Fair Division

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1 Allocation of indivisible goods

We continue were we left off in the previous lecture.

1.1 Comparison based fairness for indivisible goods

For two agents, EF1 and EFX allocations always exist, by the cut and choose protocol.

For additive valautions, the well known Round Robin (RR) picking order gives an EF1 allocation (if each agent in her turns picks her most preferred item among those available). If valuations are not additive, this is not true. Consider a setting in which $n = 3$ and the m items can be partitioned into $\frac{m}{2}$ pairs. If within each pair items are *compliments* (e.g., a pair of shoes – one needs to get both left and right shoe to get positive value), in RR an agent can be prevented from completing any pair, whereas at least some other agent completes several pairs. If within each pair items are *substitutes* (e.g., two copies of the same book – having two copies does not offer substantial value beyond having just one copy), in RR an agent can be led to ending up with a collection of items that contains several pairs, whereas the other agents receive only items from distinct pairs (the other two agets can coordinate to consume pairs by one of them selecting one item from a pair, the the other immediatly selecting the other item).

Though RR does not always produce EF1 allocations, a different allocation algorithm does.

Theorem 1.1 *For every class of monotone valuation functions, EF1 allocations always exist, and can be found using polynomially many comparison queries.*

Proof: We present a relatively simple allocation algorithm of [6] (we refer to it as the LMMS algorithm). It uses the notion of an *envy graph*. Given an allocation, each agent is a vertex in the graph, and there is a directed edge from vertex i to j if agent i envies agent j. A *unreachable vertex* in the envy graph is a vertex with no incoming edge (an agent that no other agent envies). An envy graph either has an unreachable vertex, or it has a directed cycle. If it has a directed cycle, then bundles of the allocation can be rotated one step backwards along the cycle, making every agent along the cycle happier (her bundle improves in her eyes). We call this operation *cycle elimination*, and performing it results in a new envy graph. As there cannot be an endless sequence of cycle eliminations (there are only finitely many allocations), repeated performing of cycle eliminations results in an envy graph with an unreachable vertex.

The LMMS algorithm is performed in m rounds. In each round, one allocates a yet unallocated item (chosen arbitrarily) to an agent that no one envies (chosen arbitrarily, if there are several such agents). This is followed by as many cycle elimination operations as needed, so that the new envy graph has at least one unreachable vertex.

The final allocation is EF1, as removing the last item received by agent j, no other agent envies j.

The algorithm runs in polynomial time, if agents are willing to respond to comparison queries about their valuation functions (given two bundles, which one do you prefer?). □

When valuations are additive, the agent selecting an item at a given round has a natural choice for which item to select: the one with highest value for her. For additive valautions and this version of the LMMS algorithm, if the orginal input is *identicaly ordered* (IDO - agents may have different valuations, but for every agent i and items e_j and e_k with $k > j$ it holds that $v_i(e_j) \geq v_i(e_k)$), the final allocation is EFX. Consequently, we have the following proposition.

Proposition 1.2 *For instances with additive valuations that are indentically ordered (IDO), an EFX allocation exists and can be found in polynomial time.*

It is not known whether EFX allocations always exist, and this is considered one of the major open problems in fair division. Beyond Proposition 1.2, EFX allocations are known to exist in some special cases, such as three agents with additive valuations. Here is another special case, taken from [7] (with a slightly modified proof).

Theorem 1.3 *If all agents have the same valuation function, then an EFX allocation exists.*

Proof: As all agents have the same valuation v, we may assume that for every allocation $A_1, \ldots, A_n$ it holds that if $i < j$ then $v(A_i) \leq v(A_j)$. Given two allocations $A = A_1, \ldots, A_n$ and $A' = A'_1, \ldots, A'_n$ we say that $A \geq A'$ if either $v(A_i) = v(A'_i)$ for all i, or the first i in which they differ is such that $v(A_i) > v(A'_i)$. This is referred to as the *leximin* (partial) order among allocations.

If v is such that there are no two different subsets of $\mathcal{M}$ with the same value, then the leximin order is a total order. In this case, the maximum element of this order is an EFX allocation. This is because if in an allocation A player i has more than minimal envy towards j (certifying that the allocation is not EFX), then there is some item $e \in A_j$ that can be moved to A_i while resulting in an allocation that is higher than A in the leximin ordering.

If v does have ties, the above argument need not work, because it might be that $v(A_i) = v(A_i \cup \{e\})$. Let δ denote the minimal difference in value between two bundles that do not have equal value. Change v to $v' = v + f$, where f is the additive valuation in which each item i has value $\delta 2^{-i}$. In v', no two different sets have the same value, so it has an EFX allocation. It is not hard to see that the same allocation is EFX also with respect to v. □

1.2 Share based fairness for indivisible goods

As the MMS is not feasible. We now present a fairly natural share that is feasible. This share is referred to as MXS in [4], but we shall call it the EFX-share.

Definition 1.4 *For valuation v and entitlement $\frac{1}{n}$, the* EFX-share *(EFXS) is the minimum value t such that there is an allocation $A_1, \ldots, A_n$ with $v(A_n) = t$, for which for every $j \in \{1, \ldots, n-1\}$ it holds that either $v(A_j) \le t$, or agent n that has valuation v has only minimal envy towards A_j. An allocation is* acceptable *for agent i with respect to EFXS if the agent i receives an* EFXS bundle*, namely, a bundle of value at least that of her EFX share.*

For additive valuations, feasibility of EFXS is implied by Proposition 1.2. If the instance is not IDO, first find a "virtual" allocation, pretending that the instance is IDO. Then, run a picking sequence algorithm, where for every round r, the agent holding e_r in the virtual allocation gets to pick an item. At that point, at least one of her top r items is still available, so she gets a value at least as high as in the IDO instance. The final outcome is an EFXS allocation.

We remark that in the above algorithm every agent gets value of at least $\frac{2n}{3n-1}$ of her MMS [3]. More generally, for additive valuations, every EFXS allocation offers at least a $\frac{4}{7}$ fraction of the MMS. This follows from a similar result regarding EFX allocations [2]. Possibly, the ratio of $\frac{4}{7}$ can be somewhat improved.

We now prove feasibility of MXS allocations for every class of monotone valuations. The proof is taken from [1]. It is based on the following property of EFXS (a property that does not hold for MMS).

Proposition 1.5 *For arbitrary $n \ge 2$, consider an arbitrary valuation v and let $S \subset \mathcal{M}$ be such that $v(S) < EFXS(v, \mathcal{M}, \frac{1}{n})$. Then $EFXS(v, \mathcal{M} \setminus S, \frac{1}{n-1}) \ge EFXS(v, \mathcal{M}, \frac{1}{n})$.*

Proof: Denote $EFXS(v, \mathcal{M} \backslash S, \frac{1}{n-1})$ by t_{n-1} and $EFXS(v, \mathcal{M}, \frac{1}{n})$ by t_n. We need to show that if $v(S) < t_n$ then $t_{n-1} \ge t_n$. Suppose for the sake of contradiction that $t_{n-1} < t_n$. Let $B \subset \mathcal{M} \backslash S$ be a bundle that certifies that $EFXS(v, \mathcal{M} \backslash S, \frac{1}{n-1}) = t_{n-1}$, and consider the allocaton $A = (A_1, \ldots, A_{n-2}, B)$ that certifies this. Add to A the bundle S. As $v(B) < t_n$, it must be that this new allocation does not certify that B is an acceptable bundle under $EFXS(v, \mathcal{M}, \frac{1}{n})$, implying that $v(S) > v(B)$. But then, this last allocation certifies that S is acceptable under $EFXS(v, \mathcal{M}, \frac{1}{n})$, contradicting the assumption that $v(S) < t_n$. □

Theorem 1.6 *For the class of monotone valuations, EFXS allocations always exist.*

Proof: Agent 1 proposes an EFX allocation $A = (A_1, \ldots, A_n)$ according to her valuation function v_1 (such an allocation exists, by Theorem 1.3). Every other agent j specifies which of the bundles in A are acceptable to her, and which are not. This induces a bipartite graph with agents on one sde, bundles of A on the other side, and edges connect agents to their acceptable bundles. In this graph, agent 1 is connected to all bundles. If the graph has a perfect matching, this gives an EFXS allocation. If the graph does not have a perfect matcing, then by Hall's theorem, there must be some k such that there are k bundles with at most $k-1$ neighbors ($k-1$ agents that find at least one of the bundles acceptable). Importantly, $k \geq 2$, as agent 1 finds all bundles acceptable. For the smallest such k, match the respective $k-1$ agents with $k-1$ of these bundles. Such a matching exists, by minimality of k and Hall's theorem. As none of the other agents find any of the matched bundles acceptable, Proposition 1.5 applies, and we may repeat the above argument with the remaining agents and items, until all agents receive acceptable bundles. □

We note that as EFXS is a share based fairness notion, then Theorem 1.6 implies that there are Pareto optimal allocations in which every agent gets at least her EFXS.

References

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