Characterizing the Number of m-ary Partitions Modulo m

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Abstract. Motivated by a recent conjecture of the second author related to the ternary partition function, we provide an elegant characterization of the values $b_m(mn)$ modulo m where $b_m(n)$ is the number of m-ary partitions of the integer n and $m \geq 2$ is a fixed integer.

1. INTRODUCTION. Congruences for partition functions have been studied extensively for the last century or so, beginning with the discoveries of Ramanujan [**9**]. In this note, we will focus our attention on congruence properties for the partition functions that enumerate restricted integer partitions known as m-ary partitions. These are partitions of an integer n wherein each part is a power of a fixed integer $m \geq 2$. Throughout this note, we will let $b_m(n)$ denote the number of m-ary partitions of n.

As an example, note that there are five 3-ary partitions of $n = 9$:

$$9, \quad 3+3+3, \quad 3+3+1+1+1,$$
$$3+1+1+1+1+1+1, \quad 1+1+1+1+1+1+1+1+1.$$

Thus, $b_3(9) = 5$.

In 1940, Mahler [**8**] found an asymptotic estimate of $b_m(n)$ as n tends to infinity. Mahler's estimate was improved significantly by de Bruijn [**5**] in 1948.

In the late 1960s, Churchhouse [**3, 4**] initiated the study of congruence properties of binary partitions (m-ary partitions with $m = 2$). By his own admission, he did so serendipitously. To quote Churchhouse [**4**], "It is however salutary to realise that the most interesting results were discovered because I made a mistake in a hand calculation!"

Within months, other mathematicians proved Churchhouse's conjectures and proved natural extensions of his results. These included Rødseth [**10**], who extended Churchhouse's results to include the functions $b_p(n)$ where p is any prime, as well as Andrews [**2**] and Gupta [**6, 7**], who proved that corresponding results also held for $b_m(n)$ where m could be any integer greater than 1. As part of an infinite family of results, these authors proved that, for any $m \geq 2$ and any nonnegative integer n, $b_m(m(mn-1)) \equiv 0 \pmod{m}$.

We now fast forward 40 years. In 2012, the second author conjectured the following absolutely remarkable result related to the ternary partition function $b_3(n)$.

- For all $n \geq 0$, $b_3(3n)$ is divisible by 3 if and only if at least one 2 appears as a coefficient in the base 3 representation of n.
- Moreover, $b_3(3n) \equiv (-1)^j \pmod{3}$ whenever no 2 appears in the base 3 representation of n and j is the number of 1s in the base 3 representation of n.

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This conjecture is remarkable for at least two reasons. First, it provides a complete characterization of $b_3(3n)$ modulo 3. Such **characterizations** in the world of integer partitions are rare. Secondly, the result depends on the base 3 representation of n and nothing else.

Just to "see" what the second author saw, let's quickly look at some data related to this conjecture.

n	Base 3 Representation of n	$b_3(3n)$	$b_3(3n) \pmod 3$
1	1×3^0	2	2
2	2×3^0	3	0
3	$0 \times 3^0 + 1 \times 3^1$	5	2
4	$1 \times 3^0 + 1 \times 3^1$	7	1
5	$2 \times 3^0 + 1 \times 3^1$	9	0
6	$0 \times 3^0 + 2 \times 3^1$	12	0
7	$1 \times 3^0 + 2 \times 3^1$	15	0
8	$2 \times 3^0 + 2 \times 3^1$	18	0
9	$0 \times 3^0 + 0 \times 3^1 + 1 \times 3^2$	23	2
10	$1 \times 3^0 + 0 \times 3^1 + 1 \times 3^2$	28	1
11	$2 \times 3^0 + 0 \times 3^1 + 1 \times 3^2$	33	0
12	$0 \times 3^0 + 1 \times 3^1 + 1 \times 3^2$	40	1
13	$1 \times 3^0 + 1 \times 3^1 + 1 \times 3^2$	47	2
14	$2 \times 3^0 + 1 \times 3^1 + 1 \times 3^2$	54	0
15	$0 \times 3^0 + 2 \times 3^1 + 1 \times 3^2$	63	0

In recent days, the authors succeeded in proving this conjecture. Thankfully, the proof was both elementary and elegant. After just a bit of additional consideration, we were able to alter the proof to provide a completely unexpected generalization. We describe this generalized result and provide its proof in the next section.

2. THE FULL RESULT.

Our main theorem, which includes the above conjecture in a very natural way, provides a complete characterization of $b_m(mn)$ modulo m.

Theorem 1. *If $m \geq 2$ is a fixed integer and*

$$n = a_0 + a_1 m + \cdots + a_j m^j$$

is the base m representation of n (so that $0 \leq a_i \leq m-1$ for each i), then

$$b_m(mn) \equiv \prod_{i=0}^{j}(a_i + 1) \pmod m.$$

Notice that the conjecture mentioned above is exactly the $m = 3$ case of Theorem 1.

In order to prove Theorem 1, we need a few elementary tools. We describe these tools here.

First, it is important to note that the generating function for $b_m(n)$ is given by

$$B_m(q) := \prod_{j=0}^{\infty} \frac{1}{1 - q^{m^j}}. \tag{1}$$

Note that $B_m(q)$ satisfies the functional equation

$$(1-q)B_m(q) = B_m(q^m).$$

From here, it is straightforward to prove that

$$b_m(mn) = b_m(mn+i)$$

for all $1 \leq i \leq m-1$. Thus, we see that Theorem 1 actually provides a characterization of $b_m(N) \pmod{m}$ for **all** N, not just for those N that are multiples of m.

With this information in hand, we now prove a small number of lemmas that we will use in our proof of Theorem 1.

Lemma 2. *If* $|x| < 1$, *then*

$$\frac{1-x^m}{(1-x)^2} \equiv \sum_{k=1}^{m} kx^{k-1} \pmod{m}.$$

Proof. This elementary congruence can be proven rather quickly using well-known mathematical tools. We begin with the geometric series identity

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Differentiating both sides yields

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}.$$

We then multiply both sides by $1-x^m$ and simplify as follows:

$$\begin{aligned}\frac{1-x^m}{(1-x)^2} &= \sum_{k=1}^{\infty} kx^{k-1} - x^m \sum_{k=1}^{\infty} kx^{k-1} \\ &= \sum_{k=1}^{\infty} kx^{k-1} - \sum_{k=m+1}^{\infty} (k-m)x^{k-1} \\ &= \sum_{k=1}^{m} kx^{k-1} + \sum_{k=m+1}^{\infty} mx^{k-1} \\ &\equiv \sum_{k=1}^{m} kx^{k-1} \pmod{m}.\end{aligned}$$

■

Lemma 3. *If* ζ *is the* m^{th} *root of unity given by* $\zeta = e^{2\pi i/m}$, *then*

$$\sum_{k=0}^{m-1} \frac{1}{1-\zeta^k q} = m\left(\frac{1}{1-q^m}\right).$$

Proof. Using geometric series and elementary series manipulations, we have

$$\begin{aligned}
\sum_{k=0}^{m-1} \frac{1}{1-\zeta^k q} &= \sum_{k=0}^{m-1}\sum_{r=0}^{\infty} \zeta^{kr} q^r \\
&= \sum_{k=0}^{m-1}\left(\sum_{r\,|\,m} \zeta^{kr} q^r + \sum_{r\,\nmid\,m} \zeta^{kr} q^r\right) \\
&= \sum_{k=0}^{m-1}\sum_{j=0}^{\infty} \zeta^{k(jm)} q^{jm} + \sum_{k=0}^{m-1}\sum_{r\,\nmid\,m} \zeta^{kr} q^r \\
&= \sum_{k=0}^{m-1} \frac{1}{1-q^m} \quad \text{using facts about roots of unity} \\
&= m\left(\frac{1}{1-q^m}\right).
\end{aligned}$$

■

Lemma 4. *If* $T_m(q) := \sum_{n\geq 0} b_m(mn)q^n$, *then*

$$T_m(q) = \frac{1}{1-q} B_m(q).$$

Proof. As in Lemma 3, let $\zeta = e^{2\pi i/m}$. Note that

$$\begin{aligned}
T_m(q^m) &= \sum_{n\geq 0} b_m(mn) q^{mn} \\
&= \frac{1}{m}\left(B_m(q) + B_m(\zeta q) + \cdots + B_m(\zeta^{m-1} q)\right) \\
&= \left(\prod_{j=1}^{\infty} \frac{1}{1-q^{m^j}}\right) \times \frac{1}{m}\sum_{k=0}^{m-1} \frac{1}{1-\zeta^k q} \\
&= \frac{1}{1-q^m}\prod_{j=1}^{\infty}\frac{1}{1-q^{m^j}}
\end{aligned}$$

thanks to Lemma 3. Lemma 4 then follows by replacing q^m by q. ■

We now combine these elementary facts from the lemmas above to prove one last lemma. This lemma will, in essence, allow us to "move" from considering $T_m(q)$ modulo m to a new function modulo m that makes the result of Theorem 1 transparent.

Lemma 5. *If* $U_m(q) = \prod_{j=0}^{\infty}\left(1 + 2q^{m^j} + 3q^{2m^j} + \cdots + mq^{(m-1)m^j}\right)$, *then*

$$T_m(q) \equiv U_m(q) \pmod{m}.$$

Proof. Lemma 5 will follow if we can prove that $\frac{1}{T_m(q)} \cdot U_m(q) \equiv 1 \pmod{m}$, and this will be our means of attack. Thankfully, this follows from a novel generating function manipulation that we demonstrate here. Using (1) and Lemma 4, we know that

$$\begin{aligned}
&\frac{1}{T_m(q)} \cdot U_m(q) \\
&= (1-q)^2 \prod_{j=1}^{\infty}(1-q^{m^j}) \prod_{j=0}^{\infty}\left(1+2q^{m^j}+3q^{2m^j}+\cdots+mq^{(m-1)m^j}\right) \\
&\equiv (1-q)^2 \prod_{j=1}^{\infty}(1-q^{m^j}) \prod_{j=0}^{\infty}\frac{1-q^{m^{j+1}}}{(1-q^{m^j})^2} \pmod{m} \quad \text{thanks to Lemma 2} \\
&= \frac{\prod_{j=0}^{\infty} 1-q^{m^{j+1}}}{\prod_{j=1}^{\infty} 1-q^{m^j}} \\
&= 1.
\end{aligned}$$

■

We can now utilize all of the above results to prove Theorem 1.

Proof. First, we remember that

$$\sum_{n\geq 0} b_m(mn)q^n = T_m(q) \equiv U_m(q) \pmod{m}.$$

So we simply need to consider $U_m(q)$ modulo m to obtain our proof. Note that

$$U_m(q) = \prod_{j=0}^{\infty}\left(1+2q^{m^j}+3q^{2m^j}+\cdots+mq^{(m-1)m^j}\right).$$

If we expand this product as a power series in q, then each term of the form q^n can occur at most once (because the terms $q^{i\cdot m^j}$ are serving as the building blocks for the **unique** base m representation of m). Thus, if

$$n = a_0 + a_1 m + \cdots + a_j m^j,$$

then the coefficient of q^n in this expansion is

$$\prod_{i=0}^{j}(a_i+1) \pmod{m}.$$

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