מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Thesis for the degree
Master of Science

עבודת גמר (תזה) לתואר
מוסמך למדעים

Submitted to the Scientific Council of the
Weizmann Institute of Science
Rehovot, Israel

מוגשת למועצה המדעית של
מכון ויצמן למדע
רחובות, ישראל

By
Yuval Madar

מאת
יובל מדר

משחק חלוקת עוגה עם מגבלות קיבולת

A capacitated cut and choose game

Advisor:
Prof. Uriel Feige

מנחה:
פרופ' אוריאל פייגה

August 2015

אלול ה'תשע"ה

Abstract

We consider a two player zero sum game played between a cutter and a chooser consisting of two rounds. Every instance of the game has two parameters $C < S$ known to both players. The cutter makes the first move, and cuts a cake of size S to an arbitrary number of pieces, each of size at most 1. The chooser then replies by choosing a subset of the pieces whose total size is no greater than C. The chooser's payoff is the total size of pieces captured, and maximizing his payoff involves solving the knapsack instance that results from the cutter's move. We study the optimal min-max cutter strategy.

While the set of possible cuts is infinite (as the size of the pieces are real numbers), we show that for every (S, C) an optimal cutter move exists, and that such a move may be found efficiently. Furthermore, although making an optimal reply is NP-hard, the chooser is able to efficiently find a response achieving the game's minmax value regardless of the cut chosen by the cutter.

Acknowledgments

I wish to express my deep gratitude to my advisor, Prof. Uriel Feige, for his patience, help and deep insights which have repeatedly removed obstacles from my path and made complicated ideas simple.

Contents

1 Introduction

1.1 Terminology and notation

Consider a game between two players - a *cutter* and a *chooser*.

Every instance of the game has two parameters - size of a cake S, and the capacity of the chooser C.

The cutter makes a move M, cutting the cake to an arbitrary number of pieces $\ell_1, ..., \ell_m$ of positive size no greater than 1 s.t. $\sum \ell_i = S$. Without loss of generality we assume that $\forall_{i<j} \ell_i \geq \ell_j$. We sometimes call the cutter's move a "cut". The chooser then makes a reply R, choosing a subset of the pieces in M, whose sum v is at most C. We call the unused capacity $0 \leq C - v$ the slackness achieved by R.

Value of the game: We denote by $\omega(S, C, M, R)$ the slackness for a given move M and reply R. This is the game's value, which the cutter wishes to maximize, and the chooser to minimize.

Then, we denote by $\omega(S, C, M) = \min_R[\omega(S, C, M, R)]$ the minimal slackness for a given cut M achieved by the optimal response. Lastly, we set $\omega(S, C) = \sup_M[\omega(S, C, M)]$, the minmax slackness of the game. Note that tentatively, the last definition involves supremum rather than maximum because given a cake size S, there are infinitely many possible cuts, and there may not be an optimal one. When S, C are clear from the context, we sometime omit them, and discuss the slackness $\omega(M)$ instead of $\omega(S, C, M)$.

Given C and M, we call a reply $R \subseteq M$ *feasible*, if its sum is at most C.

1.2 Research questions

In this thesis, we wish to characterize the function $\omega(S, C)$, and the optimal cuts achieving it. In particular:

1. Must there exist an optimal cut for every S, C?

2. When optimal cuts exist, characterize their form.

3. Design efficient algorithms for both players, guaranteeing for each a value no worse than $\omega(S, C)$.

1.3 Related work and concepts

The cake cutting game we consider is a *two player sequential zero sum* game of *full information*, meaning that neither player knows anything the other does not.

As discussed earlier, approaching this game, we do not know a-priori that an optimal cut must exist. To show that it does, one might attempt to use rudimentary calculus methods, such as 1860 Weierstrass' extreme value theorem, stating that a continuous function over a compact domain achieves optimal values inside it. Using the standard Euclidean metric, a-priori the set of cuts is infinite, but we show in Proposition 7 that it is sufficient to consider the compact set of standard cuts, those in which the sum of every two pieces is greater than 1.

However, even in that compact domain, the optimal slackness function $\omega(S, C, M)$ is not continuous as a function of the cutter's move M. For example, for $C = 0.75, S = 1.75$, the cut $M = \{1, 0.75\}$ has $\omega(S, C, M) = 0$, while the cut $M' = \{1-\epsilon, 0.75+\epsilon\}$ has $\omega(S, C, M') = 0.75$ for every $\epsilon < 0.25$, making that approach unfeasbile.

Several classical game theory theorems are commonly used to show the existence of optimal min-max strategies in full information games. We review some of them and explain why they do not suffice for our case.

One such theorem is the minimax theorem proved by Jon Von Neumann in 1928[13]. The theorem discusses two player simultaneous zero sum games with finitely many deterministic strategies. It defines mixed strategies to be the assignment of probabilities to the set of deterministic strategies, and picking one at random according to the resulting distribution. The theorem states that in every such a game, there exist mixed strategies for both players and some value V, such that playing these strategies, one player gains V and the other $-V$, and neither may improve his score by replacing his strategy by another.

A generalization of the minimax theorem for from 1951, called Nash's theorem[12], defines the "Nash equilibrium" solution concept, which coincides with the minmax strategy solution concept for zero-sum games. A pair of strategies for both players is called a Nash equilibrium if neither player has incentive to deviate from their strategy given the other player does not deviate from his. Nash's theorem states that in a game with finitely many deterministic strategies, there must exist a Nash equilibrium if we allow mixed strategies.

Neither the minimax nor the Nash theorems are applicable directly to our game, since the set of strategies for the cutter is uncountably infinite.

Both the minimax and Nash's theorems discuss simultaneous games, while our game is sequential. This means, since every cutter move has an optimal reply, that mixed strategies are not required in our game. Consider a mixed cutter strategy containing two cuts M, M' with $\omega(S, C, M) \geq \omega(S, C, M')$. Removing M' and increasing M's probability to the sum of the two moves' probabilities does not decrease the strategy's expected value. The chooser has no need for randomization either, since there are finitely many replies once a cut was chosen, and for every such cut there is a deterministic optimal reply.

1.3.1 Similar games

Colonel Blotto games first proposed in 1921 by Emile Borel[2], are a class of two player zero-sum games in which two players simultaneously distribute a limited number of soldiers over several battlefields, such that a player wins each battlefield iff he has chosen more soldiers than the other. The goal of each player is to maximize the number of battlefields won. While this game does not necessarily have a pure Nash, since the number of possible moves is finite, Nash's theorem states that it has an equilibrium.

An equilibrium was found by Borel in 1938 for $n = 3$ battlefields assuming the two players have symmetric resources. This result had been expanded to an arbitrary number of battlefields in 1950 by [7], still assuming symmetric resources, and the general unrestricted number of battlefields and asymmetric budgets case was handled in 2006 by [16].

The colonel blotto game has had many extensions over time. Continuous games are suggested in [7], [11], [16] among others. Asymmetric budgets are considered in [16], [11], [7] and [8], with the first three considering continuous versions and the last a discrete version. With the exception of [11], all of these had assumed that both players have the same valuation for every battlefield. In [17], a payoff function giving values to certain subsets of battlefields was suggested, instead of summing the payoff from each battlefield separately. All of these games assume a simultaneous model, while in our game one player plays first and the other reacts to his action.

Our game can be thought of as a two round, continuous Blotto game with asymmetric budgets, where the battlefields are the $2S$ pieces made by the cutter, since Corollary 8 shows that we may assume without loss of generality that the cuts contain at most $2S$ pieces, with some of the pieces may have 0 size. The contest rule in our game is that a battlefield is won if the chooser has placed forces of equal size or more in the battlefield, and his payoff is the sum of $-C$ and the sum of the pieces he chose. Note that in our case the payoff function for every battlefield is symmetrical, although determined by the cutter in his turn.

In 2009, Powell suggested a two round full information stochastic version of the Blotto game[14], with a different contest and payoff functions than ours. In Powell's game two players have asymmetric valuations for the N

battlefields, with the defender playing first, allocating defensive resources to the sites, such that the more resources allocated to a site, the lower the likelihood of a successful attack on it. The attacker plays second, picking a single target maximizing his expected value, derived from his chances of success in every site, and his valuation of a successful attack in each. Powell finds a minmax to that game. While sharing some common characteristics with our game, it differs from our game in both the contest rule and the ability to choose several battlefields in the same turn.

Other sequential colonel Blotto games exist, such as [15] which discusses a stochastic game where each player has a sequence of soldiers, according to which they will be paired to fight, where in each turn the winning soldier remains to fight the battle and the loser is taken out of the game. The player who has loses all of his soldiers first loses. The outcome of every fight between two soldiers is determined randomly with the stronger soldier more likely to win the greater the difference between them is. This game varies from our more significantly since it allows several non simultaneous contest rounds, which depend on one another, and such that in each only a single pair of soldiers compete.

Several other sequential Blotto games were considered, but none was found to directly apply to our model.

Cut and choose (for fair division) A well known somewhat similar problem is the problem of *fair cake cutting*, also known as envy-free cake cutting, cut and choose, divide and choose etc. The problem discusses a procedure for the partition of a heterogeneous resource (the cake, which may have several different toppings) regarding which the two players have different preferences.

The well known protocol is composed of two steps - the cutter first makes a cut into two pieces, and the chooser picks the piece he prefers. This protocol is envy free, since the chooser prefers the piece he chose over the piece left and the cutter was able to cut the cake into pieces he considers to have equal value and make sure he gets a piece at least as good as the one picked by the chooser. This protocol had been expanded in various ways, to an arbitrary number of players, preferences, player entitlement etc.

The protocol does not assure an efficient allocation. Considering a cake with half vanilla topping and half chocolate coating, suppose that one player is only interested in vanilla and the other only in chocolate, making two equal pieces, each containing a quarter chocolate and one of vanilla gets each player exactly half the value he could have achieved by an optimal allocation.

In our game, as in that model, we discuss two players, each attempting to maximize the other player's slackness. However, in our model, we set a maximal capacity for the chooser, to which he must abide but limited the maximal piece size to prevent the cutter from trivially preventing the chooser from getting any cake. The cutter, on the other hand picks the piece sizes in an attempting to prevent the chooser from fully utilizing his capacity.

Stackleberg games also known as Stackleberg competitions, are two player sequential zero sum games, consisting of two turns[19]. In these games, originally defined for their economic applications, the *leader* plays first, deciding the amount he wishes to produce, and the *follower* plays second, responding the the market leader's choice. Stackleberg games corresponding to various combinatorial problems, such as shortest path[9] and minimum spanning tree[3] are known to be APX-hard.

Unlike our game, the Stackleberg games are usually not zero sum. In addition, in the usual Stackleberg games model, playing first allows the market leader an advantage over the market follower, which is not true for our game, where chooser commitment to the pieces he chooses before their sizes are revealed gives the cutter an advantage.

1.3.2 Motivation - Interdiction problems

Our motivation originated in network interdiction problems, where two players interact over a flow network. One attempting to maximize the flow of some commodity through it and the other removing edges from the network in an attempt to minimize that flow. The interdictor is assumed to have limited resources preventing him from cutting the network's flow completely, which he attempts to best utilize. In addition, some of the networks edges are harder to remove than others, so a removal of some edges may cost less than others.

Among the most fundamental results in the topic of flow networks is the min-cut-max-flow theorem, proven independently in both [4] and [5] in 1956. Given a flow network with source s, sink t and a set of intermediate nodes V, the min-cut-max-flow theorem states that the max flow in a network is equal its minimal capacity s-t cut.

Given a flow network and a flow, one might consider the concept the residual network induced by the flow, the flow network composed of the residual capacities in the original network's edges. By maximizing the flow, and separating it into connected components, it is possible to find a minimal cut in the original network.

That insight is of limited use to the interdictor, since the minimal cut might not be unique, and the number of minimal cuts might even be exponential in the size of the graph.

The maximum flow reduction problem has many static variations, some discussed in [6]. One such problem is: given a flow network, and a target maximum flow value F, find the cheapest set of edges for the investigator to remove in order to decrease the maximum flow to at most F.

Among the algorithms described in [6] for this problem, are two polynomial bi-criteria algorithms, which achieve a $(1+\epsilon) \vee \left(1+\frac{1}{\epsilon}\right)$ approximation for every $\epsilon > 0$. That notation, defined in [6], means that these algorithms either decrease the maximum flow value below the threshold F, using at most $1+\epsilon$ of the optimal budget, or use at most the optimal budget for the problem, while reducing the maximum flow value to at most $\left(1+\frac{1}{\epsilon}\right)$ of the optimal maximum flow value.

In addition, [6] shows that finding exact solutions for this static problem is NP-Hard, by showing reductions to it from both Knapsack and Clique problems.

Our game may be seen as an instance of the interdiction problem in a network of two nodes with $2S$ parallel edges between them, each with a capacity of 1.

Some models of network interdiction games were discussed in the literature. One such model is was suggested in 1995 in [20] discusses a two player game with an evader attempting to get from one point to another in a given flow network without being detected and an interdictor, who attempts to discover him by inspecting the graph's edges. Another model suggested in [18] involves numerous non-cooperative independent interdictors each attempting to prevent different adversaries from traversing in a given flow network.

2 Results

Definition 1 *Consider three families of cuts:*

- *AE - Cuts composed of all equal pieces*
- *AO - Cuts composed of* $\lfloor S \rfloor$ *pieces of size 1 and at most one piece of size* $S - \lfloor S \rfloor$ *if* $S > \lfloor S \rfloor$
- *mix-1-AE - Cuts composed of pieces of two sizes, one of which is 1*

And define canonical and semi canonical cuts so:

Definition 2 *A cut is called canonical if it is either an AE or an AO cut.*

A cut is called semi-canonical if it is either canonical or an mix-1-AE cut.

Note that given S, there are at most $O(S)$ possible canonical cuts and $O(S^2)$ semi-canonical cuts.

Our main result is:

Theorem 3 *For every S, C, an optimal cut exists, and it is semi-canonical.*

From which we deduce the two corollaries:

Corollary 4 *There exists a cutter algorithm finding for every S, C the optimal cut in time $O(C \cdot S^2)$, and in particular polynomial time.*

Corollary 5 *There exists a chooser algorithm finding for every chosen cut M and capacity C, a response achieving slackness of at most $\omega(S, C)$ in time $O(C \cdot |M|^2)$, and in particular polynomial time.*

Which we prove in sections 6.1 and 6.3.

3 Proof approach

We wish to prove theorem 3. To do so, we initially restrict our attention to standard cuts, those where no two pieces may be merged into a single piece. This is justified by Proposition 7.

Then, we observe that a cut's slackness makes it sufficient to consider cuts where the difference between pieces is greater than it, meaning that cuts with few piece sizes need to be considered.

In particular, whenever the slackness is at least $\frac{1}{3}$, at most two piece sizes may be considered, since with three piece sizes, the sum of the least two piece sizes is be lesser than $\frac{1}{3} + \frac{2}{3} = 1$, meaning that the cut may be standardized to one without it. Then, with only two possible piece sizes, weight shifting arguments between the two sets of pieces are easier to analyze, and we will see that any cut may be reduced to a cut no worse than it belonging to a limited set of simpler forms.

This means that it is sufficient to prove for every $S > C$ that $\omega(S, C) \geq \frac{1}{3}$. We show that is true for every $C > \frac{4}{3}$ and $S > C$. For $C \leq \frac{4}{3}$, the slackness may be lesser than $\frac{1}{3}$, so a different argument is required. However, restricting our attention to standard cuts, the number of pieces in feasible responses is limited to either one or two, making that case easy to analyze separately.

Remark: note that the optimal cut need not be unique. Consider for example $S = 3.6, C = 2.2$. Manual calculation will show that there are 8 possible semi-canonical cut candidates, and both cuts of 4 and 6 equal pieces are optimal with slackness of $s = 2.2 - 1.8 = 0.4$.

A more detailed outline of the proof approach is:

1. For $C \leq \frac{4}{3}$, show that any cut can be replaced by a canonical cut no worse than it.

2. For any cut M with slackness $\omega(M) \geq \frac{1}{3}$ show that it can be replaced by a semi-canonical cut with a no lesser slackness.

3. For $S > C > \frac{4}{3}$, show that there is always a cut M with slackness $\omega(M) \geq \frac{1}{3}$. (This is not true for small C, for example $\omega(1, 1.5) = 0.25$, and it can be seen by separation to cases that this is the minimal slackness) This is proven separately in four different subcases, the first of which is composed of eight subcases itself.

   (a) For $C < S \leq C + \frac{7}{3}$
   (b) For $C + \frac{7}{3} < S < 2C$
   (c) For $S = 2C$
   (d) For $S > 2C$

4 Some informative observations

Consider the chooser's role in the game. His problem, given a cut, is the uniform knapsack problem, which is a well known NP-hard problem, as discussed in section 6.2. However, the existence of an optimal reply is guaranteed, since the set of replies is finite.

Furthermore, checking the feasibility of a given set of pieces may be difficult under some representations, for example, if the cake is cut into pieces whose sizes are given as sums of square roots of rational numbers, it is currently unknown whether such a decision is even in NP[1].

4.1 Standard cuts

Our first step in characterizing the problem is to define standard cuts:

Definition 6 *Given S, C, a standard cut is a partition $\sum \ell_i = S$ s.t. $\ell_i + \ell_j > 1$ for every $i \neq j$.*

A useful observation is that the number of pieces required by the cutter is at most twice the size of the cake.

Proposition 7 *Given S, C, for every cut M, there exists a standard cut M' s.t. $\omega(M') \geq \omega(M)$.*

Proof. Given a cut which is not standard, merge pieces which sum to at most 1 repeatedly, lowering the number of pieces by one at a time until a standard cut is achieved. This process does not improve the chooser's set of replies, since reply to the new cut may be transformed into a reply for the old cut, possibly by breaking some merged pieces back to their original parts. ■

Notice that since the average piece size in standard cuts containing more than a single piece is greater that 0.5, it contains less than $2S$ pieces overall.

Corollary 8 *Given $S > 1$, for every cut M, there exists a cut M' with no lesser slackness s.t. $|M'| \leq 2S$.*

4.2 Online version of the game

We define an online version of the game:

Definition 9 *The* online cake cutting game *is a sequential game with a cutter cutting pieces from a cake of initial size S at every turn, with the chooser deciding for each piece whether he wants to pick it and decrease his capacity accordingly or ignore it and use his capacity for future pieces.*

While seemingly similar to our game, the online cake cutting game might not have optimal cutter strategies.

Example 4.1 *We will show that for $S = 1.1, C = 1$, an optimal cutter strategy for the online game does not exist by showing that the chooser may always pick an amount strictly greater than 0.5, while the cutter is able to make the amount arbitrarily close to it.*

For every small $\epsilon > 0$, the cutter first cuts a piece of size $\ell_1 = 0.1 - \epsilon$. He then follows that piece by two pieces of $\ell_2 = \ell_3 = 0.5 + \frac{\epsilon}{2}$ if ℓ_1 is ignored, and by $\ell_2' = \epsilon, \ell_3' = 1$ otherwise. For such a cutting strategy, the chooser may gain at most $\max\{0.5 + \frac{\epsilon}{2}, 0.1\} = 0.5 + \frac{\epsilon}{2}$, making it impossible to choose significantly more than 0.5.

On the other hand, the chooser can always make sure an amount strictly greater than 0.5 *may be chosen, by a strategy of waiting until a piece* $\ell \geq S - C$ *is made, choosing it if* $\ell > \frac{S}{2}$*, and ignoring it and choosing all other pieces otherwise. That strategy assures at least half the remaining cake is chosen, and that is strictly greater than* $\frac{C}{2}$*, as required.*

4.3 Enforcing continuity by partitioning the set of moves

Given a cut composed of m pieces, there are 2^m subsets of these pieces, and there are 2^{2^m} subsets of the power set of the m pieces. For every cut and capacity, there is a set of subsets of the m pieces which are feasible. This may allow the separation of cuts including m pieces to 2^{2^m} classes, such that cuts of each class have the same feasible subsets. Within any such class, the value function is continuous, so we may attempt to find an optimal cut in each, and since the number of classes is finite, deduce an optimal solution for the original problem as well.

Note that these classes are not compact since the unfeasibility constraints are strict inequalities demanding that certain subsets' sums are strictly greater than the capacity. Therefore, the existence of an optimal solution is not guaranteed by Weierstrass' extreme value theorem in these domains either.

In particular, we may observe examples of instances and classes where such an optimal solution does not exist:

Example 4.2 *Given* $m = 5, S = 4, C = 1.5$*, with the feasible subsets being* $\{\{\ell_1, \ell_2\}, \{\ell_1\}, \{\ell_2\}, \{\ell_3\}, \{\ell_4\}, \{\ell_5\}\}$*. Since* $\ell_1 + \ell_3 > 1.5$*, we get that* $\ell_1 > 0.5$ *and therefore* $\ell_1 + \ell_2 > 1$*. Therefore, any cut belonging to this class has slackness strictly lesser than* 0.5*.*

It may be seen that a slackness arbitrarily close to 0.5 *may be achieved by considering, for every* $0 < \epsilon < \frac{1}{10}$*, the cut:*

$M_\epsilon = \{\frac{1}{2} + 3\epsilon, \frac{1}{2} + 3\epsilon, 1 - 2\epsilon, 1 - 2\epsilon, 1 - 2\epsilon\}$

Clearly for every such ϵ*, we get that the required subsets feasible, and the optimal subset sums to* $1 + 6\epsilon$*, arbitrarily close to* 1*, as required.*

5 Proof

5.1 Semi-canonical cuts are optimal when $\omega(S, C) \geq \frac{1}{3}$

In this subsection, we will prove the following theorem:

Theorem 10 *Given S, C, if there is a cut M with $\omega(M) \geq \frac{1}{3}$, then the optimal semi-canonical cut is also the optimal cut.*

To do so, we first show that every cut M which is not an AO cut may be assumed to have all of its pieces strictly greater than $\omega(M)$, and that the difference between every two piece sizes must also be greater than $\omega(M)$. This is be proved in Propositions 11 and 12 accordingly:

Proposition 11 *If $\omega(M) \geq \ell_m$, then there is a cut M' of slackness $\omega(M') \geq \omega(M)$ of either one of the following two forms:*

1. *Only $m-1$ pieces.*
2. *$\lfloor S \rfloor$ pieces of size 1 and one of size $S - \lfloor S \rfloor$.*

For a cut M in which not all pieces have the same size, the minimum nonzero difference between two pieces is denoted by $\delta(M)$. If all pieces are of the same size, we define $\delta(M) = 1$ (which turns out convenient for the statement of Proposition 12).

Proposition 12 *If $\delta(M) \leq \omega(M)$, then there is a cut M' of slackness $\omega(M') \geq \omega(M)$ for which $\delta(M') > \omega(M')$.*

By Proposition 12, we get that a slackness $\omega(M) \geq \frac{1}{2}$, suffices, since we only need to consider canonical cuts:

Theorem 13 *If there is a cut M with slackness $\omega(M) \geq \frac{1}{2}$, the optimal canonical cut is also the optimal cut.*

When there exists a cut M achieving slackness $\omega(M) \geq \frac{1}{3}$, we get that by Proposition 11, that it is sufficient to only consider cuts where there are no pieces of size $\frac{1}{3}$ or less. Furthermore, by Proposition 12, there are only two piece sizes, $a > b$, with $a > b + \omega(M) \geq b + \frac{1}{3} > \frac{2}{3}$.

Consider such a cut. Let t_a, t_a be the number of pieces of sizes a, b accordingly. Let r be an optimal reply, with k_a pieces of size a and k_b pieces of size b. Let $\rho_a = \frac{k_a}{t_a}$ and $\rho_b = \frac{k_b}{t_b}$.

We first show that k_a, k_b may be assumed to be unique:

Proposition 14 *W.l.o.g., the optimal reply k_a, k_b may be assumed to be unique.*

Furthermore, the possible values of k_a and k_b are limited:

Proposition 15 *If $k_a \geq 1$ then $k_b \geq t_b - 1$.*

And finish the by handling the two cases $\rho_a \geq \rho_b$ and $\rho_a < \rho_b$:

Proposition 16 *If $\rho_a \geq \rho_b$ then there is a canonical cut M' with $\omega(M') \geq \omega(M)$.*

Proposition 17 *If $\rho_b > \rho_a$ then there is a semi-canonical cut M' with $\omega(M') \geq \omega(M)$.*

Thus completing the proof of Theorem 10.

We begin by proving Proposition 11, which stated that we may assume wlog that any cut which is not an AO cut must have all pieces greater than the slackness $\omega(M)$, and in particular have $\ell_m > \omega(M)$.

Proof. Any optimal reply must contain ℓ_m, as otherwise it remains feasible by adding ℓ_m to it.

Change the cut by shifting value from ℓ_m into other arbitrary eligible (of size less than 1) pieces until one of the following two events occurs: either ℓ_m disappears, or all other pieces are of size 1. This gives the new move $M' = \{\ell'_1, \ldots \ell'_m\}$, where possibly $\ell'_m = 0$. We claim that $\omega(M') \geq \omega(M)$. Suppose otherwise, that $r(M')$ has lesser slackness than $r(M)$.

- If $\ell_m \in r(M')$, then $r(M')$ has even lesser slackness in M, implying that in M its slackness is below 0. But then $r(M') \setminus \{\ell_m\}$ must have been feasible in M and of slackness below $\omega(M)$, which is a contradiction.

- If $\ell_m \notin r(M')$, then $r(M')$ is feasible also in M, and hence of slackness at least $\omega(M)$ in M. Then the reply $r(M') \cup \{\ell_m\}$ is feasible in M and has slackness at most $\omega(M') < \omega(M)$, which is a contradiction.

■

And continue to prove Proposition 12, stating that the difference between every two pieces of different sizes may be assumed wlog to be strictly greater than the slackness.

Proof. For $i < j$ let i and j be a pair of pieces satisfying $0 < \ell_i - \ell_j \leq \omega(M)$.

Assume first for simplicity that no other piece is of size ℓ_i and no other piece is of size ℓ_j. Then change M to a new cut M' by replacing ℓ_i and ℓ_j by $\ell'_i = \ell'_j = \frac{\ell_i + \ell_j}{2}$. We claim that $\omega(M') \geq \omega(M)$. Given a move M, denote by $r(M)$ the set of optimal replies to it. Clearly, they all have the same slackness. Suppose to the contrary that a reply $r' \in r(M')$ has lesser slackness than that of $r(M)$'s replies. Then for every reply $r' \in r(M')$ either $i \in r'$ or $j \in r'$, but not both nor neither, as otherwise the slackness does not change by the averaging.

- If $i \in r'$, then r' has even lesser slackness in M, implying that it is unfeasible in M. But then $r' \cup \{j\} \setminus \{i\}$ must have been feasible in M and have slackness strictly less than $\ell_i - \ell_j \leq \delta(M) \leq \omega(M)$, which is a contradiction.
- If $j \in r'$, then r' is feasible also in M, and hence of slackness at least $\omega(M)$ in M. Then the reply $r(M') \cup \{i\} \setminus \{j\}$ is feasible in M and has slackness at most $\omega(M') < \omega(M)$, which is a contradiction.

If M' satisfies $\omega(M') < \delta(M')$ then we are done. otherwise, repeat the above process with M'. Since every repetition decreases the number of distinct piece sizes, the process must end.

We now modify the proof such that it applies also if there are n_i pieces of size ℓ_i and n_j pieces of size ℓ_j, with $n_i + n_j > 2$. Notice that our previous argument applies to any weight shift between such pairs, and not only to their averaging.

Let $\rho = \frac{n_i \ell_i + n_j \ell_j}{n_i + n_j}$. In this case we shift weights between pairs among these pieces, but always with one member of the pair reaching ρ. After at most $n_i + n_j - 1$ steps all these pieces have size ρ, and the number of distinct piece sizes decreases. ■

The intermediate Theorem 13, showing that if $\omega(S, C) \geq \frac{1}{2}$ canonical cuts are optimal follows from Propositions 12 and 11:

Proof. Given any cut $M = \{\ell_1, ..., \ell_m\}$ with $\omega(M) \geq \frac{1}{2}$, we show that there is a canonical move whose slackness no lesser than M's.

If $\ell_m \geq \frac{1}{2}$, by Proposition 12 all pieces are equal since otherwise $\ell_1 > \ell_m + \frac{1}{2} > 1$, which is a contradiction.

If $\ell_m < \frac{1}{2}$, by Proposition 11, we get that either there exists a better cut containing only $m - 1$ pieces, and repeat the argument for it, or an AO canonical cut is better than M. ■

Now, proving Proposition 14, stating that when $\omega(S, C) \geq \frac{1}{3}$, it may be assumed wlog that the optimal replies are unique:

Proof. Let $k'_a > k_a$ and $k'_b < k_b$ be another optimal reply. Then change the cut M to M' by replacing $k_b - k'_b$ pieces of size b by $k'_a - k_a$ pieces of size a. The number of pieces decreases, but the optimal reply does not improve, because any reply (k_a^*, k_b^*) to M' is either possible also in M, or can be replaced by a reply $(k_a^* + k_a - k_a', k_b^* + k_b - k_b')$ that is legal in M and has the same value. ∎

We now prove Proposition 15 stating that if $k_a \geq 1$ then $k_b \geq t_b - 1$:
Proof. Otherwise, in the optimal reply replace one a by two b. As $2b > a$ this is an improvement. This new reply is feasible because the sum of the new subset's pieces is at most $C - \delta - a + 2b \leq C + a - 3\delta \leq C$ because $\delta \geq \frac{1}{3}$. ∎

Recall that we denoted the ratios $\rho_a = \frac{k_a}{t_a}$ and $\rho_b = \frac{k_b}{t_b}$.

We now prove Proposition 16, stating that if $\rho_a \geq \rho_b$ then there is a canonical cut M' with $\omega(M') \geq \omega(M)$.
Proof. If $\rho_a \geq \rho_b$ then necessarily $k_b < t_b$ (as otherwise $\rho_a = \rho_b = 1$ implying $A \geq L$), and $k_a > 0$ (otherwise $\rho_a = \rho_b = 0$ means that $C < 1$, while we assume in this part that $C > \frac{4}{3}$), and hence by Proposition 15 we get that $k_b = t_b - 1$. Consider the canonical cut M' with $t_a + t_b$ equal pieces of size $\frac{at_a + bt_b}{t_a + t_b}$. A reply that captures $k_a + k_b$ of them has a sum of $(k_a + k_b)\frac{at_a + bt_b}{t_a + t_b} \leq ak_a + bk_b$.

No reply to M' can capture $k_a + k_b + 1$ pieces because then in M taking $k_b + 1 = t_b$ pieces of size b and k_a pieces of size a was feasible, because $ak_a + bt_b \leq (k_a + t_b)\frac{at_a + bt_b}{t_a + t_b}$ holds for $a \geq b$ and $t_a \geq k_a$. ∎

And Proposition 17:
Proof. Gradually shift weight from b to a. This lowers the value of the reply (k_a, k_b). The process of shifting weights ends at a move M' satisfying at least one of the following events.

1. $a = 1$. This is a semi-canonical move.
2. There is some new reply of value equal to the optimal reply (k_a, k_b). This case is handled as in Proposition 14.
3. There is some new reply r' of value C (that in M has value above C). We claim that r' has exactly one b item (meaning, an item of value b in M). If r' has no b item it was feasible in M (contradicting our assumption that it was not feasible in M). If r' has at least two b items, replace two b items by one a item. (There is at least one a item not in r', as otherwise the value of r' increased rather than decreased by the shift.) This gives a reply r^* that had value at least $C - 2b + a = C - b + \delta > C - \delta$ in M (because $2\delta = 2a - 2b > \frac{2}{3} > b$

when $\delta > \frac{1}{3}$), contradicting the optimality of (k_a, k_b)). (Notice that r^* has lower value than r', and hence is feasible in M', and consequently must have been feasible in M, as otherwise we could not have reached M' in the shifting process.)

Hence $r' = (k'_a, 1)$ and its value in M' is C. Given that the value of r' in M is larger than C, necessarily $\rho'_b > \rho'_a$. That means that:

$$\frac{k'_a}{t_a} < \frac{1}{t_b} \iff t_b \cdot k'_a - t_a < 0 \iff (b-a)(t_b k'_a - t_a) > 0 \iff$$

$$\iff t_b k'_a b + t_a a > t_a b + t_b k'_a a \iff$$

$$\iff t_b k'_a b + t_a a + \left(t_a k'_a a + t_b b\right) > t_a b + t_b k'_a a + \left(t_a k'_a a + t_b b\right)$$

$$\iff \left(k'_a + 1\right)(t_a a + t_b b) > (t_a + t_b)\left(k'_a a + b\right)$$

$$\iff \frac{(k'_a + 1)(t_a a + t_b b)}{t_a + t_b} > k'_a a + b$$

We know that $k'_a a + b > C$ since it was not feasible before the shift, meaning that cutting S into $t_a + t_b$ equal pieces gets at most k'_a pieces chosen, and have value of at most $ak'_a = C - b$.

If $b \geq \frac{1}{2}$ the proposition follows from Theorem 13. And if we have that $b < \frac{1}{2}$, then $t_b = 1$, and since $\rho_b > \rho_a$, we get $k_b = 1$, and we have already seen that $k'_b = 1$. As $r \neq r'$ this implies that $k'_a = k_a + 1$, and the value of r in M is smaller than $C - a < C - \frac{1}{2}$, and the proposition follows from Theorem 13.

■

That completes the proof of theorem 10.

Remark on the inequality $\omega(S, C) \geq \frac{1}{3}$. For $C \leq \frac{4}{3}$ the inequality need not hold (e.g., $C = \frac{3}{4}$ and $S = \frac{3}{2}$). For $C > \frac{3}{2}$ equality is sometimes needed (e.g., $C = 2$ and $S = \frac{5}{2}$).

5.2 $\omega(S, C) \geq \frac{1}{3}$, when $C \geq \frac{4}{3}$

5.2.1 $C = S - C$

Proposition 18 *Suppose that $C = S - C$ and $C \geq 1$. Then $\omega(S, C) \geq \frac{1}{3}$.*

Proof. Let n be the smallest odd integer satisfying $n \geq S$. Cut S into n equal pieces, each of size $x = \frac{S}{n}$. Then the slackness is $\frac{S}{2n}$. We are done unless $\frac{S}{n} < \frac{2}{3}$. Given that $S = 2C \geq 2$, the only case where $\frac{S}{n} < \frac{2}{3}$ is when $3 < S < \frac{10}{3}$. But in this case, the max-1 move ensures slackness of $C - (S-2) = 2 - C = 2 - \frac{S}{2} > 2 - \frac{5}{3} = \frac{1}{3}$ as required. ■

The condition $C \geq 1$ is needed above. For $C = \frac{3}{4}$ and $S = \frac{3}{2}$ we have that $\omega(S, C) = \frac{1}{4}$ since there must be two pieces in every standard cut, and the lesser one can be chosen, so the slackness can be at most $C - (S-1) = 0.25$.

5.2.2 $C < S - C$

Proposition 19 *Suppose that $S - C > C$, and $C \geq \frac{4}{3}$. Then $\omega(S, C) \geq \frac{1}{3}$.*

Proof. Let $c = \lceil C \rceil$ and $\ell_1 = \frac{C}{c}$. Notice that $1 \geq \ell_1 \geq \frac{2}{3}$.

Let integer $n_1 \geq 2c$ be such that $S = n_1 \ell_1 + y_1$ with $0 < y_1 \leq \ell_1$.

We separate into two cases:

Case 1 $(n_1 \geq S)$

In this case, we may cut S into n_1 equal pieces.

At most $c - 1$ pieces are chosen since:

$$c \cdot \frac{S}{n_1} = \frac{c n_1 \ell_1 + c y_1}{n_1} = C + \frac{c y_1}{n_1} > C$$

So that cut ensures a slackness of:

$$\omega(S, C) \geq C - (c-1)\frac{S}{n_1} = (c-1)\frac{n_1 \ell_1 + y_1}{n_1} = C - \ell_1 + \frac{c-1}{n_1} y_1 < C - \frac{\ell_1}{2} \leq C - \frac{1}{3}$$

Case 2 $(n_1 < S)$

Let $\ell_2 = \frac{C}{c+1}$. If $\ell_2 \geq \frac{2}{3}$ then the above argument works because $n_2 > n_1 \geq \lfloor S \rfloor$.

The only cases where $\ell_2 < \frac{2}{3}$ are the following:

Case 2.1 $(3 \leq C < \frac{10}{3})$

Then necessarily $n_1 \geq S$. Given that $c = 4$ (here we take $c = 4$ even if $C = 3$) and $n_1 \geq 2c$, the worst case is when $n_1 = 8$ and $S = \frac{9}{4} C < \frac{15}{2}$

Case 2.2 ($2 \leq C < \frac{8}{3}$)

If $C \leq \frac{18}{7}$ then necessarily $n_1 \geq S$ (here we take $c = 3$ even if $C = 2$). When $\frac{18}{7} < C < \frac{8}{3}$ and $n_1 < S$ then necessarily the max-1 move ensures $\omega(S, C) \geq \frac{1}{3}$

Case 2.3 ($\frac{4}{3} \leq C < 2$)

If $C \leq \frac{8}{5}$ then $n_1 \geq S$. When $\frac{8}{5} < C \leq \frac{16}{9}$ and $n_1 < S$ then necessarily the max-1 move ensures $\omega(S, C) \geq \frac{1}{3}$.

Hence it remains only to consider the case that $\frac{16}{9} \leq C < 2$.

Let $x = C/2$, let n be such that $(n-1)x \leq S < nx$. Observe that $n \geq 5$. We may assume that $S > C+n-\frac{7}{3}$. Otherwise either take n pieces (if they fit) or n ones and the remainder. The optimal response is 1 plus the remainder, which is at most $C - \frac{1}{3}$.

The remaining is proven separately for two cases:

Case 2.3.1 ($5 \leq n \leq 8$)

Cut S into $n+2$ pieces. The optimal response cannot take 3 pieces, since:

$3\frac{S}{n+2} > \frac{3C+3n-7}{n+2} \geq \frac{3n+6-7}{n+2} \geq 3 - \frac{7}{n+2} \geq 2$, where the last inequality holds for $n \geq 5$, which we stated to hold earlier.

The value of 2 pieces is at most $2\frac{S}{n+2} \leq \frac{n}{n+2}C < C - \frac{1}{3}$ where the last inequality holds for $n \leq 8$, and when $n = 9$ for $C \geq \frac{11}{6}$ (if $n = 9$ and $C < \frac{11}{6}$, cut into $n+1$ pieces).

Case 2.3.2 ($8 < n$)

We have that $S > 6$. In this case let n be the smallest such that $\frac{2n}{3} > S$. Cut S into n pieces. A response can capture only 2 pieces, giving at most $\omega(S, C) = C - \frac{4}{3} - \frac{1}{9} = C - \frac{13}{9} \geq \frac{1}{3}$.

■

5.2.3 $C > S - C$ and $S - C > \frac{7}{3}$

Let $b = \lceil S - C \rceil$ and $x_1 = \frac{S-C}{b}$.

Let integer $n_1 \geq 2b+1$ be such that $S = n_1 x_1 - y_1$ with $0 < y_1 \leq x_1$. Cut S into n_1 equal pieces. Then the best reply takes $n_1 - b - 1$ pieces for a value of $(n_1-b-1)\frac{n_1 x_1 - y_1}{n_1} = (n_1-b-1)x_1 - \frac{n_1-b-1}{n_1}y_1 = C - x_1 + \frac{b+1}{n_1}y_1 \leq C - \frac{b}{2b+1}x_1$. When $S - C > 3$ (and also when $S - C = 3$) then $b \geq 4$ and $x_1 \geq \frac{3}{4}$, implying that *omega* $\geq \frac{1}{3}$. Likewise, when $\frac{7}{3} \leq S - C < 3$ taking $b = 3$ implies that $\omega \geq \frac{1}{3}$.

5.2.4 $C > S - C$ and $S - C \leq \frac{7}{3}$

Let $\delta = S - C$. For a quantity $M \geq 0$ we let M_f denote its fractional part, namely, $M_f = M - \lceil M \rceil$.

Let us start with a proposition describing the cases which are handled by the AO cut.

Proposition 20 *If $0 < \delta_f \leq \frac{2}{3}$ and $\frac{1}{3} \leq C_f < 1$, then there is a cut M that assures that $\omega(M) \geq \frac{1}{3}$.*

Proof. The move is $\lfloor S \rfloor$ items of size 1, and one item of size S_f. ∎

Case 1 ($\delta \leq \frac{1}{2}$)

Proposition 20 implies that $k \leq C \leq k + \frac{1}{3}$ for some integer $k \geq 2$ (because $C > \frac{4}{3}$), and consequently $S < k + 1$.

Cut S into $k + 1$ equal pieces. The optimal reply captures at most k pieces. Hence

$$\omega(C+\delta, C) = C - \frac{k}{k+1}(C+\delta) = \frac{C}{k+1} - \frac{k\delta}{k+1} \geq \frac{k}{k+1} - \frac{1}{2}\frac{k}{k+1} = \frac{k}{2(k+1)} \geq \frac{1}{3}$$

because $c \geq k, \delta \leq \frac{1}{2}$ and $k \geq 2$ accordingly.

Case 2 ($\frac{1}{2} < \delta \leq \frac{2}{3}$)

Proposition 20 implies that $k \leq C \leq k + \frac{1}{3}$ for some integer $k \geq 2$ (because $C > \frac{4}{3}$), and consequently $k + \frac{1}{2} < S \leq k + 1$.

Consider two cuts M_1, M_2. In M_1 there are $k + 1$ equal pieces. In M_2 there are $2k + 1$ equal pieces (each of size strictly greater than $\frac{1}{2}$). If in M_2 the optimal reply captures only $2k - 1$ pieces then $\omega(M_2) \geq \frac{2}{2k+1}S - \delta > 1 - \frac{2}{3} = \frac{1}{3}$ and we are done. Hence we may assume that $C \geq \frac{2k}{2k+1}S$. But then in M_1, since at least one piece would be missed, the slackness is

$$\omega(M_1) \geq \frac{2kS}{2k+1} - \frac{kS}{k+1} = \frac{kS}{(2k+1)(k+1)} > \frac{(2k+1)k}{2(2k+1)(k+1)} = \frac{k}{2k+2} \geq \frac{1}{3}$$

because $S > k + \frac{1}{2}$, and $k \geq 2$.

Case 3 ($\frac{2}{3} < \delta \leq 1$)

Since $C \geq \frac{4}{3}$, we get that $S > 2$.

Let integer k be such that $\frac{S}{k+1} < \delta \leq \frac{S}{k}$. (Such a k exists since $S > \delta > 0$) Hence $C = S - \delta \geq \frac{k-1}{k}S$. Cut S into $k+1$ equal pieces. (This is feasible since $\frac{S}{k+1} < \delta \leq 1$) The optimal reply captures $\frac{k-1}{k+1}S$. Hence $\omega(S, C) \geq \frac{k-1}{k}S - \frac{k-1}{k+1}S = \frac{k-1}{k(k+1)}S \geq \frac{k-1}{k+1}\delta$.

For $k \geq 3$ we have $\omega(C + \delta, C) \geq \frac{1}{3}$ because $\delta \geq \frac{2}{3}$.

However, if $k < 3$, $k = 2$ because $k + 1 > \frac{S}{\delta} \geq S > 2$, and therefore $S < 3\delta \leq 3$. Three equal pieces gets at most one piece, of size at most 1 chosen, and $\omega(C + \delta, C) \geq C - 1 \geq \frac{4}{3} - 1 = \frac{1}{3}$ as required.

Case 4 ($1 < \delta \leq \frac{4}{3}$)

Proposition 20 implies that $k \leq C \leq k + \frac{1}{3}$ for some integer $k \geq 2$ because $C > \frac{4}{3}$, and consequently $k + 1 < S \leq k + \frac{5}{3}$.

Cut S into $k+2$ equal pieces. Since $1 < \delta$, the optimal reply captures only k of them. Therefore, the slackness is at most:

$$\omega(C+\delta, C) = C - \frac{k}{k+2}(C+\delta) = \frac{2C}{k+2} - \frac{k\delta}{k+2} \geq \frac{2k}{k+2} - \frac{4k}{3(k+2)} = \frac{2k}{3(k+2)} \geq \frac{1}{3}$$

because $k \leq C$, $\delta \leq \frac{4}{3}$, and $k \geq 2$.

Case 5 ($\frac{4}{3} < \delta \leq \frac{3}{2}$)

Proposition 20 implies that $k \leq C \leq k + \frac{1}{3}$ for some integer $k \geq 2$ (because $C > \frac{4}{3}$), and consequently $k + \frac{4}{3} < S \leq k + \frac{11}{6}$.

This is handled separately in 3 different subcases:

Case 5.1 ($k \geq 4$)

Cut S into $k+2$ equal pieces. Since $\delta > 1$, the optimal reply captures only k pieces, giving:
$\omega(C+\delta, C) = C - \frac{k}{k+2}(C+\delta) = \frac{2C}{k+2} - \frac{k\delta}{k+2} \geq \frac{2k}{k+2} - \frac{3k}{2(k+2)} = \frac{k}{2(k+2)}$
Since $k \geq 4$, we get that $\omega(C + \delta, C) \geq \frac{1}{3}$ as required.

Case 5.2 ($k = 3$)

Notice that $S \leq 3 + \frac{1}{3} + \frac{3}{2} < 5$, $S > 3 + \frac{4}{3} > 3.5$ and $C \leq \frac{10}{3}$.
7 equal pieces get us the required difference, since at least 3 pieces are missed as:

$$\frac{5S}{7} > \frac{5C}{7} + \frac{5}{7} \cdot \frac{4}{3} = \frac{5C}{7} + \frac{2}{7} \cdot \frac{10}{3} \geq \frac{5C}{7} + \frac{2}{7} \cdot C = C$$

And therefore:

$$\omega(C+\delta, C) = C - \frac{4S}{7} \geq C - \frac{4C}{7} - \frac{4}{7} \cdot \frac{3}{2} = \frac{3C}{7} - \frac{6}{7} \geq \frac{3 \cdot 3}{7} - \frac{6}{7} = \frac{3}{7} > \frac{1}{3}$$

Case 5.3 ($k = 2$)

Notice that $S \leq 2 + \frac{1}{3} + \frac{3}{2} < 4$, and since $\delta > 1$, four equal pieces get $\frac{S}{2}$ chosen.

If $\frac{3}{5}S \leq C$, then these four equal pieces would do since:

$\omega(C+\delta, C) = C - \frac{S}{2} \geq C - \frac{5}{6}C = \frac{C}{6} \geq \frac{2}{6} = \frac{1}{3}$

If $\frac{3}{5}S > C$, then 5 equal pieces would do, since:

$\omega(C+\delta, C) = C - \frac{2}{5}S \geq C - \frac{2}{5}C - \frac{6}{10} = \frac{3}{5}C - \frac{3}{5} \geq \frac{3}{5} > \frac{1}{3}$

Case 6 ($\frac{3}{2} < \delta \leq \frac{5}{3}$)

Proposition 20 implies that $k \leq C \leq k + \frac{1}{3}$ for some integer $k \geq 2$ (because $C > \frac{4}{3}$), and consequently $k + \frac{3}{2} < S \leq k + 2$.

Case 6.1 ($k \geq 4$)

Consider two cuts M_1, M_2, one of $2k+3$ equal pieces, and another of $k+2$ equal pieces.

If $\frac{2k}{2k+3}S > C$, we get the required slackness by $2k+3$ equal pieces, since at least 4 pieces would be missed and:

$\omega(C+\delta, C) > C - (S-2) = 2 - \delta \geq \frac{1}{3}$

Otherwise, $\frac{2k}{2k+3}S \leq C$ and considering $k+2$ equal pieces we see that at most k may be chosen (since $\delta > 1$), and therefore:

$\omega(C+\delta, C) \geq \frac{2k}{2k+3}S - \frac{k}{k+2}S = \frac{2k^2+4k-2k^2-3k}{(2k+3)(k+2)}S > \frac{k(k+\frac{3}{2})}{(2k+3)(k+2)} = \frac{k}{2(k+2)} \geq \frac{4}{2(4+2)} = \frac{1}{3}$

because $S \geq k + \frac{3}{2}$ and $k \geq 4$.

Case 6.2 ($k = 2$)

A cut of four equal pieces would get 2 pieces missed since $\delta > 1$. If that is insufficient we get:

$$\frac{S}{2} > C - \frac{1}{3} \iff \frac{C}{2} + \frac{\delta}{2} > C - \frac{1}{3} \iff \delta > C - \frac{2}{3}$$

And that means the cutting to 5 equal pieces would get at least 3 missed, since:

$$\frac{3S}{5} = \frac{3C}{5} + \frac{3\delta}{5} > \frac{3C}{5} + \frac{3}{5}(C - \frac{2}{3}) > \frac{6}{5}C - \frac{2}{5} > C$$

And that is sufficient since:

$$\omega(C+\delta, C) \geq C - \frac{2S}{5} = \frac{3C}{5} - \frac{2\delta}{5} \geq \frac{3}{5} \cdot 2 - \frac{2}{5} \cdot \frac{5}{3} = \frac{6}{5} - \frac{2}{3} > \frac{1}{3}$$

Case 6.3 ($k = 3$)

A cut of seven equal pieces would get at least 3 missed since:

$$\frac{2S}{7} = \frac{2C}{7} + \frac{2\delta}{7} \le \frac{2}{7} \cdot \frac{10}{3} + \frac{2}{7}\delta = \frac{5}{7} \cdot \frac{4}{3} + \frac{2}{7}\delta < \delta$$

And that is sufficient since:

$$\omega(C+\delta, C) \ge C - \frac{4S}{7} = \frac{3C}{7} - \frac{4\delta}{7} \ge \frac{3 \cdot 3}{7} - \frac{20}{21} = \frac{7}{21} = \frac{1}{3}$$

Case 7 ($\frac{5}{3} < \delta \le 2$)

Observe that our assumption that $C \ge \frac{4}{3}$ implies that $S > 3$.

Let integer k be such that $\frac{2}{k+1}S < \delta \le \frac{2}{k}S$. (Such a k exists since $0 < \delta < 2S$) Hence $C = S - \delta \ge \frac{k-2}{k}S$. Cut S into $k+1$ equal pieces. (That is feasible since $S < \frac{\delta}{2}(k+1) \le k+1$) The optimal reply captures $\frac{k-2}{k+1}S$ by the definition of k. Hence

$$\omega(C+\delta, C) \ge \frac{k-2}{k}S - \frac{k-2}{k+1}S = \frac{k-2}{k(k+1)}S \ge \frac{k-2}{2(k+1)}\delta$$

Case 7.1 ($k \ge 4$)

$$\omega(C+\delta, C) \ge \frac{k-2}{2(k+1)}\delta \ge \frac{4-2}{2(4+1)}\delta = \frac{1}{5}\delta > \frac{1}{3}$$

Case 7.2 ($k < 4$)

Then since $\frac{2}{k+1}S < \delta$, we get $S < \frac{\delta(k+1)}{2} \le k+1$, and since $3 < S$, we get $k = 3$, $3 < S \le 4$ and $\frac{2S}{4} < \delta$.

This means that four equal pieces would get at most one chosen, and by our assumption:

$$\omega(C+\delta, C) \ge C - 1 > \frac{4}{3} - 1 = \frac{1}{3}$$

Case 8 ($2 < \delta \le \frac{7}{3}$)

Proposition 20 implies that $k \le C \le k + \frac{1}{3}$ for some integer $k \ge 2$ (because $C > \frac{4}{3}$), and consequently $k + 2 < S \le k + \frac{8}{3}$.

Cut S into $k+3$ equal pieces. Since $\delta > 2$, the optimal reply captures at most k of them, and therefore:

$$\omega(C+\delta, C) \ge C - \frac{k}{k+3}(C+\delta) = \frac{3C}{k+3} - \frac{k\delta}{k+3} \ge \frac{3k}{k+3} - \frac{7k}{3(k+3)} = \frac{2k}{3(k+3)}$$

Case 8.1 ($k \geq 3$)

$$\omega(C+\delta, C) \geq \frac{2}{3} - \frac{6}{3(k+3)} = \frac{2}{3} - \frac{2}{k+3} \geq \frac{1}{3}$$

Case 8.2 ($k = 2$)

By the same cut and using our assumption that $S - C \leq C$ we get that $\delta \leq C$. Hence:

$$\omega(C+\delta, C) = C - \frac{2}{5}(C+\delta) = \frac{3C}{5} - \frac{2\delta}{5} \geq \frac{C}{5} \geq \frac{2}{5} > \frac{1}{3}$$

5.3 Canonical cuts are optimal for $C \leq \frac{4}{3}$

We would now prove that canonical cuts are optimal for $C \leq \frac{4}{3}$:

Proposition 21 *Canonical cuts are are optimal for $C \leq \frac{4}{3}$.*

To prove these remaining cases we begin with a proposition:

Proposition 22 *Given $S > C$, if there exists a cut assuring that at most one piece would be chosen, there exists an optimal canonical cut.*

Proof. Notice that any standard cut at most one piece to be chosen is superior to any cut allowing two pieces or more to be chosen, since the former gets at most 1 chosen, and the latter strictly more than 1.

We may assume that the cut is standard, since if after standardization two pieces may be chosen, at least two (the ones which composed them) pieces could have been chosen originally as well.

Let m be the number of pieces in it. Note that $\frac{2S}{m} > C$, since otherwise, the smallest two pieces could have been chosen. If $\frac{S}{m} > C$, we are set, since m equal pieces would get nothing chosen, and be optimal as required.

Case 1 ($C \geq 1$)

In this case, the greatest of the pieces would be chosen, so the cutter would be better off minimizing the greatest one. That would be achieved by making m equal pieces. (Making the optimal m-pieces cut the m-equal pieces cut)

Case 2 ($C < 1$)

This case is a little more complicated. First notice that the pieces may be separated into two types - those strictly greater than C and those that are not. The cutter would not be worse off if he made sure the pieces of each kind are equal. (Since in any case the greatest of the lesser pieces would best be chosen)

If there are only lesser pieces, or only greater pieces, we are done. (Nothing would be chosen, and the cut is an m equal pieces optimal cut)

If there is a single lesser piece, it may be reduced (And increase the greater pieces accordingly) until either the lesser piece reaches zero, or all other pieces reach 1. The first case would result in an all equal canonical cut, and the second to all ones canonical cut.

If there are two lesser pieces or more, they may be reduced (increasing the greater pieces accordingly) until either the lesser pieces all reach 0.5 (In which case they may be merged into pieces of size one, averaged with the other greater pieces, and revert to the single lesser piece case, or to an all greater pieces case), or all of the greater pieces reaching one. (In which case they would be broken into pieces of size 0.5, all pieces would be averaged again, decreasing the minimal lesser piece size again, as required).

In each of the cases, we arrive to a canonical cut no worse than the original one.

■

The most immediate result would be:

Corollary 23 *For every $C \leq 1$ and $C < S$, there exists an optimal canonical cut.*

Proof. Any standard cut would get at most one piece chosen, so we get the required result immediately by proposition 22.

The optimal cut would be either an AO cut if $C < 1$ or $\lceil 2S \rceil - 1$ equal pieces otherwise. ■

Corollary 24 *For every $1 < C \leq \frac{4}{3}$, there exists an optimal canonical cut.*

Proof. For $C < S \leq 2$, two equal pieces would get us the result by proposition 22.

For $2 < S$, $\lceil S \rceil$ equal pieces would do since:

$$\frac{2S}{\lceil S \rceil} > \frac{2S}{S+1} = 2 - \frac{2}{S+1} \geq 2 - \frac{2}{2+1} = \frac{4}{3} \geq C$$

and the claim follows by proposition 22 again.

The optimal cut would be a cut to $\lceil \frac{2S}{C} \rceil - 1$ equal pieces, since $1 < C$. ■

6 Algorithmic results

Recall the definition of ω from the introduction. In the following we show that for every S, C, both players can obtain the minmax value $\omega(S, C)$ in polynomial time, although given a cut M, making an optimal reply tailored especially for it and achieving $\omega(S, C, M)$ is NP-hard.

6.1 Efficient optimal cutter algorithm

We now prove corollary 4, which states that an optimal cutter algorithm taking time $O(CS^2)$ exists:

Proof. The optimal cut is obtainable by finding the optimal among all semi-canonical moves. Given a semi-canonical cut, finding the optimal choice of subset may be done in time linear in the number of pieces it contains, since for every choice of the non 1 pieces, finding how many 1's may still be chosen may be done in $O(1)$ time.

Since there are $O(S^2)$ possible semi canonical moves, and in each the number of possible 1's to be chosen is $O(C)$, finding the optimal cut would take $O(CS^2)$, as required. ■

6.2 NP Hardness of optimal reply generation

We show that finding a reply achieving $\omega(S, C, M)$ for every C, M is NP-hard, by a reduction to the uniform knapsack maximization problem.

Definition 25 *Given a finite set U of positive real numbers, and a size B, find a subset $U' \subseteq U$ maximizing $\sum_{u_i \in U'} u_i$ and satisfying $\sum_{u_i \in U'} u_i \leq B$.*

The decision version of this problem is a well known NP-hard problem[10].

We show that the maximization version reduces to optimal reply generation. The only difference between the two is the limit on the maximal piece size. A reduction from a uniform knapsack instance to the chooser's problem would be:

Given a knapsack instance U, B, pick $u = \max U$, and generate $M = \{\frac{v}{u} \mid v \in U\}$, $C = \frac{B}{u}$. Since the sum of any feasible reply to this cut corresponds to a feasible subset in the original instance and vice versa, and their values monotonically related as well, we get that an optimal reply to that cut correspond to an optimal solution for the knapsack instance.

6.3 Time efficient generation of replies assuring at least $-\omega(S, C)$

We now prove corollary 5, which states that an optimal chooser algorithm exists, achieving slackness at most $\omega(S, C)$ and taking time $O(|M|^2 \cdot C)$.

For $C \leq \frac{4}{3}$, the chooser may first standardize the cut, and find the optimal response in time $O(M^2)$, by finding the optimal among all choices of one or two pieces, since the sum of every three is greater than 1.5, and therefore unfeasible.

For $C > \frac{4}{3}$, pick k such that $0 < \frac{C}{k} = \epsilon < \frac{1}{3 \cdot |M|}$[1]. Observe that every $\ell_i \in M$ may be seen as $\ell_i = k_i \cdot \epsilon + \epsilon'_i$ with $0 \leq \epsilon'_i < \epsilon$. We refer to ϵ'_i as the fractional part of ℓ_i. Given a set $U \subseteq M$, we refer to $\sum_{i \in U} \epsilon_i$ as the fractional part of U. Every one of the pieces is replaced by $\ell'_i = k_i \cdot \epsilon$.

We find an optimal solution to that knapsack instance using dynamic programming after discarding the pieces' fractional parts. To do so, we construct a table A of size $|M| \times k$, where every cell $A[i, j]$, we consider sums of the subsets of $\ell'_1, ..., \ell'_i$ which sum to $j \cdot \epsilon$.

Filling the other columns may be done one by one, by checking whether $A[i-1, j - \frac{\ell'_i}{\epsilon}]$ contains a subset, and adding ℓ'_i to it, or picking a subset from $A[i-1, j]$ otherwise. If both cells are empty, put in an empty set.

This approach would have found an optimal subset for $M' = \{\ell'_1, ..., \ell'_m\}$. We change this algorithm to allow the extraction of a choice for the original problem as well. Instead of keeping a single subset at every cell, we keep two subsets achieving the minimal and maximal sums of fractional parts given the integral part $\epsilon \cdot j$. This does not change the base case, and the inductive updating is repeated twice - once for the subset of maximal fractional part, and again for the minimal one.

Since every sum in the matrix contains at most $|M|$ elements, the difference between the minimal and maximal fractional parts in every cell:

$$\epsilon \cdot |M| < \frac{1}{3 \cdot |M|} \cdot |M| = \frac{1}{3}$$

That is true in particular for the optimal subset's cell.

We claim that finding an optimal feasible subset among all pairs of subsets represented by the cells of the matrix would get the chooser at least $-\omega(S, C)$.

[1] Say, for $k = 3 \cdot |M|\, C + 1$.

To prove that, consider the optimal subset. It belongs in one of the matrix' cells. If the set with the maximal fractional part in this cell is feasible, it is no worse than optimal subset, and is therefore optimal itself. If it is not optimal, it must be unfeasible, so the subset with the minimal fractional parts in that cell must be strictly greater than $C - \frac{1}{3}$. The minimal subset's sum is lesser than the optimal subset's, and is therefore feasible, and achieves a slackness less than $\frac{1}{3} \leq \omega(S, C)$, as required.

The time required by that dynamic programming algorithm is $O(2 \cdot |M| \cdot k) = O(|M|^2 \cdot C)$, since updating every cell times $O(1)$ time, and there are $|M| \cdot k$ cells.

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