SIAM J. DISCRETE MATH.
Vol. 24, No. 2, pp. 570–588

COMPLEMENTARY ITERATED FLOOR WORDS AND THE FLORA GAME*

AVIEZRI S. FRAENKEL†

Abstract. Let $\varphi = (1+\sqrt{5})/2$ denote the golden section. We investigate relationships between unbounded iterations of the floor function applied to various combinations of φ and φ^2. We use them to formulate an algebraic polynomial-time winning strategy for a new four-pile take-away game *Flora*, which is motivated by partitioning the set of games into subsets *CompGames* and *PrimGames*. We further formulate recursive, arithmetic, and word-mapping winning strategies for it. The arithmetic one is based on the Fibonacci numeration system. We further show how to generate the floor words induced by the iterations using word-mappings and characterize them using the Fibonacci numeration system. We also exhibit an infinite array of such sequences.

Key words. floor function, integer part function, combinatorics of words, combinatorial game theory, Fibonacci numeration system

AMS subject classifications. 11B75, 11B39, 91A46, 05A05

DOI. 10.1137/090758994

1. Introduction.

As is customary, we denote by $\lfloor x \rfloor$ the integer part of x, commonly known as the *floor function*. It is the largest integer not exceeding x. Let $\varphi = (1+\sqrt{5})/2$ denote the golden section.

Two topics motivate this work. On the one hand, we wish to study what happens when we keep iterating the floor function with either φ or φ^2 in various ways. Are any interesting relationships between them discernible even after an unbounded number of iterations, or does total chaos take over?

On the other hand, we aim at shedding more light on the class of impartial take-away games. This class appears to be partitioned into two disjoint subclasses: those that are easy to generalize to more than one or two piles, and those for which this seems to be very hard (section 3). A well-known representative of the former is Nim [2], and of the latter there is Wythoff's game [8]. Some progress in generalizing Wythoff to multiple piles was recently made; see [14], [31], [30]. Three-pile games that are *extensions* rather than generalizations of Wythoff were also given recently [12], [5], [6].

Here we consider an extension of Wythoff to four piles. The efforts in defining a "right" extension and particularly in proving the validity of the winning strategy are considerably greater than those for three-pile extensions. We present four formulations of the winning strategy.

In section 2 we investigate unbounded iterations of the floor function and formulate a wealth of relationships and identities. In section 3 we define the subclasses *CompGames* and *PrimGames*, which motivate the definition of the four-pile game dubbed Flora. In section 3.1 we formulate an algebraic winning strategy for the game, based on the results derived in section 2, and prove that its complexity is

*Received by the editors May 13, 2009; accepted for publication (in revised form) March 17, 2010; published electronically May 26, 2010.
http://www.siam.org/journals/sidma/24-2/75899.html

†Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel (fraenkel@wisdom.weizmann.ac.il, http://www.wisdom.weizmann.ac.il/~fraenkel).

polynomial-time. In section 3.2 we formulate a recursive winning strategy which appears very simple, but its polynomiality is implied only by a recent result [17]. We end in section 3.3 with a polynomial-time arithmetic winning strategy, based on the Fibonacci numeration system. In section 4 we indicate how to generate sequences induced by iterations of the floor function using word-mappings. We apply it to one of the sequences in section 4.1. In section 4.2 we present our fourth winning strategy for the Flora game, which is also polynomial-time; it stems from a word-mapping viewpoint. In section 4.3 we use results from sections 3.3 and 2 and make minor use of the language of section 4 to characterize the representations of general cases of the special sequences playing a major role in the algebraic formulation of the winning strategy of Flora. In section 5 we show, by means of an example, how to produce infinite complementary arrays using sequences induced by the iterations. In the conclusion we wrap up and indicate natural further directions of research.

Let $n \in \mathbb{Z}_{\geq 1}$. Let $a(n) = \lfloor n\varphi \rfloor$, $b(n) = \lfloor n\varphi^2 \rfloor$. It is well known that the sequences a and b split the positive integers [8, section 3]. An example of an iterated identity is $a(b(n)) = a(n) + b(n)$. It can be abbreviated as $ab = a + b$, where the product means iteration (composition) and the suppressed variable n is assumed to range over all positive integers, unless otherwise specified. We also write a^2 for aa, ab^3a^2 for $abbbaa$, etc. An example of four iterated complementary sequences is $a^2 = b - 1$, $ab = a + b$, $ba = a + b - 1$, $b^2 = a + 2b$, since every positive integer is in precisely one of these four sequences. We use the notation $w = w_1w_2 \dots w_k$ to denote the word w as well as the (iterated) sequence $w(n)$. If the sequence is intended, we sometimes write $w(n)$ rather than only w. Notice that the product, though not commutative, is associative. A general reference on combinatorics of words is [25].

Let $h = b$, $u = a$, and, for $k \geq 2$, $h^k = a^{k-1}b$, $u^k = ba^{k-1}$. Let $\Delta a^k(n) = a^k(n+1) - a^k(n)$, $\Delta h^k(n) = h^k(n+1) - h^k(n)$. For technical reasons we put

$$a^0(n) = n, \qquad h^0(n) = a(n).$$

Further, let $F_{-1} = 1$, $F_0 = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n \geq 1)$ be the Fibonacci sequence.

Notation 1. For $k \geq 0$, $s \in \mathbb{Z}$, let $G_k = \cup_{n=1}^{\infty} a^k(n)$, $H_k = \cup_{n=1}^{\infty} h^k(n)$, $U_k = \cup_{n=1}^{\infty} u^k(n)$, $V_2 = \cup_{n=1}^{\infty} b^2(n)$, and $G_k - s = \cup_{n=1}^{\infty} (a^k(n) - s)$ (subtracting s from every element of G_k).

In particular, $G_0 = \mathbb{Z}_{\geq 1}$, and $H_0 = G_1 = U_1$.

Note. In our applications, $s \in \{-2, -1, 0, 2\}$, most often 0.

2. Identities. After multiplying by the irrational φ and then throwing out the fractional part for an unbounded number of times, one might expect complete chaos among relationships involving a^k, h^k, u^k, and b^k. It is thus surprising that there are many striking identities and relationships among them. Our purpose in this section is to prove a selection of them.

THEOREM 1. *For every $k \in \mathbb{Z}_{\geq 1}$ and every $n \in \mathbb{Z}_{\geq 1}$ the following hold:*

(a) *The $k+1$ sequences $G_k, H_k, H_{k-1}, \dots, H_2, H_1$ partition $\mathbb{Z}_{\geq 1}$.*

(b) $u^{k+1} = a^k + a^{k+1} = a^{k+2} + 1$.

(c) $h^k = a^{k+1} + F_{k-1}$.

(d) $u^{k+1} = h^{k+1} - F_k + 1 = a^{k+2} + 1$.

(e) (e1) $h^{k+1} - h^k = a^k + F_{k-2} - 1$.

(e2) $h^{k+1} - a^{k+1} = a^k + F_k - 1$.

(e3) $au^{k+1} = u^{k+2} + 1$.

(f) (f1) *Let*

$$S_1 = \{n \in \mathbb{Z}_{\geq 1} : \Delta a(n) = F_0\}, \quad S_2 = \{n \in \mathbb{Z}_{\geq 1} : \Delta a(n) = F_1\}.$$

Then S_1 and S_2 split $\mathbb{Z}_{\geq 1}$; and each of S_1 and S_2 is an infinite set.

(f2) *For all $k \in \mathbb{Z}_{\geq 1}$ the following holds: $\Delta a^k(n) = F_{k-1}$ for all $n \in S_1$ and $\Delta a^k(n) = F_k$ for all $n \in S_2$.*

(f3) (i) *If $\Delta a^k(n+1) = F_{k-1}$ for some $n \in \mathbb{Z}_{\geq 1}$, then $\Delta a^k(n) = \Delta a^k(n+2) = F_k$.*

(ii) *If $\Delta a^k(n+1) = \Delta a^k(n+2) = F_k$ for some $n \in \mathbb{Z}_{\geq 1}$, then $\Delta a^k(n) = \Delta a^k(n+3) = F_{k-1}$.*

(f4) *$\Delta a^k(n) = \Delta h^{k-1}(n) \in \{F_{k-1}, F_k\}$, and each of F_{k-1} and F_k is assumed for infinitely many n.*

(f5) (i) *$\Delta a^k(0) = 1$.*

(ii) *$\Delta h^k(0) = F_{k-1} + 1$.*

(g) *$a^k(h(n)) = h^{k+1}(n)$ (due to Lior Goldberg).*

(h) (h1) *$(G_2 + 2) \subset G_1$.*

(h2) *$G_2 \cup (G_2 + 2) = G_1$.*

(h3) *$U_2 \subset (G_1 - 2) \subset G_1 \cup U_2$.*

(h4) *$(V_2 - 1) \subset G_2$.*

We begin by recalling some elementary properties of the floor function. Let x, y be any real numbers. Denote by $\{x\}$ the fractional part of x, so $x = \lfloor x \rfloor + \{x\}$. Then we have the following:

- $0 \leq \{x\} < 1$, $x - 1 < \lfloor x \rfloor \leq x$. Replacing x by $-x$, $-x - 1 < \lfloor -x \rfloor \leq -x$, hence $-1 \leq \lfloor x \rfloor + \lfloor -x \rfloor \leq 0$ and $\lfloor x \rfloor + \lfloor -x \rfloor = 0$ if and only if x is an integer. For example, $\lfloor \varphi \rfloor = 1$, $\lfloor -\varphi \rfloor = -2$, $\lfloor \varphi \rfloor + \lfloor -\varphi \rfloor = -1$; and $\varphi^2 = \varphi + 1$ implies $\{\varphi\} = \varphi^{-1} = \varphi - 1$.
- $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$. This follows immediately from $\lfloor x + y \rfloor = \lfloor \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor$.

Lemma 1. (i) *Let $s \in \mathbb{Z}$. Each of the sequences $G_k + s$, $H_k + s$, $U_k + s$, and $V_2 + s$ is strictly increasing for every $k \geq 1$.*

(ii) *The sequences G_k, H_k split G_{k-1} for every $k \geq 1$.*

Proof. (i) The proof follows from the fact that $\varphi^2 = \varphi + 1 > \varphi > 1$.

(ii) Since $\varphi^{-1} + \varphi^{-2} = 1$, the sequences G_1 and H_1 split $\mathbb{Z}_{\geq 1} = G_0$ (see, e.g., [8, section 3]), so the result holds for $k = 1$. For any $k \geq 1$, assume that G_k, H_k split G_{k-1}. Then

$$\begin{aligned} G_{k+1} \cup H_{k+1} &= \bigcup_n (a^{k+1}(n) \cup h^{k+1}(n)) = \bigcup_n (aa^k(n) \cup ah^k(n)) \\ &= \bigcup_n aa^{k-1}(n) \text{ (by induction)} = \bigcup_n a^k(n) = G_k. \qquad \square \end{aligned}$$

Proof of Theorem 1(a). We noted that G_1 and H_1 split $\mathbb{Z}_{\geq 1}$. Suppose that $G_k, H_k, H_{k-1}, \ldots, H_2, H_1$ partition $\mathbb{Z}_{\geq 1}$. Then $G_{k+1}, H_{k+1}, H_k, \ldots, H_2, H_1$ partition $\mathbb{Z}_{\geq 1}$, since G_{k+1}, H_{k+1} split G_k by Lemma 1. $\square$

Note. It follows from Lemma 1(ii) (or from Theorem 1(a)) that for any positive integers m, n, $a(m) \neq b(n)$. This property will be referred to in what follows as *disjointness.*

Proof of Theorem 1(b). By definition,

$$a^{k+2} = a^2 a^k = \lfloor \varphi \lfloor \varphi a^k \rfloor \rfloor \leq \lfloor \varphi^2 a^k \rfloor = ba^k = u^{k+1}.$$

By disjointness, $u^{k+1} = ba^k \geq a^{k+2} + 1$. Conversely, multiply $\varphi a^k < a^{k+1} + 1$ by φ to get $\varphi^2 a^k < \varphi(a^{k+1} + 1)$; hence $ba^k \leq \lfloor \varphi(a^{k+1} + 1) \rfloor$. By disjointness this inequality is strict, so

$$u^{k+1} \le \lfloor \varphi(a^{k+1}+1) \rfloor - 1 \le a^{k+2} + \lfloor \varphi \rfloor = a^{k+2} + 1.$$

On the other hand, $u^{k+1} = \lfloor (\varphi+1)a^k \rfloor = a^k + a^{k+1}$. ∎

LEMMA 2. *For every* $k \in \mathbb{Z}_{\ge 1}$,

(i) $\lfloor \varphi F_{k-1} \rfloor \in \{F_k - 1, F_k\}$;

(ii) $\lfloor \varphi^2 F_{2k-2} \rfloor = F_{2k} - 1$, $\lfloor F_{2k-1}\varphi^2 \rfloor = F_{2k+1}$.

Proof. (i) The ratios F_k/F_{k-1} are the convergents of the simple continued fraction expansion of $\varphi = [1, 1, 1, \ldots]$. Therefore $|\varphi F_{k-1} - F_k| < F_{k-1}^{-1}$ (see, e.g., [19, Chap. 10]), so $\varphi F_{k-1} - F_k = \delta$, where $-F_{k-1}^{-1} < \delta < F_{k-1}^{-1}$. Thus $\lfloor \varphi F_{k-1} \rfloor = F_k + \lfloor \delta \rfloor$. The result follows if $|\delta| < 1$, which is the case for all $k \ge 1$, since $F_{k-1} \ge F_0 = 1$.

(ii) The ratios F_{k+2}/F_k are the convergents of the simple continued fraction expansion of $\varphi^2 = [2, 1, 1, 1, \ldots]$. In fact, $F_{2k+1}/F_{2k-1} < \varphi^2 < F_{2k}/F_{2k-2}$. This follows easily from [19, Chap. 10]. Then $\varphi^2 F_{2k-1} - F_{2k+1} = \delta$, where $0 < \delta < F_{2k-1}^{-1}$; hence $\lfloor \varphi^2 F_{2k-1} \rfloor = F_{2k+1}$, since $0 < \delta < 1$ for all $k \ge 1$. Similarly, $\varphi^2 F_{2k-2} - F_{2k} = \delta$, where $-F_{2k-2}^{-1} < \delta < 0$. Thus $\lfloor \varphi^2 F_{2k-2} \rfloor = F_{2k} - 1$, since $-1 < \delta < 0$ for all $k \ge 1$. ∎

LEMMA 3. $h^2 - a^3 = 2$.

Proof. In Lemma 9 of [12] we proved the special case $k = 1$ of Theorem 1(d), namely, $h^2 = u^2 + 1$. Thus $h^2 - a^3 = u^2 - a^3 + 1$. Clearly $a^3 = \lfloor \varphi \lfloor \varphi \lfloor n\varphi \rfloor \rfloor \rfloor \le \lfloor \varphi^2 a \rfloor = u^2$. But this inequality is strict by disjointness. Thus $h^2 - a^3 \ge 2$.

Conversely, multiply the inequality $\varphi a < a^2 + 1$ by φ to get $\varphi^2 a < \varphi(a^2 + 1)$. Therefore $\lfloor \varphi^2 a \rfloor \le \lfloor \varphi(a^2 + 1) \rfloor$. Again by disjointness, this inequality is strict; i.e., $u^2 \le a^3 + 1$. As we saw, Lemma 9 of [12] asserts that $u^2 = h^2 - 1$. Therefore, $h^2 - a^3 \le 2$. ∎

Notation 2. For any positive integer N, denote by $R(N)$ the representation of N in the Fibonacci numeration system. It has the form $R(N) = (d_m, \ldots, d_0)$, where $N = \sum_{i=0}^{m} d_i F_i$, $d_i \in \{0, 1\}$, $d_i = 1 \implies d_{i-1} = 0$, $i \ge 1$ [9]. The *position* of a representation is the subscript i of d_i. Thus, d_0 is in position 0, d_1 in position 1, etc.

Proof of Theorem 1(c). For $k = 1$, this is Lemma 5 of [12, section 5]. For $k = 2$, it is Lemma 3 above. Suppose that $h^k = a^{k+1} + F_{k-1}$ for some arbitrary $k \ge 2$. Multiply by φ and take the floor of both sides. This gives, by Lemma 2, $h^{k+1} = \lfloor \varphi(a^{k+1} + F_{k-1}) \rfloor \le a^{k+2} + \lfloor \varphi F_{k-1} \rfloor + 1 \le a^{k+2} + F_k + 1$. Now [12, section 6] implies that $R(a^2)$ ends in 01. By Lemma 1, the same holds for a^k and h^k for every $k \ge 3$ (but it does not hold for h^2). Since $R(F_k)$ ends in 00 for $k \ge 2$, $R(a^{k+2} + F_k)$ also ends in 01 for $k \ge 2$, and so does h^{k+1} for $k \ge 2$. But $R(a^{k+2} + F_k + 1)$ ends in 10. Hence $h^{k+1} = a^{k+2} + F_k$. ∎

Proof of Theorem 1(d). From (b) and (c), $u^{k+1} = a^{k+2} + 1 = h^{k+1} - F_k + 1$. The second equality follows once more from (c). ∎

We note that inspection shows that Theorem 1(d) does not hold for $k < 1$.

Proof of Theorem 1(e1). Subtracting (c) from (c) with k replaced by $k + 1$ gives $h^{k+1} - h^k = a^{k+2} - a^{k+1} + F_{k-2}$. Substituting the value of a^{k+2} from (b) yields the desired result. ∎

(e2) The proof follows from (e1), where we replace h^k by its value from (c). ∎

(e3) We have

$$\begin{aligned} au^{k+1} &= \lfloor \varphi \lfloor \varphi^2 a^k \rfloor \rfloor = \lfloor \varphi(a^k + a^{k+1}) \rfloor \\ &\le a^{k+1} + a^{k+2} + 1 = u^{k+2} + 1, \end{aligned}$$

where the last equality follows from (b). On the other hand,

$$\lfloor \varphi(a^k + a^{k+1}) \rfloor \ge a^{k+1} + a^{k+2} = u^{k+2}.$$

Hence by disjointness, $\lfloor \varphi u^{k+1} \rfloor = u^{k+2} + 1$. □

We recall the following special case of Lemma 2 of [8].

LEMMA I. *For integers $i > j \geq 0$ and integer $N_{i+1} \in \mathbb{Z} \geq 1$, let $N_{i+1} = F_i + F_{i-2} + \cdots + F_j$, where $j = 0$ if i is even, and $j = 1$ if i is odd. Then $N_{i+1} = F_{i+1} - 1$.*

This is the analogue in the Fibonacci numeration system of the decimal $99 \ldots 9$.

Proof of Theorem 1(f1). For any $n \in \mathbb{Z}_{\geq 1}$, clearly $\varphi - 1 < \Delta a(n) < \varphi + 1$, so $\Delta a(n) \in \{1, 2\} = \{F_0, F_1\}$. This shows already that S_1, S_2 split $\mathbb{Z}_{\geq 1}$. Moreover, if $\Delta a(n) = 1$ for all large n, then, since $h(n)$ is increasing, we would have $a(n) \cap h(n) \neq \emptyset$ for infinitely many $n \in \mathbb{Z}_{\geq 1}$, contradicting the complementarity of the two sequences. If $\Delta a(n) = 2$ for all large n, then also $\Delta h(n) = 2$ for all large n by complementarity. But a direct computation shows that $\Delta a(n) = 2 \implies \Delta h(n) = 3$, another contradiction. Thus each of S_1 and S_2 is infinite as claimed. □

Proof of Theorem 1(f2). We proceed by induction on k. Suppose that for some $k \geq 1$, $\Delta a^k(n) = F_{k-1}$ for all $n \in S_1$, and $\Delta a^k(n) = F_k$ for all $n \in S_2$. This holds for $k = 1$ by (f1). For now let us assume that $n \in S_1$. Then

$$\Delta a^{k+1}(n) = \lfloor \varphi a^k(n+1) \rfloor - \lfloor \varphi a^k(n) \rfloor < \varphi a^k(n+1) - \varphi a^k(n) + 1 = \varphi F_{k-1} + 1$$

by the induction hypothesis. Also,

$$\Delta a^{k+1}(n) > \varphi a^k(n+1) - \varphi a^k(n) - 1 = \varphi F_{k-1} - 1.$$

So $\lfloor \varphi F_{k-1} \rfloor \leq \Delta a^{k+1}(n) \leq \lfloor \varphi F_{k-1} \rfloor + 1$. Then Lemma 2 implies that $\Delta a^k(n) \in \{F_k - 1, F_k, F_k + 1\}$.

In the proof of Theorem 1(c), it was mentioned that $R(a^2(n+2))$ ends in 01. The same thus holds for $R(a^{k+1}(n+1))$ and $R(a^{k+1}(n))$ for all $k \geq 1$, since G_{k+1} is a subsequence of G_2 for all $k \geq 1$. Therefore $R(\Delta a^{k+1}(n))$ ends in 00, the same as $R(F_k)$. But $R(F_k + 1)$ ends in 01, and Lemma I implies that $R(F_k - 1)$ ends in 10, or in 01, depending on whether k is even or odd. Hence $\Delta a^{k+1}(n) = F_k$ for all $n \in S_1$. The same proof shows that $\Delta a^{k+1}(n) = F_{k+1}$ for all $n \in S_2$. □

Proof of Theorem 1(f3). This follows easily for $k = 1$ by considering the size of φ. For all $k \geq 1$ it follows from Theorem 1(f2). □

Proof of Theorem 1(f4). This follows directly from (f1) and (f2). □

Note. Part of the proof of (f4) follows directly from (d):

$$h^{k+1}(n+1) - a^{k+2}(n+1) = h^{k+1}(n) - a^{k+2}(n) = F_k.$$

Hence $\Delta h^{k+1}(n) = \Delta a^{k+2}(n)$. But this establishes the equality part of Theorem 1(f4) only for $k \geq 3$ and does not prove the membership part.

Proof of Theorem 1(f5). (i) The proof follows by induction on k.

(ii) By definition, $\Delta h^k(0) = h^k(1) - h^k(0) = h^k(1)$. The result for $h^k(1)$ follows directly from (i) and Theorem 1(c). □

Proof of Theorem 1(g). For $k = 1$, $h^2(n) = \lfloor \varphi h(n) \rfloor = a(h(n))$. If the assertion holds for any $k \geq 1$, then $h^{k+2}(n) = \lfloor \varphi h^{k+1}(n) \rfloor = \lfloor \varphi a^k(h(n)) \rfloor = a^{k+1}(h(n))$. □

Proof of Theorem 1(h1). Clearly $G_2 \subset G_1$, so for every $n \in \mathbb{Z}_{\geq 1}$, $a^2(n) = a(m)$ for some $m \in \mathbb{Z}_{\geq 1}$. By (c), $a^2(n) + 1 = h(n) \notin G_1$. But then $a^2(n) + 2 = a(m+1) \in G_1$ by (f4) for $k = 1$. □

The following is a special case of Property 1 of [8, section 5].

LEMMA II. *The set of numbers $\{R(N) : N \in G_1\}$ ends in an even (possibly 0) number of 0's; hence the complementary set of numbers $\{R(N) : N \in H_1\}$ ends in an odd number of 0's.*

Proof of Theorem 1(h2). By (h1), $G_2 \cup (G_2 + 2) \subseteq G_1$. Choose any $a(n) \in G_1$. If $a(n) \in (G_2 + 2)$, we are done. So suppose that $a(n) = a^2(m) + 2$ for no $m \in \mathbb{Z}_{\geq 1}$. By (c), $a^2(m) + 1 = h(m)$ for all $m \in \mathbb{Z}_{\geq 1}$, so by disjointness, $a(n) = a^2(m) + 1$ for no $m \in \mathbb{Z}_{\geq 1}$. But then $a(n) = g^2(m)$ for some $m \in \mathbb{Z}_{\geq 1}$ by (f4) for $k = 1$, so $a(n) \in G_2$. □

Proof of Theorem 1(h3). The following is immediately implied by Theorem 1(f3): (a) if $a(n) - 1 \notin G_1$, then $a(n) - 2 \in G_1$, and, conversely, (b) if $a(n) - 1 \in G_1$, then $a(n) - 2 \notin G_1$. Consider case (b). Lemma II then implies that $R(a(n) - 1)$ ends in an even *positive* number of 0's, and $R(a(n))$ ends in 01. By Lemma I, $R(a(n) - 2)$ then ends in 10. We now show that $R(\lfloor \varphi^2 a(n) \rfloor)$ ends in 10 for all $n \in \mathbb{Z}_{\geq 1}$.

Now $R(a(n))$ ends in F_{2k-2} for some $k \in \mathbb{Z}_{\geq 1}$. By Lemma 2(ii), $\lfloor \varphi^2 F_{2k-2} \rfloor = F_{2k} - 1$, and $R(F_{2k} - 1)$ ends in 10 by Lemma I, the same as $R(a(n) - 2)$ for case (b). This proves that $R(\lfloor \varphi^2 a(n) \rfloor)$ ends in 10 for all $n \in \mathbb{Z}_{\geq 1}$ and proves the right-hand side of (h3). On the other hand, let $N \in U_2$. Then $R(N)$ ends in 10, and so $N + 1$ and $N + 2$ are both in G_1. Thus $N \in G_1 - 2$, proving the left-hand side of (h3). □

Proof of Theorem 1(h4). In the proof of Theorem 1(h3) we showed that $R(\lfloor \varphi^2 a(n) \rfloor)$ ends in 10 for all $n \in \mathbb{Z}_{\geq 1}$. Since $R(h(n))$ ends in an odd number of 1's for all $n \in \mathbb{Z}_{\geq 1}$ by Lemma II, $R(v^2(n))$ ends in an odd number $N \geq 3$ of 1's. Then Lemma I implies that $R(v^2(n) - 1)$ ends in 01. Theorem 3 of [12] states that $R(G_2)$ is the set of all numbers whose representation ends in 01, so $(V_2 - 1) \subset G_2$. □

Remark. Consider the word $w = \ell_1 \ell_2 \ldots \ell_k$ of length k over the binary alphabet $\{a, b\}$. The number m of occurrences of the letter b is the *weight* of w. We also put $F_{-2} = 0$. Recently, Kimberling [24] proved the following nice and elegant result.

Theorem I. *For $k \geq 2$, let $w = \ell_1 \ell_2 \ldots \ell_k$ of length k be any word over $\{a, b\}$ of length k and weight m. Then $w = F_{k+m-4}a + F_{k+m-3}b - c$, where $c = F_{k+m-1} - w(1) \geq 0$ is independent of n.*

Notice that in the theorem—where $w(1)$ is w evaluated at $n = 1$—only the weight m appears, not the locations within w where the b's appear. Their locations, however, obviously influence the behavior of w. This influence is hidden in the "constant" $c = c_{k,m,w(1)}$.

We could have used Theorem I to prove most of the results of Theorem 1 simply by expressing both sides of an identity as in Theorem I and verifying that they are identical. This verification, however, seems less satisfactory than the above proofs, which shed some light on the nature of the identities. In a recent book review it says, "but it is fair to say that while it is a proof, it is not an explanation" [7, p. 660]. Hardy [18], writing about seven proofs of the Rogers–Ramanujan identities, put it this way: "None of these proofs can be called 'simple' and 'straightforward,' since the simplest are essentially verifications." I got the Hardy reference from opinion 90 on the webpage of my esteemed opinionated friend Doron Zeilberger (http://www.math.rutgers.edu/~zeilberg/OPINIONS.html). Moreover, the computation of c is not, generally, so easy, as acknowledged by Kimberling. For example, we can show that for $w = h^k$ $(m = 1)$ we get $w(1) = h^k(1) = F_{k-1} + 1$, so $c_k = F_{k-2} - 1$. The proof depends on Lemma 2 and the Fibonacci numeration system.

3. An application: The Flora game.

Let G be a take-away game on m piles. A *generalization* of G is any game G' on $> m$ piles such that when G' is reduced to m piles, G' becomes identical to G. An *extension* of G is defined similarly, except that when G' is reduced to m piles, it is not identical to G.

The class of impartial take-away games appears to be partitioned into two disjoint subclasses:

- *CompGames* (composite games) and
- *PrimGames* (prime games).[1]

Informally, CompGames are games that are easy to generalize to more than one or two piles; PrimGames are those for which this seems to be very hard. A well-known representative of the former is Nim, and of the latter, Wythoff's game. Some progress in generalizing Wythoff to multiple piles was recently made. Two three-pile games that are extensions rather than generalizations of Wythoff were also given recently. It appears that, largely, a game belongs to class CompGames if it decomposes into a *disjunctive* sum of subgames, such as Nim, which is the Nim-sum of its pile sizes, and it belongs to class PrimGames if it is not decomposable. Hence the names CompGames (*composite* games) and PrimGames (not decomposable—*prime*). Whereas for the former there are theories both for the impartial as well as for the partisan case, there is no general theory for the latter yet, and we believe that these "lone wolf" games should be investigated more seriously.

Here we study an extension of Wythoff to *four* piles, which appears to be a PrimGame. The efforts in defining a "right" extension, and particularly in proving the validity of the winning strategy for this apparent PrimGame, are considerably greater than those for *three*-pile extensions. We present four winning strategies: algebraic, recursive, arithmetic, and word-mapping. The recursive is the easiest to describe, though it seems to be the hardest computationally. Actually it is also polynomial-time [17]. The algebraic depends on iterations of the floor function, the arithmetic on the Fibonacci numeration system, and the word-mapping on a morphism-like mapping. All are polynomial-time winning strategies.

The Flora game is a two-player game played on four piles of tokens. We denote positions of Flora by (a_1, a_2, a_3, a_4) with $0 \le a_1 \le a_2 \le a_3 \le a_4$. It goes without saying that every pile must contain a nonnegative number of tokens at all times.

The end position is $T_0 := (0, 0, 0, 0)$. The first player unable to move (because the present position is T_0) loses; the opponent wins.

There are three rules of move:

I. Arbitrary positive numbers of tokens from up to three piles may be removed.

II. From a nonzero position one can move to T_0 if any of the following three conditions holds: (i) Two piles have the same size (possibly empty).

(ii) $a_3 - a_2 = 1$.

(iii) $a_1 = h(n)$ and $a_2 < h^2(n) - 2$ for some $n \in \mathbb{Z}_{\ge 1}$.

III. If $0 < a_1 < a_2 < a_3 < a_4$, one can remove $p > 0$ from a_3, $q > 0$ from a_4, and arbitrary nonnegative integers from a_1 and a_2, subject to

(i) $q = p$ if $a_4 - a_3 \notin V_2$, except for the proviso that if $a_3 - p$ is the second smallest component in the quadruple moved to, then $p \ne 5$;

(ii) $q = p + 1$ if $a_4 - a_3 \in V_2$.

We say that a move in Flora is *legal* if it is consistent with rules (I)–(III).

Note. If the position moved to under rule III(i) is (b_1, b_2, b_3, b_4) (where of course $0 \le b_1 \le b_2 \le b_3 \le b_4$), then $a_3 - p = b_i$, $a_4 - p = b_j$ for some $1 \le i < j \le 4$. Then $a_4 - a_3 = b_j - b_i = t$ for some $t \in \mathbb{Z}_{\ge 1}$, and normally $t \ne p$.

3.1. Algebraic formulation of the *P*-positions. The set of P-positions of a game is the set of game positions from which the second (Previous) player can force a win. The set of all P-positions of a game is denoted $\mathcal{P}$. In particular, for Flora, $T_0 \in \mathcal{P}$.

[1] This is different from the partition into MathGames and PlayGames defined in [10].

Let

$$A_n = h(n), \quad B_n = a^3(n), \quad C_n = h^2(n), \quad D_n = h^3(n),$$

$$A = \cup_{n=1}^{\infty} A_n, \quad B = \cup_{n=1}^{\infty} B_n, \quad C = \cup_{n=1}^{\infty} C_n, \quad D = \cup_{n=1}^{\infty} D_n,$$

$$T_n := (A_n, B_n, C_n, D_n), \quad T = \cup_{n=0}^{\infty} T_n.$$

A prefix of T of size 19 is shown in Table 1. We shall presently show that T constitutes the set of P-positions of Flora. Assuming the truth of this assertion, we illustrate simple moves in instances of Flora.

Examples. (i) From $(6, 7, 9, 14)$, one can move to $(5, 6, 8, 12) \in \mathcal{P}$ by rule I.

(ii) From each of the positions $(4, 6, 9, 9)$, $(5, 8, 9, 14)$, $(7, 8, 11, 20)$, one can move to $T_0 \in \mathcal{P}$ by rule II.

(iii) From $(19, 21, 22, 32)$, one can move as follows: $19 \to 9$, $21 \to 11$, $22 \to 7$, $32 \to 17$, resulting in $(7, 9, 11, 17) \in \mathcal{P}$ by rule III(i).

(iv) From $(24, 29, 32, 37)$, one can move to $(5, 6, 8, 12) \in \mathcal{P}$ by rule III(ii) (since $37 - 32 = 5 \in V_2$).

TABLE 1
P-positions of Flora.

n	$h(n)$	$a^3(n)$	$h^2(n)$	$h^3(n)$
0	0	0	0	0
1	2	1	3	4
2	5	6	8	12
3	7	9	11	17
4	10	14	16	25
5	13	19	21	33
6	15	22	24	38
7	18	27	29	46
8	20	30	32	51
9	23	35	37	59
10	26	40	42	67
11	28	43	45	72
12	31	48	50	80
13	34	53	55	88
14	36	56	58	93
15	39	61	63	101
16	41	64	66	106
17	44	69	71	114
18	47	74	76	122

Notation 3. For $n \in \mathbb{Z}_{\geq 1}$, let $\Delta_{DC}(n) := h^3(n) - h^2(n)$, $\Delta_{DB}(n) := h^3(n) - a^3(n)$, $\Delta_{DA}(n) := h^3(n) - h(n)$, $\Delta_{CB}(n) := h^2(n) - a^3(n)$, $\Delta_{CA}(n) := h^2(n) - h(n)$, $\Delta_{BA}(n) := a^3(n) - h(n)$, $\Delta(n) = \Delta_{DC}(n) \cup \Delta_{DB}(n) \cup \Delta_{DA}(n) \cup \Delta_{CB}(n) \cup \Delta_{CA}(n) \cup \Delta_{BA}(n)$, $\Delta = \cup_{n=1}^{\infty} \Delta(n)$.

LEMMA 4. (i) $\Delta_{DC}(n) = a^2(n)$.

(ii) $\Delta_{DB}(n) = a^2(n) + 2$.

(iii) $\Delta_{DA}(n) = u^2(n)$.

(iv) $\Delta_{CB}(n) = 2$.

(v) $\Delta_{CA}(n) = a(n)$.

(vi) $\Delta_{BA}(n) = a(n) - 2$.

(vii) $\Delta = \mathbb{Z}_{\geq 1} \setminus V_2$.

(viii) $\Delta = \cup_{n=1}^{\infty} (\Delta_{DC}(n) \cup \Delta_{DB}(n) \cup \Delta_{DA}(n))$.

Proof. (i) This is Theorem 1(e1) for $k = 2$.

(ii) This is Theorem 1(e2) for $k = 2$.

(iii) $\Delta_{DA}(n) = (h^3(n) - h^2(n)) + (h^2(n) - h(n)) = a^2(n) + a(n) = u^2(n)$ by Theorem 1(e1) and (b).

(iv) This is Theorem 1(c).

(v) This is Theorem 1(e1).

(vi) By Theorem 1(c) and (e1), $\Delta_{BA}(n) = (a^3(n) - h^2(n)) + (h^2(n) - h(n)) = a(n) - 2$.

(vii) Notice that for every $n \in \mathbb{Z}_{\geq 1}$, $a^2(n) \in G_1$, $a^2(n) + 2 \in G_1$ (by Theorem 1(h1)), $2 \in U_2$, and $a(n) - 2 \in G_1 \cup U_2$ (by Theorem 1(h3)). It then follows from (iii) and (v) that $\Delta = G_1 \cup U_2$. The result follows since the sets G_1, U_2, V_2 clearly partition $\mathbb{Z}_{\geq 1}$.

(viii) The proof follows from (i)–(iii), (vii), and Theorem 1(h2). ∎

LEMMA 5. *For fixed $n \in \mathbb{Z}_{\geq 1}$, let $0 < t < a^2(n)$, $t \notin V_2$. Then there exists $0 \leq m < n$ such that $t \in \Delta_{DC}(m) \cup \Delta_{DB}(m) \cup \Delta_{DA}(m)$.*

Proof. We have $t < a^2(n) = \Delta_{DC}(n) < \Delta_{DB}(n) < \Delta_{DA}(n)$. It then follows from Lemma 4(viii) that there must be some $m < n$ for which $t \in \Delta_{DC}(m) \cup \Delta_{DB}(m) \cup \Delta_{DA}(m)$. ∎

THEOREM 2. *The set T constitutes the set of P-positions of the game Flora.*

Proof. To begin with we note the following facts:

- Lemma 1 implies that each of the sequences A_n, B_n, C_n, D_n is increasing.
- A, B, C, D partition $\mathbb{Z}_{\geq 1}$ (Theorem 1(a)).

It evidently suffices to prove the following two statements:

(A) Every move from any position in T results in a position outside T.

(B) For every position outside T there is a move into a position in T.

(A) Clearly there is no legal move $T_1 \to T_0$. Suppose that there are positions T_n, T_m with $m < n$, $n \geq 2$ such that there is a legal move $T_n \to T_m$. This move must be of type III, since A, B, C, D partition $\mathbb{Z}_{\geq 1}$, from which it follows easily, using Lemma 4, that $A_n < B_n < C_n < D_n$ for $n \geq 2$.

By Lemma 4(vii), $\Delta_{DC}(n) \notin V_2$, so we have to consider only move III(i). We first show that $D_n - p$ can only be D_m. It cannot be A_m, since then $C_n - p < A_m$ has no place in row m of T. Suppose $D_n - p = B_m$. Then $C_n - p = A_m$. But $\Delta_{BA}(m) < \Delta_{BA}(n) = a(n) - 2 < \Delta_{DC}(n) = a^2(n)$, contradicting move rule III(i). Suppose $D_n - p = C_m$. Since $C_n - B_n = 2$ for all $n \in \mathbb{Z}_{\geq 1}$, we have $C_n - p = A_m$. But $\Delta_{CA}(m) = \Delta_{BA}(m) + 2 \leq \Delta_{BA}(n) + 1 = a(n) - 1 < \Delta_{DC}(n) = a^2(n)$, again contradicting move rule III(i). Thus indeed $D_n - p = D_m$.

Suppose $C_n - p = C_m$. Subtracting, $\Delta_{DC}(n) = \Delta_{DC}(m)$, so $a^2(n) = a^2(m)$, which is impossible for $m < n$ since the sequence $g^2(\ell)$ is strictly increasing. Suppose $C_n - p = B_m = C_m - 2$. Then $\Delta_{DC}(n) = \Delta_{DB}(m) = \Delta_{DC}(m) + 2$, so $a^2(n) = a^2(m)+2$ by Lemma 4. Hence $m = n-1$. Since $D_n - D_{n-1} = p$, Theorem 1(f) implies $p = 5$. But this case is excluded by the proviso. Finally, suppose that $C_n - p = A_m$. Then $\Delta_{DC}(n) = \Delta_{DA}(m)$. By Lemma 4 this is equivalent to $a^2(n) = u^2(m)$. This is possible for no $m < n$ by disjointness.

(B) Let $(a_1, a_2, a_3, a_4) \notin T$, $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4$. If there is equality in any of these or $a_3 - a_2 = 1$, a move of type I or II leads to T_0. So we may assume $0 < a_1 < a_2 < a_2 + 1 < a_3 < a_4$. By the complementarity of A, B, C, D, a_1 appears in precisely one component of precisely one T_n, $n \geq 1$. If $a_1 = D_n$, move $a_2 \to A_n$, $a_3 \to B_n$, $a_4 \to C_n$.

Now suppose that $a_1 = C_n$.

If $a_4 \geq D_n$, move $a_2 \to A_n$, $a_3 \to B_n$, $a_4 \to D_n$. Assume $a_4 < D_n$. Let

$$t := a_4 - a_3.$$

We consider two cases.
(a) $t \notin V_2$ and
(b) $t \in V_2$.
(a) $t \notin V_2$. We have

$$0 < t = a_4 - a_3 < D_n - a_3 < D_n - a_1 = D_n - C_n = \Delta_{DC}(n) = a^2(n).$$

By Lemma 5, there exists $m < n$ such that either (i) $t = \Delta_{DC}(m)$, or (ii) $t = \Delta_{DB}(m)$, or (iii) $t = \Delta_{DA}(m)$.

For case (i), move $(a_1, a_2, a_3, a_4) \to (A_m, B_m, C_m, D_m)$. This is a legal move: $a_1 = C_n > A_n > A_m$, $a_2 > a_1 = C_n > B_n > B_m$, $a_3 > a_1 = C_n > C_m$, $a_4 = a_3 + D_m - C_m > D_m$, so this move (as well as in the remainder of this proof) is of form III.

For case (ii), move $a_1 \to A_m$, $a_2 \to C_m$, $a_3 \to B_m$, $a_4 \to D_m$. This is a legal move: $a_1 = C_n > A_n > A_m$, $a_2 > a_1 = C_n > C_m$, $a_3 > a_1 = C_n > C_m > B_m$, $a_4 = a_3 + D_m - B_m > D_m$.

For case (iii), move $a_1 \to B_m$, $a_2 \to C_m$, $a_3 \to A_m$, $a_4 \to D_m$. This is a legal move: $a_1 = C_n > B_n > B_m$, $a_2 > a_1 = C_n > C_m$, $a_3 > a_1 = C_n > C_m > A_m$, $a_4 = a_3 + D_m - A_m > D_m$.

(b) $t \in V_2$. Thus $t > 1$. To remind ourselves, $t = a_4 - a_3$, and we have $a_1 = C_n$, $a_4 < D_n$. Now $t - 1 \in (V_2 - 1)$. Since $(V_2 - 1) \subset G_2$ (Theorem 1(h4)), we have $t - 1 = a^2(m)$ for suitable $m \in \mathbb{Z}_{\geq 1}$. Also $\Delta_{DC}(m) = a^2(m)$ (Lemma 4(i)). So we move $(a_1, a_2, a_3, a_4) \to (A_m, B_m, C_m, D_m)$. This is a legal move:

- $m < n$, since $\Delta_{DC}(m) = a_4 - a_3 - 1 < D_n - a_1 = D_n - C_n = \Delta_{DC}(n)$.
- $a_1 = C_n > A_n > A_m$, $a_2 > a_1 = C_n > B_n > B_m$, $a_3 > a_1 = C_n > C_m$, $a_4 = a_3 + 1 + D_m - C_m > a_1 + D_m - C_m = C_n + D_m - C_m > D_m$.

So suppose that $a_1 = B_n$.

If $a_4 \geq D_n$, then move $a_2 \to A_n$, $a_3 \to C_n$, $a_4 \to D_n$. This is a legal move, since $a_2 > a_1 = B_n > A_n$ and

$$a_3 \geq a_2 + 1 \geq a_1 + 2 = B_n + 2 = C_n.$$

Therefore we may assume $a_4 < D_n$. The proof is similar to the above case $a_1 = C_n$. We have $a_3 \geq C_n$, and $0 < t - 1 < t = a_4 - a_3 < D_n - a_3 \leq \Delta_{DC}(n)$. Hence by Lemma 5 there is $m < n$ such that, for case (a), either (i) $t = \Delta_{DC}(m)$, or (ii) $t = \Delta_{DB}(m)$, or (iii) $t = \Delta_{DA}(m)$. For case (b) we have $t - 1 = a^2(m)$ for some $m \in \mathbb{Z}_{\geq 1}$.

(a) $t \notin V_2$.

For case (i), move $a_2 \to A_m$, $(a_3, a_4) \to (C_m, D_m)$. This is a legal move: $a_2 > a_1 = B_n > A_n > A_m$, $a_3 \geq C_n > C_m$,

$$a_4 = a_3 + D_m - C_m \geq C_n + D_m - C_m > D_m.$$

For case (ii), move $a_1 \to A_m$, $a_2 \to C_m$, $a_3 \to B_m$, $a_4 \to D_m$. This is a legal move: $a_1 = B_n > A_n > A_m$, $a_2 \geq a_1 + 1 = B_n + 1 = C_n - 1 \geq C_m$, $a_3 > a_1 = B_n > B_m$, $a_4 = a_3 + D_m - B_m \geq C_n + D_m - B_m > D_m$.

For case (iii), move $a_1 \to B_m$, $a_2 \to C_m$, $a_3 \to A_m$, $a_4 \to D_m$. This is a legal move: $a_1 = B_n > B_m$, $a_2 \geq a_1 + 1 = B_n + 1 = C_n - 1 \geq C_m$, $a_3 > a_1 = B_n > B_m > A_m$, $a_4 = a_3 + D_m - A_m > D_m$.

(b) $t \in V_2$. We have $t = a_4 - a_3$, $a_1 = B_n$, $a_4 < D_n$. As in case (b) above, we move $(a_1, a_2, a_3, a_4) \to (A_m, B_m, C_m, D_m)$. This is a legal move:

- $m < n$, since $\Delta_{DC}(m) = a_4 - a_3 - 1 < D_n - a_3 - 1 \leq D_n - C_n - 1 < \Delta_{DC}(n)$.
- $a_1 = B_n > A_n > A_m$, $a_2 > a_1 = B_n > B_m$, $a_3 \geq C_n > C_m$, $a_4 = a_3 + 1 + D_m - C_m \geq C_n + 1 + D_m - C_m > D_m$.

Finally, we consider the case $a_1 = A_n = h(n)$.

If $a_2 < h^2(n) - 2$, we can move to T_0 (rule II(iii)). Otherwise, $a_2 \geq h^2(n) - 2 = B_n$. Since $a_3 - a_2 > 1$, we have $a_3 \geq B_n + 2 = C_n$. If $a_4 \geq D_n$, then at least one of the inequalities for a_2, a_3, a_4 is strict, since $(a_1, a_2, a_3, a_4) \notin T$. Then move $(a_1, a_2, a_3, a_4) \to (A_n, B_n, C_n, D_n)$. If $a_4 < D_n$, then for case (a) there is $m < n$ such that $0 < t = a_4 - a_3 < \Delta_{DC}(n)$. Hence by Lemma 5, there is $m < n$ such that either (i) $t = \Delta_{DC}(m)$, or (ii) $t = \Delta_{DB}(m)$, or (iii) $t = \Delta_{DA}(m)$. For case (b), $0 < t - 1 = a^2(m)$ for some $m \in \mathbb{Z}_{\geq 1}$.

(a) $t \notin V_2$. For case (i) move $(a_1, a_2, a_3, a_4) \to (A_m, B_m, C_m, D_m)$. This is a legal move, since $m < n$ implies $a_1 = A_n > A_m$, $a_2 \geq B_n > B_m$, $a_3 \geq C_n > C_m$, $D_n > D_m$. For case (ii), move $a_1 \to A_m$, $a_2 \to C_m$, $a_3 \to B_m$, $a_4 \to D_m$. This is a legal move: $a_1 = A_n > A_m$, $a_2 \geq B_n = C_n - 2 > C_m$, where the strict inequality follows since $C_n - C_m \geq 3$ (Theorem 1(f4) for $k = 3$). Also $a_3 \geq C_n > B_n > B_m$, $a_4 = a_3 + D_m - B_m > D_m$. For case (iii) move $a_1 \to B_m$, $a_2 \to C_m$, $a_3 \to A_m$, $a_4 \to D_m$. We have to prove the legality of this move. We begin by showing that $a_1 = A_n > B_m$. Notice that

$$\begin{aligned} t &= a_4 - a_3 = \Delta_{DA}(m) = u^2(m) \quad \text{(Lemma 4)} \\ &< \Delta_{DC}(n) = a^2(n) \quad \text{(Lemma 4)} \\ &= h(n) - 1 \quad \text{(Theorem 1(c))}. \end{aligned}$$

Thus $h(n) > u^2(m) + 1$. But $h(n) = A_n$ and $u^2(m) = a^3(m) + 1$ (by Theorem 1(d)) $= B_m + 1$, so indeed $A_n > B_m + 2 > B_m$. Next,

$$\begin{aligned} a_2 &\geq B_n = a^3(n) = h^2(n) - 2 \quad \text{(Theorem 1(d))} \\ &> h^2(n-1) \quad \text{(Theorem 1(f4))} \\ &= C_{n-1} \geq C_m. \end{aligned}$$

Also $a_3 \geq C_n > C_m > A_m$ and $a_4 = a_3 + D_m - A_m \geq C_n + D_m - A_m > D_m$.

(b) $t \in V_2$. We have $t = a_4 - a_3$, $a_1 = A_n$, $a_4 < D_n$, $t - 1 = a^2(m) = \Delta_{DC}(m)$ for some $m \in \mathbb{Z}_{\geq 1}$. As above we move $(a_1, a_2, a_3, a_4) \to (A_m, B_m, C_m, D_m)$. This is a legal move:

- $m < n$, since $\Delta_{DC}(m) = a_4 - a_3 - 1 < D_n - C_n - 1 < \Delta_{DC}(n)$.
- $a_1 = A_n > A_m$, $a_2 \geq B_n > B_m$, $a_3 \geq C_n > C_m$, $a_4 = a_3 + 1 + D_m - C_m > C_n + D_m - C_m > D_m$. ∎

THEOREM 3. *The algebraic winning strategy of Flora precipitates a polynomial-time algorithm for consummating a win.*

Proof. Given a position (a_1, a_2, a_3, a_4) of Flora with $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4$, its input size is $O(\log a_1 + \log a_2 + \log a_3 + \log a_4)$. Whether or not move rules II(i) and II(ii) apply can be checked trivially. We know (Theorem 1(a)) that a_1 is precisely one of $a^3(n)$, $h^3(n)$, $h^2(n)$, $h(n)$. We have to find out which it is and the corresponding value of n.

Suppose first that $a_1 = a^3(n) = \lfloor \varphi \lfloor \varphi \lfloor n\varphi \rfloor \rfloor \rfloor$. Using the inequality $x - 1 < \lfloor x \rfloor \leq x$, a straightforward computation shows that $\lfloor a_1 \varphi^{-3} \rfloor + 1 \leq n \leq \lfloor (a_1 + 1)\varphi^{-3} \rfloor + 1$. Computing φ to $O(\log a_1)$ places gives the range for the candidate values of n, and for each of them (one or two), we have to compute $a^3(n)$, comparing it with a_1. A similar computation can be done for $h^3(n)$, $h^2(n)$, $h(n)$. (Notice that there is not necessarily an integer candidate n for some of these ranges. For example, if we suppose that $a_1 = h^3(n)$, then we get $\lfloor a_1 \varphi^{-4} \rfloor + 1 \leq n \leq \lfloor a_1 \varphi^{-4} + 2\varphi^{-2} \rfloor$.) The same method also indicates whether or not $a_4 - a_3 \in V_2$, or whether move rule II(iii) applies. All these computations can be done in linear time in the input size.

Finally, we use a binary search to find $m \in [0, n]$ such that if $a_4 - a_3 \notin V_2$, then $h^3(m) - h^2(m) = a_4 - a_3$ or some other difference of the columns in the mth row is $a_4 - a_3$. This is done similarly for the case $a_4 - a_3 \in V_2$. ∎

3.2. Recursive formulation of the *P*-positions. Let $S \subsetneq \mathbb{Z}_{\geq 1}$ and $\overline{S} = \mathbb{Z}_{\geq 1} \setminus S$. The "Minimum EXcludant" of S is defined by

$$\operatorname{mex} S = \min \overline{S} = \text{least positive integer not in } S.$$

In particular, the mex of the empty set is 1. (This somewhat nonstandard definition of the mex function is needed for section 5.)

Let $T'_0 = (0, 0, 0, 0)$, $T'_1 = (2, 1, 3, 4)$. If $T'_m := (A'_m, B'_m, C'_m, D'_m)$ already has been defined for all $m < n$ ($n \geq 2$), then let

$$A'_n = \operatorname{mex}\{A'_i, B'_i, C'_i, D'_i : 0 \leq i < n\},$$

$$B'_n = \begin{cases} B'_{n-1} + 3 & \text{if } A'_n - A'_{n-1} = 2, \\ B'_{n-1} + 5 & \text{otherwise,} \end{cases}$$

$$C'_n = B'_n + 2,$$

$$D'_n = \begin{cases} D'_{n-1} + 5 & \text{if } A'_n - A'_{n-1} = 2, \\ D'_{n-1} + 8 & \text{otherwise.} \end{cases}$$

Let $T' := \cup_{n=0}^{\infty} T'_n$.

THEOREM 4. *The set T' constitutes the set of P-positions of the game Flora.*

Proof. We show that for all $m \in \mathbb{Z}_{\geq 0}$, $A'_m = A_m$, $B'_m = B_m$, $C'_m = C_m$, $D'_m = D_m$. Suppose this holds for all $m < n$ ($n \geq 1$). Let $E = \operatorname{mex}\{A_i, B_i, C_i, D_i : 0 \leq i < n\}$. The value E cannot have been assumed in any of the four sequences for $m < n$, since A, B, C, D split $\mathbb{Z}_{\geq 1}$, so $E \geq A_n$. If $E > A_n$, then A_n would never be assumed since the sequences are strictly increasing, again contradicting the complementarity of the sequences. Thus $A_n = E = A'_n$, and the other three equalities follow from Theorem 2. ∎

The definition of the set T' is straightforward; it does not use the functions $h(n)$, $a(n)$ used for defining T. Thus the recursive computation of T' looks easier than that of the set T. Moreover, the proof of Theorem 4 is very short, and that of Theorem 2 is long.

However, the proof of Theorem 4 relies heavily on Theorems 2 and 1. If the initial position of the game is (a_1, a_2, a_3, a_4), the input size is $O(\log a_1 + \log a_2 + \log a_3 + \log a_4)$. The time needed to compute whether the position is a P-position or not seems to be proportional to $a_1 + a_2 + a_3 + a_4$, because the unwieldy mex function appears to require scanning previous entries of the sequences A_n, B_n, C_n, D_n. However, a new method [17] shows that actually the algorithm implied by Theorem 4 is also polynomial.

3.3. Arithmetic formulation of the *P*-positions. For $N \in \mathbb{Z}_{\geq 1}$, let $R(N) = (d_m, \ldots, d_0)$ be the representation of N in the Fibonacci numeration system (recall Notation 2). Then $(d_m, \ldots, d_0, 0)$ is the *left shift* of $R(N)$.

THEOREM 5. *$R(A)$ is the set of all representations that end in an odd number of 0-bits in the Fibonacci numeration system, $R(B)$ the set of all representations that end in 001, $R(C)$ the set of all representations that end in a positive even number of 0-bits, and $R(D)$ the set of all representations that end in 101. Moreover, for every $n \in \mathbb{Z}_{\geq 1}$, $R(C_n)$ is the left shift of $R(A_n)$.*

See Table 2 for an example.

TABLE 2
Representation of the P-positions in the Fibonacci numeration system.

21	13	8	5	3	2	1	A_n	n	B_n	34	21	13	8	5	3	2	1
					1	0	2	1	1								1
			1	0	0	0	5	2	6					1	0	0	1
			1	0	1	0	7	3	9				1	0	0	0	1
		1	0	0	1	0	10	4	14			1	0	0	0	0	1
	1	0	0	0	0	0	13	5	19			1	0	1	0	0	1
	1	0	0	0	1	0	15	6	22		1	0	0	0	0	0	1
	1	0	1	0	0	0	18	7	27		1	0	0	1	0	0	1
	1	0	1	0	1	0	20	8	30		1	0	1	0	0	0	1
1	0	0	0	0	1	0	23	9	35	1	0	0	0	0	0	0	1
1	0	0	1	0	0	0	26	10	40	1	0	0	0	1	0	0	1

34	21	13	8	5	3	2	1	C_n	n	D_n	55	34	21	13	8	5	3	2	1
					1	0	0	3	1	4							1	0	1
			1	0	0	0	0	8	2	12					1	0	1	0	1
			1	0	1	0	0	11	3	17				1	0	0	1	0	1
		1	0	0	1	0	0	16	4	25			1	0	0	0	1	0	1
	1	0	0	0	0	0	0	21	5	33			1	0	1	0	1	0	1
	1	0	0	0	1	0	0	24	6	38		1	0	0	0	0	1	0	1
	1	0	1	0	0	0	0	29	7	46		1	0	0	1	0	1	0	1
	1	0	1	0	1	0	0	32	8	51		1	0	1	0	0	1	0	1
1	0	0	0	0	1	0	0	37	9	59	1	0	0	0	0	0	1	0	1
1	0	0	1	0	0	0	0	42	10	67	1	0	0	0	1	0	1	0	1

Proof. The proof is similar to that of Theorem 3 of [12]. For every $m \in \mathbb{Z}_{\geq 1}$, $R(\lfloor m\varphi \rfloor)$ ends in an even number of 0-bits (including 0 0-bits), and $R(\lfloor m\varphi^2 \rfloor)$ ends in an odd number of 0-bits [8, section 4]. Hence $R(A)$ is the set of all numbers that end in an odd number of 0-bits in the Fibonacci numeration system, whereas each of the other three representations ends in an even number of 0-bits. Now $R(C)$ is the set of all numbers that end in a positive even number of 0-bits [12]; hence $R(B)$ and $R(D)$ each end in a 1-bit. Recall that $C_n = B_n + 2$. If $R(B)$ contained a number with representation ending in 101, then adding 2 to it would end in 1 (since $2 + 3 = 5$ is the next Fibonacci number), contradicting the form of $R(C_n)$. Therefore $R(B)$ is the set of all numbers ending in 001. By complementarity, $R(D)$ is therefore the set of all numbers ending in 101.

Since $R(A)$ is the set of all representations ending in an odd number of 0-bits, and $R(C)$ is the set of all representations ending in a positive even number of 0-bits, the latter is the left shift of the former. Suppose that $R(C_m)$ is the left shift of $R(A_m)$ for every $m < n$. If $R(C_n)$ were not the left shift of $R(A_n)$, then it would be assumed later on (by complementarity), contradicting the strict increase of C. ☐

This formulation of the P-positions is also easily seen to lead to a polynomial-time winning strategy.

4. The word-mapping approach. In this section we show how to construct G_k recursively by a word-map for every $k \in \mathbb{Z}_{\geq 1}$. Similar methods can be used to construct other functions defined in section 2. We also present our fourth formulation of the game Flora, which stems from the word-mapping approach. The length of any (sub)word w is denoted by $|w|$.

4.1. Word-mapping for G_k. Define the morphism $1 \mapsto 10$, $0 \mapsto 1$. Its fixed point is the word $F = 1011010110110\ldots$, also known as the Fibonacci word. For $k \geq 1$, the characteristic function χ_k of G_k is defined by

$$\chi_k(m) = \begin{cases} 1 & \text{if } \exists\, n \text{ such that } a^k(n) = m, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1. *Given a binary word W, a* run *of* 0*'s is any (possibly empty) subword of W consisting solely of* 0*'s, flanked on the left and right by a* 1*-bit. A* block *is any subword consisting of a* 1*-bit followed by a run of* 0*'s.*

Theorem 6. *For every $k \geq 1$, the word-mapping for producing the characteristic function χ_k of G_k, beginning with $10^{F_{k-1}-1}$, is*

$$10^{F_{k-1}-1} \mapsto 10^{F_k-1}, \quad 0^{F_{k-2}} \mapsto 10^{F_{k-1}-1}.$$

Proof. Notice that for G_1, the word-mapping is simply the well-known morphism $1 \mapsto 10$, $0 \mapsto 1$, which produces F. Moreover, $\chi_1 = F$. See, e.g., [1, Chap. 9] and [16].

The word-map is well defined. Indeed, the initial block of length F_{k-1} is mapped into a block B_1 of length F_k. In the second iteration, the prefix of length F_{k-1} of B_1 is again mapped into B_1. The remaining abutting suffix of B_1 consists of F_{k-2} 0's, so it is mapped into a block B_2 of length F_{k-1}. In the third iteration, B_1 and B_2 are generated again, and then the block B_2 of length F_{k-1} generates a block B_1. Thus for all subsequent iterations only blocks of the form B_1 and B_2 are generated, and there is never any parsing conflict.

Since $\chi_1 = 1011010110110\ldots$, Theorem 1(f) implies that $\chi_2 = 10010100100\,1010010100\ldots$, where we inserted into χ_1 $F_1 - F_0 = 1$ zero to each run of $F_0 - 1 = 0$ zeros (i.e., one 0 between every pair of consecutive 1's), and $F_2 - F_1 = 1$ zero to each run of $F_1 - 1 = 1$ zeros. Doing this yields distances between consecutive 1's in χ_2 of F_1 and F_2, precisely at the locations where the distances between consecutive 1's of χ_1 are F_0 and F_1, respectively. Similarly, $\chi_3 = 1000010010000100001001000010010000\ldots$, where we inserted into χ_2 $F_2 - F_1 = 1$ zero to each run of $F_1 - 1 = 1$ zero, and $F_3 - F_2 = 2$ zeros to each run of $F_2 - 1 = 2$ zeros.

In general, for producing χ_{k+1} from χ_k, we add to χ_k $F_k - F_{k-1} = F_{k-2}$ zeros to each run of $F_{k-1} - 1$ zeros and $F_{k+1} - F_k = F_{k-1}$ zeros to each run of F_k zeros. This yields blocks of sizes F_k and F_{k+1}, respectively, at the locations specified by Theorem 1(f).

Assume inductively that the word-mapping

$$10^{F_{k-1}-1} \mapsto 10^{F_k-1}, \quad 0^{F_{k-2}} \mapsto 10^{F_{k-1}-1}$$

produces χ_k, so it generates distances between consecutive 1's of F_{k-1} and F_k at the locations specified by Theorem 1(f). Then the word-mapping

$$10^{F_k-1} \mapsto 10^{F_{k+1}-1}, \quad 0^{F_{k-1}} \mapsto 10^{F_k-1}$$

produces χ_{k+1}, since it adds $F_{k+1} - F_k$ to the F_k 0's of the long 0-runs of χ_k, and $F_k - F_{k-1}$ 0's to the F_{k-1} short 0-runs of χ_k. □

4.2. Word-mapping formulation of the *P*-positions. Denote terms of A_n, B_n, C_n, D_n by a, b, c, d, respectively.

THEOREM 7. *The word-mapping*

$$bac \mapsto bacda, \qquad da \mapsto bac,$$

beginning with bac*, generates the characteristic function of the P-positions of the Flora game.*

Proof. The proof is rather similar to that of Theorem 6, and is therefore omitted. □

This theorem also leads to a polynomial-time winning strategy, since induction shows that for every $k \in \mathbb{Z}_{\geq 1}$, the kth application of the word-mapping generates a word of length F_{k+2}.

Notice that if we replace *bac* by 1 and *da* by 0, we get back our old morphic friend $1 \mapsto 10$, $0 \mapsto 1$.

4.3. Characterization of the sequences G, H by the Fibonacci numeration system. We know from Lemma II and section 3.3 that $R(a(n))$ ends in an even number of 0's, $R(h(n))$ in an odd number of 0's, $R(h^2(n))$ in an even positive number of 0's, $R(h^3(n))$ in 101, and $R(g^3(n))$ in 10^s1, $s \geq 2$. What is the general pattern?

THEOREM 8. (i) *$R(G_1)$ is the set of all representations that end in an even number of 0's, $R(H_1)$ is the set of all representations that end in an odd number of 0's, $R(G_2)$ is the set of all representations that end in a 1-bit, and $R(H_2)$ is the set of all representations that end in an even positive number of 0's.*

(ii) *For every $n \in \mathbb{Z}_{\geq 1}$, $R(h(n))$ is the left shift of $R(a(n))$, and $R(h^2(n))$ is the left shift of $R(h(n))$.*

(iii) *For every $k \in \mathbb{Z}_{\geq 3}$ and all $n \in \mathbb{Z}_{\geq 1}$, $R(G_k)$ is the set of all representations that end in the word 10^s1 for all $s \geq k-1$, and $R(H_k)$ is the set of all representations that end in the word $10^{k-2}1$ (left 1-bit in position $k-1$).*

Proof. Items (i) and (ii) are already known from Theorem 5 and Lemma II, and are included here only for the sake of completeness. We only have to point out the statement about $R(G_2)$, which follows from the fact that G_2, H_2, H_1 split the positive integers (see also [12, Theorem 3]).

(iii) Induction on k. The base case $k = 3$ was proved in Theorem 5. For $k \geq 3$, suppose that we already proved that $R(G_k)$ is the set of all representations that end in 10^s1 for all $s \geq k-1$, and that $R(H_k)$ is the set of all representations that end in $10^{k-2}1$. It clearly remains only to show that $R(G_{k+1})$ is the set of all representations that end in 10^s1 for all $s \geq k$, and that $R(H_{k+1})$ is the set of all representations that end in the word $10^{k-1}1$. Recall Theorem 1(c): $h^k(n) = g^{k+1}(n) + F_{k-1}$. If $R(G_{k+1})$ were to contain a number, say $a^{k+1}(n)$, with representation ending in $10^{k-1}1$ (with leftmost 1-bit in position k), then adding F_{k-1} to it would result in a word with representation ending in 0^k1 because $F_{k-1} + F_k = F_{k+1}$ is the *next* Fibonacci number. But then $R(h^k(n))$ would end in 10^s1 for some $s \geq k$, contradicting the induction hypothesis. Thus $R(H_{k+1})$ is the set of all representations that end in the word $10^{k-1}1$. Since $G_{k+1}, H_{k+1}, H_k, \ldots, H_2, H_1$ split the integers, $R(H_{k+1})$ is the set of all representations that end in the word $10^{k-1}1$. □

Notes. (1) The general pattern of the representation of the suffixes of H_k for $k \geq 3$ is quite different from that of H_1 and H_2, and both of these are different from each other. The same holds for G_k, $k \geq 3$, and G_1 and G_2. Therefore the induction proof could not have begun with $k = 1$ or 2.

(2) The statement in (i) about $R(G_1)$ and $R(H_1)$ is Theorem 9.1.15 (see also Corollary 9.1.14) in [1], credited there to [15]. (It is also Lemma II above.) The proof method of [1] follows that of [3].

Membership problem. Fix some $k \in \mathbb{Z}_{\geq 1}$. Given $N \in \mathbb{Z}_{\geq 1}$, N is in precisely one of G_k, H_k, $H_{k-1}, \ldots$, H_2, H_1 (Theorem 1(a)). Can the following problem be solved in polynomial time?

Problem. Determine the set in which N lies.

COROLLARY 1. *For every $k \in \mathbb{Z}_{\geq 1}$, the membership problem can be solved in linear time.*

Proof. This can be proved by generalizing the method for computing n in the proof of Theorem 3 to the case of arbitrary k. But a more "elegant" method is to compute the Fibonacci representation of N, which can be done in linear time in the input size $\Theta(\log N)$. Theorem 8 then implies that the membership problem can be solved by scanning the suffix of $R(N)$, at most all of its $\Theta(\log N)$ bits. □

5. Infinite complementary arrays. The doubly infinite *Stolarsky Array* A with entries $A(i,j)$, $i,j \geq 1$ [29], is defined as follows: for every $m \geq 1$, $A(m,1) = \text{mex}\{A(i,j) : i < m,\ j \geq 1\}$, $A(m,2) = \lfloor \varphi A(m,1) + 1/2 \rfloor$, and for all $i \geq 1$, $j \geq 3$, $A(i,j)$, $A(i,j) = A(i,j-1) + A(i,j-2)$. Then every positive integer appears precisely once in A. A beginning portion is exhibited in Table 3. Many variations, interspersions, and dispersions have since been given; see, e.g., [21], [26]. All are doubly infinite, $\lim_{j\to\infty}(A(i,j+1) - A(i,j)) = \infty$ for every $i \geq 1$, and every positive integer appears precisely once in A.

TABLE 3
The Stolarsky Array.

1	2	3	5	8	13	...
4	6	10	16	26	42	...
7	11	18	29	47	76	...
9	15	24	39	63	102	...
⋮	⋮	⋮	⋮	⋮	⋮	

For every $k \geq 1$, define the Flora Array L_k with the $k+1$ rows $H_1, H_2, \ldots, H_k, G_k$. This array has a different character. By Theorem 1(a), this *singly* infinite array also has the property that every positive integer appears precisely once. Moreover, $A(i,j+1) - A(i,j) \in \{F_i, F_{i+1}\}$ is bounded for every fixed i and all $j \geq 1$, but $\lim_{i\to\infty}(A(i,j+1) - A(i,j)) = \infty$. Table 4 depicts the case $k = 6$. The last two lines, below the horizontal line, illustrate the fact that G_7, H_7 split G_6, so replacing G_6 by H_7 and G_7 constitutes L_7.

6. Conclusion. We have generated sequences consisting of nested arbitrary applications of the floor function to φ and φ^2, established many identities and relationships involving them, and then applied them to formulate an algebraic winning strategy to the game Flora. We also presented recursive, arithmetic, and word-mapping formulations of the winning strategy. In addition, we characterized the main sequences by means of the Fibonacci numeration system and generated infinite complementary arrays of the sequences.

Can some of the relationships of the iterated floor functions be generalized to reals other than φ and φ^2? As a first step, we could interchange φ with φ^2, studying the ensuing sequences and the games implied by them. Specifically, define $g'(n) = \lfloor n\varphi^2 \rfloor$, $h'(n) = \lfloor n\varphi \rfloor$, and, for $k \geq 2$, $g'^k(n) = \lfloor \varphi^2 g'^{k-1}(n) \rfloor$, $h'^k(n) = \lfloor \varphi^2 g'^{k-1}(n) \rfloor$. If we

TABLE 4
A complementary Flora Array L_6 with seven rows.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
H_1	2	5	7	10	13	15	18	20	23	26	28	31	34
H_2	3	8	11	16	21	24	29	32	37	42	45	50	55
H_3	4	12	17	25	33	38	46	51	59	67	72	80	88
H_4	6	19	27	40	53	61	74	82	95	108	116	129	142
H_5	9	30	43	64	85	98	119	132	153	174	187	208	229
H_6	14	48	69	103	137	158	192	213	247	281	302	336	370
G_6	1	22	35	56	77	90	111	124	145	166	179	200	221
H_7	22	77	111	166	221	255	310	344	$\cdots$				
G_7	1	35	56	90	124	145	179	200	234	268	289	323	$\cdots$

define G'_i, H'_i in the obvious way, then it is straightforward to see that the first item of Theorem 1 is preserved, namely, that $G'_k, H'_k, H'_{k-1}, \ldots, H'_2, H'_1$ partition $\mathbb{Z}_{\geq 1}$. What games can be spawned from this partition?

More generally, it would be well to investigate which of the above results hold for which classes of positive reals beyond φ. For example, Lemma 1 and Theorem 1(a) clearly hold if we replace φ by any irrational $\alpha \in (1, 2)$ and φ^2 by $\beta = \alpha/(\alpha - 1)$. Perhaps large parts of Theorem 1 can be generalized for the case where $\alpha = (2-t+\sqrt{t^2+4})/2$, $\beta = \alpha+t$, where t is any given positive integer, since then the simple continued fraction of α is $[1, t, t, t, \ldots]$, so the numeration system arguments used in the proofs of (c) and (d) of Theorem 1 carry over in a simple way. What games are induced by these relationships?

The notion of arbitrary iterations of the floor function appeared in [28] and [20]. In the former, the iterations are with *rational* numbers whose sizes depend on the iteration depth; in the latter, the aim is to represent the positive integers in the form of iterated floor functions involving φ and φ^2. These are quite different from our iterations of the floor function. However, in [24] iterations of the form considered here were studied, as pointed out at the end of section 2. There is some relationship to [23], [22].

The Raleigh game [12] is a three-pile extension—not generalization—of Wythoff's game. Flora is an extension of Raleigh. Although Flora appears not to be decomposable into sums of more elementary games, we were able to formulate for it three polynomial-time winning strategies. The one based on the Fibonacci numeration system is of particular interest. It demonstrates once again that numeration systems can make strategies of games in PrimGame efficient, similarly to appropriate data structures; see [27].

We can also define a five-pile extension of Flora, but in the sequence of games with increasing number of piles, both the definition of the games and the validity proof of their strategies seem to become more difficult. For example, whereas the union of the differences Δ between the three columns of the P-positions of Raleigh covers all of $\mathbb{Z}_{\geq 1}$, the same union for the four columns of the P-positions of Flora leaves out V_2. But perhaps a pattern for these games will emerge. This possibility may not be so far-fetched, since, as we saw, e.g., in section 4.3, the general behavior begins only with $k = 3$ (corresponding to a game with five piles).

Acknowledgments. Thanks to Herb Wilf, who had written to me that he defined the sequence $a^k(n)$ for solving part (a) of [11], and conjectured that $\Delta a^k(n)$ assumes only two values for every fixed k and $n \in \mathbb{Z}_{\geq 1}$. His communication and earlier work of mine—including joint work with Eric Duchêne, Richard Nowakowski, and

Michel Rigo [4]—motivated this paper. I am also obliged to the anonymous referees for their useful comments and for finding typos—and then I found some more. I am sure that additional ones lurk at unsuspected spots.

REFERENCES

[1] J.-P. Allouche and J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, Cambridge, UK, 2003.

[2] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, Vol. 1, 2nd ed., A K Peters, Natick, MA, 2001.

[3] T. C. Brown, *Descriptions of the characteristic sequence of an irrational*, Canad. Math. Bull., 36 (1993), pp. 15–21.

[4] E. Duchêne, A. S. Fraenkel, R. J. Nowakowski, and M. Rigo, *Extensions and restrictions of Wythoff's game preserving its P-positions*, J. Combin. Theory Ser. A, 117 (2010), pp. 545–567.

[5] E. Duchêne and M. Rigo, *A morphic approach to combinatorial games: The Tribonacci case*, Theor. Inform. Appl., 42 (2008), pp. 375–393.

[6] E. Duchêne and M. Rigo, *Cubic Pisot unit combinatorial games*, Monatsh. Math., 155 (2008), pp. 217–249.

[7] N. Falkner, *Basic real analysis. Advanced real analysis* (book reviews), Amer. Math. Monthly, 116 (2009), pp. 657–664.

[8] A. S. Fraenkel, *How to beat your Wythoff games' opponent on three fronts*, Amer. Math. Monthly, 89 (1982), pp. 353–361.

[9] A. S. Fraenkel, *Systems of numeration*, Amer. Math. Monthly, 92 (1985), pp. 105–114.

[10] A. S. Fraenkel, *Complexity, appeal and challenges of combinatorial games*, Theoret. Comput. Sci., 313 (2004), pp. 393–415.

[11] A. S. Fraenkel, *Problem* 11238, Amer. Math. Monthly, 113 (2006), p. 655; *Solution to Problem* 11238, 115 (2008), p. 667.

[12] A. S. Fraenkel, *The Raleigh game*, in Combinatorial Number Theory, Proceedings of the Integers Conference 2005 in Celebration of the 70th Birthday of Ronald Graham, Carrollton, GA, B. Landman, M. Nathanson, J. Nešetřil, R. Nowakowski, and C. Pomerance, eds., de Gruyter, Berlin, 2007, pp. 199–208; reprinted in Integers, 7(2) (2007), A13.

[13] A. S. Fraenkel, *From enmity to amity*, Amer. Math. Monthly, to appear.

[14] A. S. Fraenkel and D. Krieger, *The structure of complementary sets of integers: A 3-shift theorem*, Int. J. Pure Appl. Math., 10 (2004), pp. 1–49.

[15] A. S. Fraenkel, J. Levitt, and M. Shimshoni, *Characterization of the set of values* $f(n) = [n\alpha]$, $n = 1, 2, \cdots$, Discrete Math., 2 (1972), pp. 335–345.

[16] A. S. Fraenkel, M. Mushkin, and U. Tassa, *Determination of* $\lfloor n\theta \rfloor$ *by its sequence of differences*, Canad. Math. Bull., 21 (1978), pp. 441–446.

[17] A. S. Fraenkel and U. Peled, *Harnessing the Unwieldy MEX Function*, preprint.

[18] G. H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, AMS Chelsea Publishing, Providence, RI, 1999 (reprinted with corrections from the 1978 edition).

[19] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Clarendon Press, Oxford, UK, 1960.

[20] V. E. Hoggatt and M. Bicknell-Johnson, *Representations of integers in terms of greatest integer functions and the golden section ratio*, Fibonacci Quart., 17 (1979), pp. 306–318.

[21] C. Kimberling, *Interspersions and dispersions*, Proc. Amer. Math. Soc., 117 (1993), pp. 313–321.

[22] C. Kimberling, *A self-generating set and the golden mean*, J. Integer Seq., 3 (2000), article 00.2.8.

[23] C. Kimberling, *Complementary equations*, J. Integer Seq., 10 (2007), article 07.1.4.

[24] C. Kimberling, *Complementary equations and Wythoff sequences*, J. Integer Seq., 11 (2008), article 08.3.3.

[25] M. Lothaire, *Algebraic Combinatorics on Words*, Encyclopedia Math. Appl. 90, Cambridge University Press, Cambridge, UK, 2002.

[26] D. R. Morrison, *A Stolarsky array of Wythoff pairs*, in A Collection of Manuscripts Related to the Fibonacci Sequence, Fibonacci Association, Santa Clara, CA, 1980, pp. 134–136.

[27] B. Obrenić, *Hyperbaric Numeration Systems and Improved Fibonacci Coding*, preprint.

[28] K. A. Redish and W. F. Smyth, *Closed form expressions for the iterated floor function*, Discrete Math., 91 (1991), pp. 317–321.

[29] K. B. Stolarsky, *A set of generalized Fibonacci sequences such that each natural number belongs to exactly one*, Fibonacci Quart., 15 (1977), p. 224.

[30] X. Sun, *Wythoff's sequence and N-heap Wythoff's conjectures*, Discrete Math., 300 (2005), pp. 180–195.

[31] X. Sun and D. Zeilberger, *On Fraenkel's N-heap Wythoff's conjectures*, Ann. Comb., 8 (2004), pp. 225–238.