. Consider a 3-face $p = \{a, b, c, d\}$. It contains four triangles and six edges. Moreover, no matter when the value of f on the edges is, the number of triangles that belong to T^* must be even. So if it is non-zero, it must be larger than one.

Now consider an upper random walk

$$t_1 \to t_2 \to t_3 \to t_4$$



where every consecutive pair of triangles belong together to a 3-face. (This implies that they intersect on an edge). If we know that $t_1 \in T^*$, there is a probability of at least 1/3 that $t_2 \in T^*$, because in the 3-face $t_1 \cup t_2$ there are at least two triangles for which $\delta f \neq 0$. For the same reason, there is probability $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$ that $t_3 \in T^*$ and probability $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$ that $t_4 \in T^*$.

If this were the case *even conditioned on* $t_1 \cap t_4 = \phi$, then we would get, via swap-walk arguments, that the set T^* is large.

Claim 2.4. If $\mathbb{P}_{t_1,t_2,t_3,t_4}[t_4 \in T^* | t_4 \cap t_1 = \phi] \geqslant \mathbb{P}[t_1 \in T^*] \cdot \alpha$ then $|T^*| = \Omega(X(2))$.

The reason is that the graph whose nodes are the triangles, and where we connect two triangles t_1, t_4 according to the rule above, is an expander graph. So by applying the Alon-Chung lemma we get that a subset of nodes with average degree above α must be linearly large as long as $\alpha > \lambda$.

However, the proof is more complicated, because there is a constant fraction of random walks $t_1 \to t_2 \to t_3 \to t_4$ in which all of t_1, t_2, t_3, t_4 share an edge, or a vertex. If $t_1 \in T^*$ it could a priori be that all of the walks in which also $t_4 \in T^*$ are such walks. In that case the above reasoning would fail. However, when this happens it means that either

– One of the edges touching t_1 is heavy (touches many triangles in T^*), or

– One of the vertices touching t_1 is heavy (touches many triangles in T^*)

So to complete the proof we show that the set of heavy vertices and edges is tiny, even with respect to T^* and therefore, most of the time the walks don't get stuck.

Why are there few heavy vetices, for example? For this we must look at the triangles in its link

$$T_v^* = \{t \in T^* \mid \ t \cup v \in X(3)\}\,.$$

Claim 2.5. If T_v^* is small, we can use coboundary expansion of X_v to find a better assignment to the edges touching v.

In addition, we can show, using the fact that the bipartite graph connecting a vertex v to a triangle $t \in T_v$ is a very strong expander, that only very few vertices v can have large T_v^* (since all in all the set T^* is by assumption not too large). So only those vertices can be heavy vertices, and these make up a tiny fraction of all vertices.

A similar, but more subtle, argument can be made for edges. (We show that except for a tiny minority, a heavy edge almost always touches a heavy vertex).

3 Is the extra dimension really needed?

One might conjecture that link-expansion suffices for proving coboundary expansion. This would give a "Cheeger inequality" in dimensions higher than 0. However, Gundert and Wagner [2] give a counterexample. They describe a randomized construction of a 2-dimensional simplicial complex whose links are excellent spectral expanders; and yet the complex has poor coboundary expansion.

References

[1] Shai Evra and Tali Kaufman. Bounded degree cosystolic expanders of every dimension. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 36–48, 2016. 1, 2

[2] Anna Gundert and Uli Wagner. On eigenvalues of random complexes. *Israel Journal of Mathematics*, 216(2):545–582, 2016. 1, 4

[3] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. Ramanujan complexes and bounded degree topological expanders. In *Proc. 55th IEEE Symp. on Foundations of Comp. Science (FOCS)*, pages 484–493, 2014. 1