Selfish Rounting and Price of Anarchy

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1 Introduction

The topics discussed in this lecture are covered in full in [Roughgarden16], chapter 11, parts of chapter 12, and 13.2. Students are advised to consult that reference.

The notes here were made for my own use, to remind me during the lecture what to cover, and to remind me in the future what was covered.

2 Notes on chapter 11

Waze. Drivers get a prediction of travel time on each route. Does Waze lead to good outcome (on average)?

Routing from s to t in directed network, with cost functions $c(x)$ on edges. Each driver chooses a path of minimum cost. Cost of path affected by others. Hence, a game. In equilibrium, no one wants to change path.

Large flow assumption - each driver has infinitisimal effect on others. **Nonatomic.** Has a pure Nash, as we can distribute fractional values for fractional Nash.

Braess's Paradox. Total flow 1. $c(x) = x$ followed by $c(x) = 1$, in parallel to $c(x) = 1$ followed by $c(x) = x$. Average cost $\frac{3}{2}$. Add intermediate $c(x) = 0$ road. Cost jumps to 2.

Optimal cost is monotone in resources. Equilibrium cost is not.

Equilibrium cost higher than optimal cost.

Price of Anarachy (POA). $\frac{4}{3}$ in our case.

Are there worst examples?

Answer depends on family C of cost functions. Here we assume that C contains the constant functions (such as $c(x) = 1$), in addition to other functions, and is nondecreasing.

Pigou network. Top edge $c(x) = 1$. Bottom edge $c(x) = x$, or for nonlinear versions, can be $c(x) = x^p$, or other. As p grows, POA grows.

Are there worse examples? Generalized Pigou: flow r (rather than 1), worst $c \in C$, top edge is $c(r)$, bottom edge is $c(x)$.

Equilibrium $rc(r)$. Opt $\inf_{0 \leq x \leq r} xc(x) + (r - x)c(r)$

Define:

$$\alpha(C) = \sup_{c \in C} \sup_{r \geq 0} \sup_{x \geq 0} \frac{rc(r)}{xc(x) + (r-x)c(r)}$$

Note that by monotonicity of c, optimum at $x \leq r$.

Theorem: $POA \leq \alpha(C)$.
$\mathcal{P}$ set of paths from s to t.
Flow f.
$\{f_P\}$: nonnegative flows on paths.

$$f_e = \sum_{P|e \in P} f_P$$

$$c_P(f) = \sum_{e \in P} c_e(f_e)$$

$$c(f) = \sum_P c_P(f) = \sum_e f_e c_e(f_e)$$

Equilibrium: $f_p > 0$ only if P is one of the paths of minimum $c_p(f)$.
All equilibrium paths have the same cost.
Proof of theorem.
r total traffic. f equilibrium flow. f^* opt. L lowest cost path in equilibrium.

$$\sum_P f_P c_P(f) = rL$$

$$\sum_P f_P^* c_P(f) \geq rL$$

$$\sum_e (f_e^* - f_e) c_e(f_e) \geq 0$$

$$\alpha(C) \geq \frac{f_e c_e(f_e)}{f_e^* c_e(f_e^*) + (f_e - f*_e) c_e(f_e)}$$

$$f_e^* c_e(f_e^*) \geq \frac{f_e c_e(f_e)}{\alpha(C)} + (f_e^* - f_e) c_e(f_e)$$

$$c(f^*) \geq \frac{c(f)}{\alpha(C)} + \sum_e (f_e^* - f_e) c_e(f_e) \geq \frac{c(f)}{\alpha(C)}$$

3 Notes on chapter 12, and 13.2

3.1 Resource augmentation

Theorem. Cost of equilibrium with traffic rate r is at most optimal cost with traffic rate $2r$.

$$\sum_e f_e c_e(f_e) = rL$$

$$\sum_e f_e^* c_e(f_e) \geq 2rL$$

We claim:

$$\sum_e f_e^* c_e(f_e^*) \geq \sum_e f_e^* c_e(f_e) - \sum_e f_e c_e(f_e) \geq 2rL - L = rL = \sum_e f_e c_e(f_e)$$

Proof of claim is term by term:

$$f_e^*(c_e(f_e) - c_e(f_e^*)) \leq f_e c_e(f_e)$$

3.2 Atomic selfish routing (also Chapter 13.2)

Equilibrium: for agent i with path P_i and alternative path P

$$\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in P_i \cap P} c_e(f_e) + \sum_{e \in P \setminus P_i} c_e(f_e + 1)$$

Pigou network with $r = 2$, $c(x) = 2$ at top edge and $c(x) = x$ at bottom edge. Both drivers at bottom edge is a pure Nash, and one on each edge is another pure Nash. Their values are different.

Hence, distinguish between POA (worst Nash) and **price of stability** (best Nash).

Theorem. Pure equilibrium exists.

Proof by a **potential function** argument. Each improving move for a driver improves the potential function:

$$\phi(f) = \sum_e \sum_{i=1}^{f_e} c_e(i)$$

As ϕ can receive only one of finitely many values (there are finitely many paths), it has a minimum. Every local minimum (with respect to moves by single drivers) is a Nash equilibrium.

For non-atomic case, the inner sum is an integral, and all local minima have same value (by a convexity argument). For atomic, local minima may have different values (as in the Pigou example above).

POA still bounded, but worse than in nonatomic case.

References

[Roughgarden16] Tim Roughgarden. Twenty Lectures on Algorithmic Game Theory. Cambridge University Press, 2016.