High Dimensional Expanders and Property Testing

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§1. Expanders: one dimensional to high dimensional

Expander graphs can be defined by a number of equivalent definitions.
Let $X = (V, E)$ be a k-regular graph

(I) **Combinatorial** X is ε-expander $(\varepsilon > 0)$ if $h(X) \geq \varepsilon$
where

$$h(X) = min\Big\{ \frac{|\partial Y|}{|Y|} \,\big|\phi \neq Y \subset V,\ Y < \tfrac{1}{2}|V| \Big\}$$

is the Cheeger constant.

(II) **Spectral** X is λ-(spectral) expander $(\lambda < 1)$ if the 2nd largest e.v. $\lambda_1(X)$ of the random walk operator $(\Delta = \frac{1}{k}Adj)$ is at most λ.

Various methods to get expanders:

(i) random

(ii) Kazhdan property (T)

(iii) Zig-Zag

(iv) Ramanujan graphs ($\equiv$ optimal from spectral point of view, i.e., $\lambda_1 \leq 2\sqrt{k-1}$) were obtained as $X = \Gamma(m) \setminus SL_2(F)/K$ where $F = \mathbb{Q}_p$ or $\mathbb{F}_p((t)),\ K =$maximal compact and $\Gamma(m) =$ congruence subgroup of an arithmetic lattice.

(v) Interlacing polynomial

§2. What are high dimensional expanders (HDX)?

These are supposed to be simplicial complexes which resemble one (or more) of the properties of expander graphs.

Various definitions have been suggested. Let's start with the **least** intuitive one, which generalizes the Cheeger constant.

For it, one needs the language of cohomology.

Simplicial complexes and their cohomology

Let X be a (uniform) d-dimensional simplicial complex, i.e., a collection of (finite) subsets of a set of vertices V closed under inclusion. If $Y \in X$, $dim Y = |Y| - 1$ and all maximal elements of X are of size $d+1$. Let $X(i) =$ simplices of dim i, so $X(0) = V$, $X(-1) = \{\emptyset\}$. Let $C^n = C^n(X) = \{f : X(n) \to \mathbb{F}_2 = \{0,1\}\}$ an $\mathbb{F}_2$-vector space of dim $|X(n)|$.

Let $d = d_n : C^n \to C^{n+1}$, $(df)(\sigma) = \sum\limits_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} f(\tau)$ for $\sigma \in X(n+1)$.

Ex: $d \circ d = 0$

Denote $Z^n = Z^n(X) = \mathrm{Ker}(d_n)$, $B^n = Im(d_{n-1})$
by the Ex, $B^n \subseteq Z^n$ and let $H^n := H^n(X, \mathbb{F}_2) = Z_n / B_n$.
The n-cohomology group of X over $\mathbb{F}_2$

For every $\sigma \in X(n)$,

$$wt(\sigma) = \frac{1}{|X(d)|} \#\{\tau \in X(d) | \tau \supseteq \sigma\}$$

and for $f \in C^n$, $\|f\| = \frac{1}{|X(n)|} \sum_{\sigma \in \text{Support}(f)} wt(\sigma)$.

Def. (Gromov, Linial-Meshulam)

"Coboundary expansion"

$$h_i(X) = \min\left\{ \frac{\|df\|}{dist(f, B^i)} \,\middle|\, f \in C^i \setminus B^i \right\}$$

EX: **(i)** For a graph X, $h_0(X) = h(X)$ - the Cheeger constant

(ii) For the complete d-dim complex in n vertices $h_i(X_{d,n}) \gg 0$.

- [LM] for studying random complexes
- [G] for "topological overlapping" and topological/geomeric expanders

§. 3 Combinatorial HDX and property testing

Kaufman-Lubotzky: $h_i > 0$ means property testing!

namely: Given $f \in C^i$, is $f \in B^i$?

Answer: Pick a random cell σ in $X(i+1)$ and calculate $df(\sigma)$.

If $df(\sigma) = 0$ answer YES

If $df(\sigma) = 1$ answer NO

Note: to calculate df one reads f at only $i+1$ places, (read $f(\tau)$ for $\tau \subset \sigma$).

• Several applications, e.g., given ± 1 $n \times n$ matrix A, with 1's along the diagonal.
Is $A = v \otimes v$ for some ± 1 n-vector?

Answer: Choose i, j, k and check $a_{ij} a_{jk} a_{ki} \stackrel{?}{=} 1$

Pf: apply $h(X_{2,n}) > 0$ $\boxtimes$

§4. Spectral HDX

Given a s.c. X and $\tau \in X(i)$, $-1 \leq i \leq d-2$.

The link $X_\tau = \{\sigma \setminus \tau \,|\, \tau \subseteq \sigma \in X\}$ is a s.c. of dimension $d-i-1$.

X is spectral HDX if $\exists \lambda < 1$, s.t. $\forall \tau$, the 2nd e.v. of the random walk operator on the 1-skeleton of X is at most λ.

Garland Theory (Garland 1972, Oppenheim - recent years)

If there is a very good bound for all $\tau \in X(d-2)$, then X is spectral HDX, i.e., if X_τ is an excellent expander graph, then X is spectral HDX.

Note: This is a "local to global" property (which was also the inspiration toward the recent LTC of [DELLM]).

Many applications (Serre conj/Property T/Unitary stability, etc.)

§5. Random walks on spectral HDX

Dinur $\&$ Kaufman initiated the study of random walks on spectral HDX.

They gave sharp estimates on

- up-down walk i.e. fix $i < d$: random move from $\tau_0 \in X(i)$ means: move first to a random $\sigma \in X(i+1)$ containing τ_0 and then delete a random vertex from σ to get $\tau_1 \in X(i)$
- down-up walk

§6. Random walks on matroids

The work of [OK] inspired [ALOGV] (Anari, Liu, Oveis Gharan, Vinzant) and led to a breakthrough on random walks on matroids.

An application: Let $G = (V, E)$ - we want to sample a random spanning tree T (they form a matroid).

The alg: Start with any such T_0, delete a random edge from it and add a random edge among the ones which will make it back connected. This will be T_1.

To analyze it: Define a s.c. X, by $X(0) = E$ and $X(n-2) =$ the spanning trees (close downward to make it s.c. - give weights according to the number of span trees containing it). This is s.c. of dim $d = n-2$, $n = |V|$.

By Garland look at $\tau \in X(n-4)$: This is a spanning tree minus two edges. This divides V into 3 connected components.

The vertices of $X_\tau =$ edges between 2 out of the 3.

The edges of $X_\tau =$ are all pairs from above which do **not** connect the same two components.

This is an excellent expander (check!) and hence the RW converges fast poly(n) to the uniform distribution (independent of G!)

§7. Construction of HDX

Unlike expanders, so far only two methods to construct HDX **of bounded degree**.

(I) Ramanujan complexes (Lubotzky-Samuels-Vishne 2006)

$$X = \Gamma(m) \setminus B(PGL_d(\mathbb{F}_q((t))))/K$$

B = Bruhat-Tits building.

K = maximal compact subgroup of $G = PGL_d(\mathbb{F}_q((t)))$

$\Gamma(m)$ = Congruence subgroup of arithmetic lattice Γ.

(II) Coset geometries (Kaufman-Oppenherm 2018, generalization by Pratt 2022)

Coset geometries which are complexes of cosets of Chevalley groups (e.g. SL_n) over rings (e.g. $\mathbb{Z}$).

Major open problem: Is there a "good" random model for HDX?

Fox-Gromov-Lafforgue-Naor-Pach gave a random model which gives "geometric expanders" but not more.