Coboundary Expansion of Spherical Building

Yotam Dikstein*

May 2, 2023

Based on joint work with Irit Dinur [DD23]. Today we will show that the $SL_n(\mathbb{F}_q)$-spherical building is a coboundary expander.

1 Introduction

Let X be a simplicial complex. Recall that $C^i = \{f : X(i) \to \mathbb{F}_2\}$. We defined the *coboundary* maps $d = d_i : C^i \to C^{i+1}$ to be

$$df(t) = \sum_{s \subseteq t, |s|=|t|-1} f(s). \tag{1.1}$$

We also defined coboundary expansion.

Definition 1.1 (Coboundary expansion)**.** Let X be a d-dimensional simplicial complex, let $\beta > 0$. We say that X is a β-coboundary expander for on level i (or just write $h^i(X) \geqslant \beta$), if the following holds. For every $f : X(i) \to \mathbb{F}_2$ there exists $g : X(i-1) \to \mathbb{F}_2$ such that

$$\beta \operatorname{dist}(f, d_{i-1}g) \leqslant wt(df) = \mathop{\mathbb{P}}_{t \in X(i+1)} [d_i f(t) \neq 0].$$[1]

This is a *property testing* notion. In property testing we have a subset $P \subseteq C^i$. We are given some $f \in C^i$, and we want to determine whether $f \in P$ or f is far away from P. We typically want to do so while reading as few entries of f as possible.

- The property P we are testing is $B^i = \mathrm{I}md_{i-1}$.
- The tester gets $f : X(i) \to \mathbb{F}_2$ as input, samples $t \in X(i+1)$ and reads f on all faces $\{s \subseteq t\}$.
- The test accepts if $df(t) = 0$ and rejects otherwise. Indeed, every $dg \in B^i$ has that $d(dg) \equiv 0$ so this test is complete for B^i.
- The probability $\mathbb{P}_{t \in X(i+1)} [d_i f(t) \neq 0]$ is the test rejection probability.
- The soundness promise we have is that if the test rejects with small probability then f is close to some $dg \in B^i$ ($\beta > 0$ is the soundness parameter).

*Weizmann Institute of Science, ISRAEL. email: yotam.dikstein@weizmann.ac.il.

[1] Recall that we have probability distributions on $X(j)$ where $\mathbb{P}_{s \in X(j)} [s] \propto \#\{t \in X(d), t \supseteq s\}$. Thinking about it too much makes your head hurt, so we usually just imagine that all probabilities are equal.

1.1 Motivation

We will see more motivation for coboundary expansion by the end of this seminar. Here are some slogans that will do for now.

- As we saw last week, it generalizes graph connectivity to higher dimension.
- This notion relates to testability and robustness of *topological properties* of a simplicial complex: topological overlap [2] [Gro10] and cover stability [DM22].
- Locally testable codes and quantum codes could be phrased in a similar way as coboundary expansion. Techniques for proving coboundary expansion could be used in other settings as well. This is viewed by some as a clean and simple setting to test such techniques.

We stress that the fact that $df(t) = 0$ on *almost all* triangles $t \in X(i+1)$, doesn't necessarily mean that f itself is close to a function $h \in C^i$ such that $dh(t) = 0$ on *all* $t \in X(i+1)$.

Example 1.2. Here is an example for a non-coboundary expander. Let X be the path over $2n$ vertices (a 1-dimensional simplicial complex). That is $X(0) = [2n]$, and $X(1) = \{\{v_i, v_{i+1}\} \mid \ i = 1, 2, ..., 2n-1\}$. The coboundary map defines a set of equations:

$$\overbrace{f(v_i) + f(v_{i+1})}^{df(v_i v_{i+1})} = 0.$$

The only functions that satisfy all equations are the constant functions. However, the function

$$f(v_i) = \begin{cases} 1 & 1 \leqslant i \leqslant n \\ 0 & i > n \end{cases},$$

is far from both constant functions. Yet, it satisfies all but one equation (i.e. $df(v_i v_{i+1}) = 0$ except when $i = n$). In the language, $\mathrm{dist}(f, dg) \geqslant \frac{1}{2}$ for any $g \in B^0$, but $\mathbb{P}_i\left[df(v_i v_{i+1}) \neq 0\right] = \frac{1}{2n-1}$.

1.2 Coboundary Expansion via Decoding Cones

For the rest of this talk, we restrict ourselves to coboundary expansion for level 1. However, all results generalize to arbitrary levels. We now develop a method to decode a function $f : X(1) \to \mathbb{F}_2$ such that $df \approx 0$, to some $dg \approx f$. This method will work on symmetric simplicial complexes. The following appeared in [Gro10] implicitly, spelled out explicitly by [LMM16], and then abstracted by [KM19] and independently by [KO21].

Definition 1.3 (Decoding cone)**.** A decoding cone is a triple $C = (v_0, \{P_u\}_{u \in X(0), v_0 \neq u}, \{T_{uw}\}_{uw \in X(1)})$ such that

1. $v_0 \in X(0)$.
2. For every $v_0 \neq u \in X(0)$ $P_u \subseteq X(1)$ is (edges of) a path from v_0 to u.
3. For every $uw \in X(1)$, $T_{uw} \subseteq X(2)$ is a "tiling" of the cycle $P_u \circ uw \circ P_w$. More precisely, $T_u w$ is a set of triangles, such that the set of edges that appear an odd number of times in the multi-set $\bigcup_{xyz \in T_u w} \{xy, yz, zx\}$ is $P_u \circ uw \circ P_w$[3].

For a function f we define its C-decoding $g_C : X(0) \to \mathbb{F}_2$ by

$$g_C(v_0) = 0 \text{ and } g_C(u) = \sum_{e \in P_u} f(e).$$

If $f = dh \in B^1$, then it is easy to see that $g_C(u) = h(u) + h(v_0)$ and in particular $f = dg_C$ as well, so the decoding is "complete". The following observation connects the tilings to the "soundness".

[2] Loosly, how many collisions must there be in a low-dimensional embedding of the simplicial complex?

[3] Actually, it should be the edges that appear an odd number of times in the cycle. We ignore this subtle point.

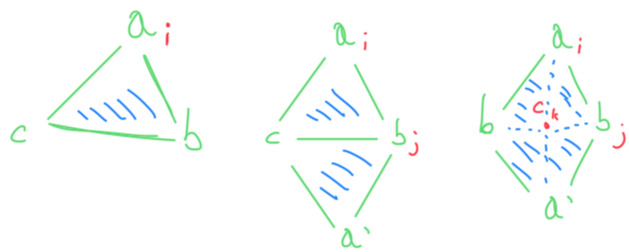

Figure 1: Tiling the complete partite complex

Observation 1.4. Assume that $df = 0$ on every triangle in T_{uw}. Then $f(uw) = g_C(u) + g_C(w) = dg(uw)$.

Proof of Observation 1.4. We need to prove $f(uw) = g_C(u) + g_C(w)$, which is equivalent to $f(uw) + \sum_{e\in P_u} f(e) + \sum_{e\in P_w} f(e) = 0$. These are exactly the edges that appear an odd number of times in all the triangles of T_{uw}. Hence this is equal to $\sum_{t\in T_{uw}} df(t) = 0$. We assumed that $df = 0$ on all these triangles so the observation follows. □

In light of Observation 1.4, it seems as though the less triangles we have to sum the better. In light of this we define

$$diam(C) = \max_{uw} |T_{uw}|.$$

1.3 Cones ⇒ Coboundary Expansion of Complete Partite Complex

The three partite complete complex is the complex whose vertices are three copies of $[n]$, $X(0) = A \cup B \cup C$. Every possible $\{a, b, c\}$ (for $a \in A, b \in B, c \in C$) is a triangle in $X(2)$.

Claim 1.5. Let X be the three-partite complete complex. Then $h^1(X) \geqslant \frac{1}{4}$.

Proof of Claim 1.5. Fix $f : X(1) \to \mathbb{F}_2$. We need to find some $g \in C^0$ such that $\mathrm{dist}(f, dg) \leqslant 4\,\mathbb{P}[df \neq 0]$. Let us construct a set of cones, such that one of the cones will be a good decoding of f. The idea is that the tiling in these cones samples random triangles in the complex; so if $\mathbb{P}_{t\in X(2)}[df \neq 0]$ is small, then $df = 0$ on most triangles of a tiling of one of the cones. Details follow.

For $i, j, k \in [n]$, let $C_{i,j,k} = (a_i, \{P_u^{ijk}\}_u, \{T_{uw}^{ijk}\}_{uw})$ where:

1. $P_b = \{a_i b\}, P_c = \{a_j c\}$ and $P_{a'} = \{\{a_i, b_j\}, \{b_j, a'\}\}$.
2. We tile as depicted in Figure 1.

We will show that one decoding $g_{ijk} = g_{C_{i,j,k}}$ is a good decoding by showing it is good in expectation.

$$\mathop{\mathbb{E}}_{(i,j,k)}\left[\mathrm{dist}(f, dg_{ijk})\right] = \mathop{\mathbb{E}}_{(i,j,k)}\left[\mathop{\mathbb{P}}_{e}\left[f(e) \neq dg_{ijk}(e)\right]\right].$$

By Observation 1.4, this is at most

$$\mathop{\mathbb{E}}_{e\in X(1),(i,j,k)}\left[\mathbb{P}\left[df \neq 0 \text{ on one of } C_e^{ijk}\text{'s triangles}\right]\right].$$

There are at most four triangles in every T_{uw} so this is at most 4 times the probability of sampling a triangle $t \in T_{uw}$ uniformly such that $df(t) \neq 0$, i.e.

$$\leqslant 4 \mathop{\mathbb{E}}_{e\in X(1),(i,j,k)}\left[\mathop{\mathbb{P}}_{t\in T_e^{ijk}}[df \neq 0]\right],$$

where T_e^{ijk} is the tiling of e for C_{ijk}. Taking expectation over e, i, j, k, $t \in T_e^{ijk}$ is just a uniformly random triangle so this is at most

$$4\,\mathbb{P}[df \neq 0].$$

In particular there is one decoding g_{ijk} such that $\mathrm{dist}(f, g_{ijk}) \leqslant 4\,\mathbb{P}[df \neq 0]$. □

1.4 Generalizing the Technique

Let us generalize this technique. An automorphism of a simplicial complex is a bijection $\sigma : X(0) \to X(0)$ such that $s \in X(i) \Leftrightarrow \sigma(s) \in X(i)$. The set (group) of automorphisms is denoted by $Aut(X)$. We say that $Aut(X)$ is transitive on $X(k)$ if for every pair of k-faces s, s' there is some $\sigma \in Aut(X)$ such that $\sigma(s) = s'$.

Lemma 1.6. *Let X be a simplicial complex such that $Aut(X)$ is transitive on k-faces. Assume there exists a cone C with diameter R. Then $h^1(X) \geqslant \frac{1}{\binom{k+1}{3}R}$.*

The proof for this lemma is just an abstraction of what we did above. The only difference is that instead of taking $\{C_{ijk}\}$ we take $\{C_\sigma\}_{\sigma \in Aut(X)}$ where C_σ is the cone obtained by applying σ to every vertex, edge and triangle in C (i.e. v_0 becomes $\sigma(v_0)$, the path P_u becomes $P_{\sigma(u)}$, a path from $\sigma(v_0)$ to $\sigma(u)$ etc.).

Example 1.7. Let σ_{ijk} be the automorphism on the complete bipartite complex such that $a_\ell \overset{\sigma_{ijk}}{\mapsto} a_{\ell+i}$, $b_\ell \overset{\sigma_{ijk}}{\mapsto} b_{\ell+j}$ and $c_\ell \overset{\sigma_{ijk}}{\mapsto} c_{\ell+k}$ (summation is modulo n). The set of automorphisms σ_{ijk} show that $Aut(X)$ is transitive on $X(1)$.

Lemma 1.6 is one of the most powerful tools in our arsenal for coboundary expansion, however it has a few caveats.

1. It only applies to symmetric complexes.
2. The dependence on diameter leads to bad constants if the diameter is large[4].
3. If $Aut(X)$ is only transitive on $X(k)$ for large k, the constant gets even worse.

It was used by the above authors to show coboundary expansion of most simplicial complexes that are known to be coboundary expanders, including the spherical building. The work in [DD23] was mainly to overcome the last two points.

2 Spherical Building

We won't define spherical buildings in full generality. The $SL_n(\mathbb{F}_q)$-building is the simplicial complex whose vertices are non-trivial subspaces of $\mathbb{F}_q^n$ and whose faces are partial flags. That is

$$S(k) = \{\{W_0, W_1, ..., W_k\} \mid \ W_0 \subset W_1 \subset ... \subset W_k\}.$$

This complex is very important for the study of high dimensional expansion. The reason is that [LSV05] (and others) showed how to construct bounded-degree families of simplicial complexes that locally resemble these objects. These sparse complexes have given us

Theorem 2.1 ([DD23], informal)**.** *The spherical building S is a coboundary expander with $h^k(S) = \exp\left(-O\left(k^6 \log(1+k)\right)\right)$. In particular $h^1(x) = \Omega(1)$.*

As a corollary, we can also show that the complexes of [LSV05] are *cosystolic expanders*. This corollary uses local-to-global techniques and builds upon previous work by [KKL14] and [EK16].

2.1 Coboundary Expansion of the Spherical Building

In order to prove Theorem 2.1 (for $k = 1$) we need two components. The first component is a reduction between the coboundary expansion of S to that of its low-dimensional analogues. For $i \in \{1, 2, ..., n-1\}$, let $S[i] = \{W \in S(0) \mid \ \dim(W) = i\}$. For a *set* of dimensions $D \subseteq \{1, 2, ..., n-1\}$ let S^D be the induced complex on vertices $S^D(0) := \bigcup_{d \in D} S[D]$.

Lemma 2.2. *Let $p > 0$. Assume that there is a set $\mathcal{D} \subseteq \binom{[n]}{5}$ such that $|\mathcal{D}| \geqslant p\binom{n}{5}$ and such that $h^1(S^D) \geqslant \beta$ for every $D \in \mathcal{D}$, then $h^1(S) \geqslant \Omega(p\beta)$.*

[4]Open question: can we strengthen this technique to get a constant closer to one?

Figure 2: Tiling for Cones in the Spherical Building

The family $\mathcal{D}$ we take is all $\{i_1 < i_2 < i_3 < i_4 < i_5\}$ such that $i_j \geqslant 10 i_{j-1}$. It is direct to show that $|\mathcal{D}| \geqslant 10^{-10}\binom{n}{5}$.

The rest of the talk will focus on showing the second component, that is, that for every $D \in \mathcal{D}$, $h^1(S^D) \geqslant \beta$ using cones.

Claim 2.3. Let $D \in \mathcal{D}$. Then $h^1(S^D) \geqslant \frac{1}{110}$.

Proof. Fix $D \in \mathcal{D}$. It is well known that $SL_n(\mathbb{F}_q)$ acts transitively on 4-faces of S^D. So we just need to find a short cone, and then the claim shall follow from Lemma 1.6.

We define $C = (v_0, \{P_u\}, \{T_{uw}\})$ as follows.

1. $v_0 \in S[i_1]$ be any lowest dimension subspace.

2. Let $u \in X(0)$. If $u \in S[i_1]$ there is some $s \in S[i_2], s \supseteq u + v_0$. We take $P_u = \{v_0 s, su\}$. For the other cases we take $P_u = \{v_0 s, sr, ru\}$ where $r \subseteq u$ is a i_1-dimensional subspace and $s \supseteq r + v_0$.

The tiling here is more involved, but still simple enough. It is recommended to stare at Figure 2 while reading the text. There are three cases. Let $uw \in S^D(1)$.

I If $\dim(u), \dim(w) \leqslant i_4$. In this case, there exists some $x \in S[i_5]$ such that x contains *all subspaces* in $P_u \cup P_w$. In particular, the 7-triangle star is a tiling of $P_u \circ uw \circ P_w$.

II The second case is where $\dim(u) = i_5$ and $\dim(w) \leqslant i_3$. In this case we first find some $u' \subseteq u$ such that $\dim(u') = i_4$ and u' contains the neighbours of u in the cycle. We tile using two triangles $uu'y$ (for both neighbours y of u). We now have the same cycle, on where we replaced u by a low-dimensional vertex, thus reducing to the first case.

III The final case is where $\dim(u) = i_5$ and $\dim(w) = i_4$. In this case we first replace w by some w' of dimension i_3 as before, and then reduce to the second case.

□

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A Sketch of Lemma 2.2

We probably will not get to this in the talk itself, so feel free to contact me for further discussion. For those interested, we give a short description of the iterative decoding underneath Lemma 2.2. The decoding for an $f : S(1) \to \mathbb{F}_2$ is as follows. We first choose a sub-complex S^D such that $wt(f|_{S^D}) \leqslant p^{-1} wt(f)$ (this is just another averaging argument where the size of $\mathcal{D}$ comes into play). We decode f to some dg for some $g : S^D(0) \to \mathbb{F}_2$.

Next we set g for vertices $v \notin S^D(0)$ by $g(v) = \max_{u \in S^D, u \sim v}(f(uv) + g(u))$.

Showing that $f \approx dg$ when $wt(f) \approx 0$, is done by an interative argument, starting from edges $uv \in S^D$ (where it follows from coboundary expansion of g), then on edges where $u \in S^D, v \notin S^D$, and finally for edges uv where $u, v \notin S^D$. Every step builds upon the previous ones. See the technical overview in [DD23] for more details.