1. **Stabilizer code** is a (simultaneous) +1 eigenspace of an abelian subgroup $S$ of the Pauli group $P_n$.

$$P_n = \langle x_i = \prod_{j=1}^{n} X_{i,j}, \quad z_i = \prod_{j=1}^{n} Z_{i,j} \rangle_{i=1}^{n}$$

**Example** (Not good LDPC) **surface code**

- qubit on every edge
- $\mathbf{+T}$ - x-checks
- $\mathbf{+C}$ - z-checks

$$S = \langle +, T, C \rangle \subseteq P_n, \quad n = l^2 + (l+1)^2$$

codespace = +1 eigenspace of $S$.

Qubits $\in (C^2)^\otimes n$.

dim = 1 = n - |S|.

distance = $L \approx \sqrt{n}$.  

2. **CSS codes**: subclass of stabilizer codes.

Choose two linear binary classical codes $C_0, C_1 \subseteq F_2^n$.

$C_0 \subseteq C_1$. Then

- z-checks $Z^c = \prod_{i=1}^{n} Z^c_i$ c-row of $H_0$
- x-checks $X^d = \prod_{j=1}^{n} X^d_j$ d-row of $H_1$

$$S = \langle \text{z-checks}, \text{x-checks} \rangle \subseteq P_n$$

codespace = +1 eigenspace of $S$.

**Defn**: A CSS code is LDPC if both $H_0$ and $H_1$ are sparse matrices, i.e., each row and column has constant weight independent of code length $n$.

In other words, each check needs constant # of qubits & each qubit is part of constant # of checks.
A code family $\{C_n\}_n$ is asymptotically good if
\[ \lim_{n \to \infty} r(n) = \lim_{n \to \infty} d(n) = \Theta(n) \]
(i.e., relative rate $r$ and relative distance are bounded below by a constant).

**Remark:** CSS codes can be described via length 2 chain complexes.

### 3. Classical Tanner Codes

Given a $D$-regular graph $G$ and a code $C \subset \mathbb{F}_2^D$, define the (classical) Tanner code, denoted $T(G, C)$, by putting bits on the edges of the graph with the constraint that the local view of a vertex $v := \phi$ edges incident at $v$ is a codeword in $C$.

Then, $T(G, C) \subset \mathbb{F}^n$; $n = \text{Total \# of edges} = \frac{D \cdot |V|}{2}$.

**Theorem (Sipser-Spielman ’96)**

Let $C$ be $[D, \Delta, \delta \Delta]$. Then,
- $\dim T(G, C) \geq (2\delta - 1)n$
- $\text{distance } T(G, C) \geq \delta(\delta - \frac{\lambda(G)}{\Delta})n$.

where $\lambda(G)$ is the "expansion factor" of the regular graph $G$.

**Remark:** If $G$ is an expander graph, then distance $T(G, C) \approx \delta^2 n$.

### Structure of the good $Q$-LDPC code

- **Graph** $\rightarrow$ classical Tanner code $\rightarrow$ CSS code $\rightarrow$ $Q$-LDPC code
- **Graph** $\rightarrow$ classical Tanner code
4. **Tensor Codes**

Given two classical codes, $C_A$, $C_B \subseteq F^n$

- **Tensor code** $C_A \otimes C_B \subseteq F^{n \times n}$
  
  $$
  C_A \otimes C_B := \{ x \in \text{Matrices}_{n \times n}(F) : \text{col} x \subseteq C_A, \text{row} x \subseteq C_B \}.
  $$

- **Dual tensor code** $(C_A \otimes C_B)^\perp \subseteq F^{n \times n}$
  
  $$
  (C_A \otimes C_B)^\perp = C_A \otimes F^n + F^n \otimes C_B
  $$

  $$
  \{ x \in \text{Matrices}_{n \times n}(F) : H_A \cdot H_B = 0 \text{ mod } 2 \}.
  $$

**Proposition:** If $C_A = [n_A, v_A, d_A]$ and $C_B = [n_B, v_B, d_B]$.

Then, $C_A \otimes C_B = [n_A n_B, v_A v_B, d_A d_B]$, and

$$(C_A \otimes C_B)^\perp = [n_A n_B, n_A v_B + n_B v_A - v_A v_B, \min(d_A, d_B)]$$

and $C_A \otimes C_B \subseteq (C_A \otimes C_B)^\perp$ - This will be used later to check the CSS cond.

**Remark:** To achieve linear distance for quantum code LZ construct, they need to impose a cond called $w$-robustness on the dual tensor code. This property is related to the support of the non-zero codewords of weight $\leq w$.

Dinur et al show that when $C_A$ and $C_B$ of appropriate parameters are randomly chosen, then the dual tensor code $C_A \otimes F^n + F^n \otimes C_B$ is $\Delta^2$-robust with high probability, $w = \Delta^2$ is sufficient for the LZ construction.

Nir will talk more about the robustness of tensor codes.
5. **Left-Right Cayley Complex**: (Let's first see the ingredients & then see the nec. cond's we need to impose & later some candidates)

**Ingredients:**

- G - finite group
- Generating sets A, B of G s.t.
  
  \[ A^{-1} = A, \ B^{-1} = B, \ |A| = |B|, \] and
  
  A, B satisfy the TNC - Total Non-Conjugacy cond.
  
  i.e., \( a^g \neq g^b \) & \( (a,g,b) \).

\( \text{Cay}_2(A,G,B) \) - double cover, left-right Cayley complex is a 2D complex consisting of

a) **Vertices** \( V = V_0 \cup V_1 \) (we want the graph to be bipartite)

\[ V_0 = \{0,3 \cup G \times 1\} \]

b) **Edges** \( E = E_A \cup E_B \), where

\[ E_A = \{ (g,0), (a^g,1) \} \Rightarrow \text{ag} \]

\[ E_B = \{ (g,0), (gb,1) \} \Rightarrow \text{gb} \]

|\( |E_A| = |G| = |E_B| \). Therefore, \( |E| = 2|G| \). |

c) **Squares** \( Q = \{ g, a^g, gb, agb \} \Rightarrow \text{3a,3g,3b} \)

\[ |Q| = \frac{|G|^2}{2} \]

Local view of a vertex

**TNC** \( \Rightarrow \)

\[ \text{Cay}_2(A,G,B) \]
The graphs that we'll define our Tanner codes on, are:

\[ Y_0^\square = (V_0, \mathcal{Q}) \quad \text{and} \quad Y_1^\square = (V_1, \mathcal{Q}) \]

**Note:** A square \( \mathcal{Q} = \{ g, ag, gb, agb \} \) is identified with its diagonal \((g, agb)\) (resp. \((ag, gb)\)) and considered an edge in \( Y_0^\square \) (resp. \( Y_1^\square \)).

\[ TNC \Rightarrow \]

\begin{itemize}
  \item \( Y_1^\square \) is a \( \Delta^2 \)-regular graph
  \item i.e. every vertex is incident to exactly \( \Delta^2 \)-squares.
  \item each square contains exactly 4 distinct vertices
  \item local view of a vertex: \( Q(v) \sim A \times B \)
  \[ (g, ag, gb, agb) \leftrightarrow (a, b) \]
  \end{itemize}

is a bijection.

Define \( \Phi_v : Q(v) \to A \times B \) as the inverse of the above map.

Then,

\[ \Phi_v(Q(v) \cap Q(ub)) = A \times B, \quad \Phi_v(Q(v) \cap Q(uv)) = A \times B \]

\[ \Phi_v(Q(v) \cap Q(au)) = A \times B, \quad \Phi_v(Q(v) \cap Q(au)) = A \times B \]

In other words,

all rows of \( A \times B \) are shared by its (the fixed vertex) \( A \)-neighbours

all columns of \( A \times B \) are shared by its \( B \)-neighbours.

6. **Construction of Q-LDPC**

Let

\[ c_0 = C_A \otimes C_B \]

\[ c_1 = C_A^\perp \otimes C_B^\perp \]

Define \( e_0 = T(Y_0^\square, C_0^\perp) \) \{ and \( \mathcal{Q} = CSS(e_0, e_1) \)

\[ e_1 = T(Y_1^\square, C_1^\perp) \]
put qubits on squares (≡ diagonals). The $z$-checks will be on $V_0$ and the $x$-checks on $V_1$.

Let $p_0$ be a basis of $C_0$ and $p_1$ be a basis of $C_1$.

To define $z$-check generators:

For each vertex $v \in V_0$

Define a codeword $x \in \mathbb{F}^{Q^4}$ by requiring that

$\begin{align*}
\text{local view} \quad & \in p_0 \\
\text{of } v \\
\text{x} & = 0 \quad \text{elsewhere.}
\end{align*}$

The corresponding stabilizer (of length $|Q|$) is weight $\leq \Delta^2$

$\hat{Z}_v = \prod z$ if $Q_{ab} = 1$

$I$ if $Q_{ab} = 0$

To define $x$-check generators:

For each vertex $v \in V_1$

Define a codeword $x' \in \mathbb{F}^{Q^4}$ of type $x$ by requiring that

$\begin{align*}
\text{local view} \quad & \in p_1 \\
\text{of } v' \\
\text{x} & = 0 \quad \text{elsewhere.}
\end{align*}$

The corresponding $x$-stabilizer (of length $|Q|$) is weight $\leq \Delta^2$

$\hat{X}_v = \prod X$ if $Q_{ab} = 1$

$I$ if $Q_{ab} = 0$

# of $z$-generators = $|\text{dim } C_0| \cdot |V_0|$

# of $x$-generators = $|\text{dim } C_1| \cdot |V_1|$

remark (imp): codewords in (*) and (*)' are not codewords of the quantum code.
Recap:

\[ C_0 = C_A \otimes C_B \quad \text{and} \quad C_1 = (C_A^\perp \otimes C_B^\perp)^\perp \]

\[ T(G_0^D, C_0) \quad \text{and} \quad c_{\text{LS}}(e_0, e_1) = q-LDPC \]

Claims to prove:

1. \( e_0^\perp \subseteq e_1 \)
2. \( \mathcal{A} = c_{\text{LS}}(e_0, e_1) \) is LDPC.
3. Relative rate of \( \mathcal{A} = \frac{\dim \mathcal{A}}{\text{code length}} \geq (2e-1)^2 \)
4. Distance \( d \geq \frac{\delta n}{A \Delta^{-1}} \)

5. Conditions to impose candidates:

- Choice of Cayley complex: Let \( q \) be an odd prime power. For \( \Delta = q+1 \), \( \mathcal{A} \) an infinite family over \( G = PSL_2(q^2) \) satisfying all the conditions required by \( A, B \), and \( G \).

Further, \( \text{Cay}(G, A) \) and \( \text{Cay}(G, B) \) are Ramanujan graphs and hence expanding. Thus, both \( G_0^D \) and \( G_1^D \) are expander graphs.

[Theorem 17] (LZ '22) Fix \( p, \varepsilon \in (0,1/2) \) s.t.

- \( \dim C_A = p \Delta \), \( \dim C_B = (1-p) \Delta \)
- \( \min \) distance \( C_A, C_B, C_A^\perp, C_B^\perp \), all \( \geq \delta \Delta \)
- \( C_0^\perp = (C_A \otimes C_B)^\perp \) and \( C_1^\perp = (C_A^\perp \otimes C_B^\perp) \) are \( \Delta^2 \)-robust

Then,

\[ \mathcal{A} = c_{\text{LS}}(e_0, e_1) \] is \( n, k \geq (1-2p)^2 n, \quad d \geq \frac{\delta}{4 \Delta^{-1} + \varepsilon} \)
[Theorem 18] (LZ'22) ~ Recipe for explicit construction

1. Take family of highly expanding left-right Cayley complex.

2. For fixed ρ ∈ (0, 1/2),
   - $c_A \leftarrow \text{uniform } \Delta \times \Delta \text{ matrix}$
   - $c_B \leftarrow \text{uniform } (1-\rho) \Delta \times \Delta \text{ matrix}$

   w.h.p robustness cond' is satisfied

3. Construct CSS code.