Linear Programming – Handout 1

November 9, 2022

Parts of the course will overlap the book: Jiri Matousek, Bernd Gartner: Understanding and Using Linear Programming (Springer 2007), which is available freely on the web.

http://link.springer.com/book/10.1007%2F978-3-540-30717-4.

Topics covered in first lectures: formulating problems as a linear program, transformations among various forms, basic feasible solutions, the Beck-Fiala theorem

Homework. Hand in by November 24, in English. (Extension given until November 30.) Grader: Vadim Grinberg. vadim.grinberg@weizmann.ac.il.

1. Suppose that you are given a system of linear equations $Ax = b$, and are asked to find a solution that minimizes a linear objective function $c^t x$. (This is similar to an LP in standard form, but without the nonnegativity constraints.) Give a polynomial time algorithm that determines whether the problem has a bounded optimal solution. (The algorithm should just answer *yes* of *no*. It need not find a bounded optimal solution, if there is one.)

2. Consider the problem of minimizing the ratio

$$\frac{c^t x}{f^t x}$$

   of two linear functions, subject to all the following constraints:

   - $Ax \geq b$,
   - $f^t x \geq 1$,
   - $c^t x \geq -8$,
   - $c^t x \leq 8$.

   Show how linear programming can be used as a subroutine so as to find the value of the optimal solution within any degree of accuracy (the running time may depend on the degree of accuracy desired).

3. Consider an LP in general form, and suppose that:

   (a) The LP is feasible.

   (b) The value of the optimal solution is bounded.

(c) Among the constraints of the LP there are n constraints that are linearly independent. (Here we do not distinguish between main constraints and nonnegativity constraints.)

A basic feasible solution (bfs) is one that satisfies n linearly independent constraints with equality. Show that the LP has a bfs that is optimal.

4. Let s be a slackness parameter which in this homework you can set to be a fixed positive integer of your choice. (A choice of $s = 2$ suffices, but if you find it easier to do the homework with a larger value of s, then you may do so.) Let G be an arbitrary graph and let $\alpha_1, \ldots, \alpha_k$ be nonnegative with $\sum_{i=1}^{k} \alpha_i = 1$. Show that one can color the edges of the graph such that for every vertex v and every color class i, the number of edges of color i incident with vertex v is between $\lfloor \alpha_i d_v \rfloor - s$ and $\lceil \alpha_i d_v \rceil + s$, where d_v is the degree of vertex v.

   [Some remarks. The slackness term $s = 2$ can be improved when $k = 2$, and it is an open question whether it can be improved in general. For the special case of bipartite graphs, if $k = 2$ no slackness term is needed at all (namely, $s = 0$), and it is conjectured that this is true for every k.]

Hints, by question number.

1. This question has two parts. One is to determine whether the problem has a feasible solution. Students should be familiar with algorithms for testing this (from their previous studies). The other part, needed only if the problem is feasible, is to certify that there is no sequence of feasible solutions whose value tends to $-\infty$. One way (though not the only way) of certifying this is by a simple (though perhaps clever) reduction to a test of whether a related system of linear equations is feasible.

2. Consider the problem of deciding whether the objective function is at most a given value.

3. The following two statements are equivalent:

   - There are n constraints that are linearly independent.
   - The feasible region of the LP does not contain an infinite line (that is, there are no two $u, v \in R^n$, with $v \neq 0$, such that $u + t \cdot v$ is feasible for every $t \in R$).

   Using this equivalence (better still, after proving it) can help in the proof.

4. For every edge e and color class i introduce a variable x_{ei} whose intended value is 1 if edge e is colored by color i, and 0 otherwise. Then follow the proof technique used in the proof of the Beck-Fiala theorem.