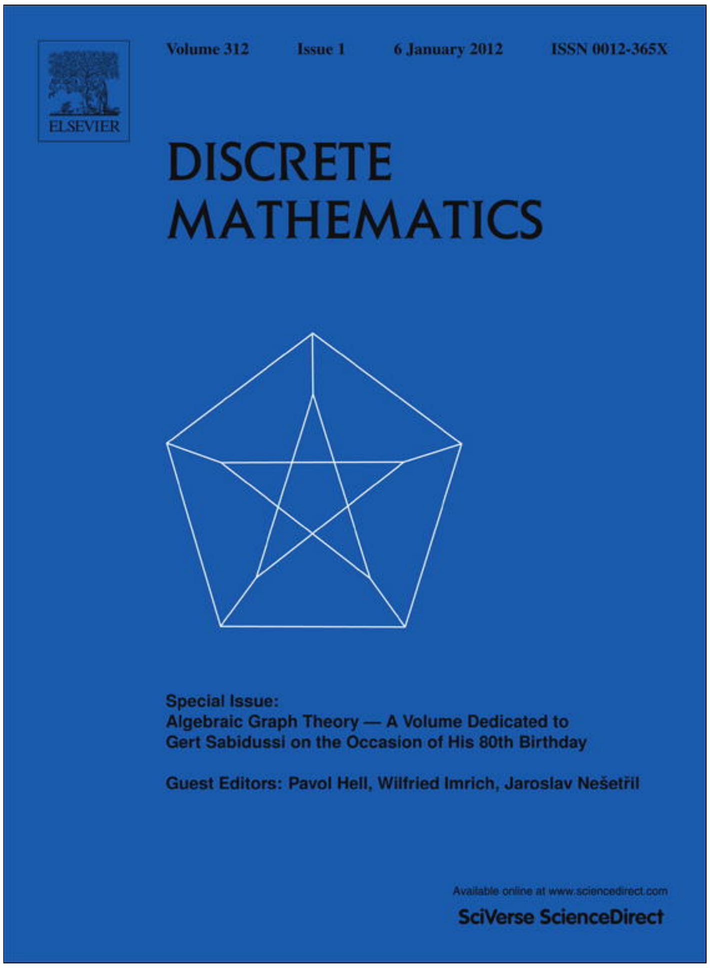
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The vile, dopey, evil and odious game players

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ABSTRACT

Many of my friends celebrate anniversaries these days, be it 50th birthday, or 60th, 70th, 80th, even 90th. It is to be deplored that they all accumulate together, rather than distribute themselves uniformly over my lifetime! It is now Gert Sabidussi's turn, who just joined the octogenarians club, and it is a pleasure to dedicate a paper to him, since he did, and continues to do, excellent algebraic graph theory, including insights into the automorphism group of graphs, studies of stable graphs, Sabidussi representation theorems for symmetric graphs, Sabidussi's compatibility conjecture, Sabidussi graphs, etc. Many leading mathematicians throughout the world are working on problems and insights initiated by Gert. He does all this in a relaxed playful way, as I witnessed when I acquired my own Sabidussi number 1. Hence it is only natural to relate here the fun that Gert and his friend Wil had while playing games in Dubrovnik, where a grand conference took place in 2009 honoring Gert. Unfortunately, I had to skip that conference, but, unknown to them, I planted a listening device.

Wil: We have worked and published jointly on the complexity of graph focality. Now that one of us is already an octogenarian and the other is only a decade away, let us have some fun; let us play a game.

Gert: I'm game.

Wil: In the Fall 2009 issue of the MSRI gazette Emissary, Elwyn Berlekamp and Joe Buhler proposed the following puzzle: "Nathan and Peter are playing a game. Nathan always goes first. The players take turns changing a positive integer to a smaller one and then passing the smaller number back to their opponent. On each move, a player may either subtract one from the integer or halve it, rounding down if necessary. Thus, from 28 the legal moves are to 27 or to 14; from 27, the legal moves are to 26 or to 13. The game ends when the integer reaches 0. The player who makes the last move wins. For example, if the starting integer is 15, a legal sequence of moves might be to 7, then 6, then 3, then 2, then 1, and then to 0. (In this sample game one of the players could have played better!) Assuming both Nathan and Peter play according to the best possible strategy, who will win if the starting integer is 1000? 2000?"

Let us dub it the Mark game, since it is due to Mark Krusemeyer according to Berlekamp and Buhler.

Gert: To get a feel for the Mark game, I'd construct a small table listing its P-positions (Previous player wins) and N-positions (Next player wins). For example, $0 \in \mathcal{P}$, since the Next (first) player cannot move, so the Previous (second) player wins by default, $1 \in \mathcal{N}$ since Next can move to $0 \in \mathcal{P}$; and $2 \in \mathcal{P}$. In general, every position that has a follower in $\mathcal{P}$ is in $\mathcal{N}$, and every position all of whose followers are in $\mathcal{N}$ is in $\mathcal{P}$ ($\mathcal{P}$ and $\mathcal{N}$ are the set of all P- and N-positions respectively).

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WIL: (Extracting his palm computer)…Here it is!

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
P/N	P	N	P	N	N	N	P	N	P	N	P	N	N	N	P	N	N	N	P	N	N	N	P	N

GERT: It is hard to see what's going on…It might be useful to separate out the P-positions and the N-positions into two sequences.

WIL: Alright, the rearranged table below suggests that $b_n = 2a_n$ for every nonnegative integer n, $\mathcal{P} = \cup_{n\geq 0} b_n$, $\mathcal{N} = \cup_{n\geq 1} a_n$, where $N_n = a_n, n \geq 1$; $P_n = b_n, n \geq 0$. But what's N_n?…Oh I see, $N_n = \text{mex}\{P_i, N_i : 0 \leq i < n\}$ for every $n \geq 0$, where the mex of a finite subset of nonnegative integers is the least nonnegative integer not in the set. In particular, the mex of the empty set is 0. Notice that the sequences (for $n \geq 1$) are complementary: they split the positive integers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	1	3	4	5	7	9	11	12	13	15	16	17	19	20	21	23	25	27	28	29	31	33	35
b_n	0	2	6	8	10	14	18	22	24	26	30	32	34	38	40	42	46	50	54	56	58	62	66	70

n	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
a_n	36	37	39	41	43	44	45	47	48	49	51	52	53	55	57	59	60	61	63	64	65	67	68	69
b_n	72	74	78	82	86	88	90	94	96	98	102	104	106	110	114	118	120	122	126	128	130	134	136	138

n	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71
a_n	71	73	75	76	77	79	80	81	83	84	85	87	89	91	92	93	95	97	99	100	101	103	105	107
b_n	142	146	150	152	154	158	160	162	166	168	170	174	178	182	184	186	190	194	198	200	202	206	210	214

GERT: Since $15 \in \mathcal{N}$ which has the follower $14 \in \mathcal{P}$, we indeed see, as hinted by Berlekamp–Buhler, that Nathan could have played better by moving $15 \to 14$ rather than $15 \to 7$, thus securing his win! But for deciding 1000 and 2000, the above recursive computation of the P- and N-positions is not too convenient. Is there a "closed form" formula for them, I wonder?

WIL: Well, the second sequence is not a "spectrum", i.e., there exist no real α, γ such that $b_n = \cup_{n\geq 0}\lfloor n\alpha + \gamma \rfloor$ ($\lfloor x \rfloor$ is the integer part of the real number x), since a necessary — though not sufficient — condition for that is that for all $n \geq 0$, $b_{n+1} - b_n \in \{k, k+1\}$ for some integer k, and here the differences are 2 and 4. Since the sequences are complementary, also the first sequence is not a spectrum…However, the fact that there are two followers and one of them is halving, suggests to consider some sort of binary numeration system.

GERT: The simplest such system is the ordinary positional binary system…Indeed, it appears that $\mathcal{N}$ is the set of all vile numbers, and $\mathcal{P}$ is the set of all dopey numbers.

WIL: What are vile and dopey numbers?

GERT: The vile numbers are those whose binary representations end in an even number of 0s, and the dopey numbers are those that end in an odd number of 0s.

WIL: …No doubt their names are inspired by the evil and odious numbers, those that have an even and an odd number of 1's in their binary representation respectively. To indicate that we count 0s rather than 1s, and only at the tail end, the "ev" and "od" are reversed to "ve" and "do" in "vile" and "dopey". "Evil" and "odious" where coined by Elwyn Berlekamp, John Conway and Richard Guy while composing their famous book *Winning Ways*.

GERT: Precisely. Indeed, the sequence $\{a_n\}_{n\geq 1}$ consists of all alternately evil and odious numbers: a_{2n-1} odious, a_{2n} evil ($n \geq 1$); the same holds for $\{b_n\}_{n\geq 1}$, which is just a shift of $\{a_n\}_{n\geq 1}$…Talking about shifts, let $R_B(m)$ denote the representation of a number m in the numeration system to base B. Since $b_n = 2a_n$, $R_2(b_n)$ is just a *left shift* of $R_2(a_n) : \mathcal{L}R_2(a_n) = R_2(b_n)$ for every $n \geq 0$. Thus $a_9 = 13$, $R_2(a_9) = 1101$, $b_9 = 26$, $R_2(26) = 11010\ldots$ Now the binary representations of 1000 is 1111101000, and therefore that of 2000 is its left shift 11111010000. Therefore Peter, who moves second, can win 1000 and Nathan, who moves first, can win 2000. This solves the puzzle for every positive integer k with a linear-time algorithm in its input size $O(\log k)$. Incidentally we have shown that $\{a_n\}_{n\geq 1}$ is all vile if and only if $\{b_n\}_{n\geq 1}$ is all dopey. Anything else?

WIL: Yes, but before that my palm notices that the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are the sequences A003159 and A036554 respectively in the famous NESLOPEDIA.

GERT: NESLOPEDIA?…Oh, you mean Neil Sloane's Encyclopedia of Integer Sequences, indeed a wonderful tool for doing math, especially discrete math.

WIL: Yes! As to your question "anything else?", it would be nice to play this game in a *sum* of games.

GERT: You mean, given a finite collection of games, Nathan and Peter each select a game at each of their turns and make a legal move therein? The player making the last move in the last surviving game wins?

WIL: Yes on both counts…I think that the P, N tool is not strong enough to decide sums of games.

GERT: …Right. For example, the sum of two MARK games with value $n = 1 \in \mathcal{N}$ is clearly a P-position in their sum, yet the sum of two MARK games with values $n = 1 \in \mathcal{N}$ and $3 \in \mathcal{N}$ is an N-position in their sum. Indeed, the first player can move to (1, 1) and win. However, the sum of P-positions is always a P-position in the sum.

WIL: To analyze sums, it is helpful to compute the Sprague–Grundy function of the component-games, g-function for short, and then compute the Nim-sum (or XOR, or sum over GF(2)) of their g-values: Nim-sum 0 is equivalent to a P-position of a single game; nonzero Nim-sum—to an N-position of a single game…My palm computer produced the following table.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
g	0	1	0	2	1	2	0	1	0	2	0	1	2	1	0	2	1	2	0	1	2	1	0	2

GERT: Notice that there can be no g-value larger than 2, since each game position has only two followers. All the 0s of g (beyond 0) clearly comprise all the dopey numbers — the P-positions. But which are 1s and which are 2s?…Oh, the odious and evil numbers raise their revolting heads again! All the 1s of g comprise all the vile-odious numbers and all the 2s of g comprise all the vile–evil numbers.

WIL: So if I leave you in the 3-game position (1, 3, 4) where will you move to?

GERT: Well, $1 + 3 + 4 = 8$, so I see that you left me in a P-position from which, whatever I do, I'll lose. I expected a more gentlemanly gesture from you.

WIL: You are indeed vile *and* evil!

GERT: I think that we won't be friends anymore after all these unwarranted insults!

WIL: All I meant is that the position I suggested to you is vile–evil: $g(1) \oplus g(3) \oplus g(4) = 1 \oplus 2 \oplus 1 = 2$ (where $\oplus$ denotes Nim-sum), and $g(n) = 2$, is equivalent to a single game where n is vile–evil, as you had pointed out previously. I suggested to you a position from which you can win, namely, with the move $3 \to 2$. Then $g(1) \oplus g(2) \oplus g(4) = 1 \oplus 0 \oplus 1 = 0$.

GERT: Indeed, this move makes the position dopey. I apologize profusely. I erred twice: Instead of *Nim-summing* the g-values, I *summed* their arguments…Sequence A091855 of the NESLOPEDIA seems to confirm that $\{n : g(n) = 1\}$ comprises all vile-odious numbers, and sequence A091785 seems to confirm that $\{n : g(n) = 2\}$ comprises all vile–evil numbers, so $A091855 \cup A091785 = A003159\ldots$

Incidentally, until now we considered *normal* games, that is, the player making the last move wins. What about *misère* play of MARK, where the player making the last move loses?

WIL: Then we can assume that 1 is the last move, so $1 \in \mathcal{P}$. This, with the help of my palm, produces the following results, where $N_n = a_n$ and $P_n = b_n$, $n \geq 0$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	2	3	5	7	8	9	11	12	13	15	17	19	20	21	23	25	27	28	29	31	32	33	35
b_n	1	4	6	10	14	16	18	22	24	26	30	34	38	40	42	46	50	54	56	58	62	64	66	70

GERT: So for misère play of MARK, we seem to have $a_n = \text{mex}\{a_i, b_i : 0 \leq i < n\}$ for $n \geq 0$, $b_0 = 1$ and $b_n = 2a_n$ for $n \geq 1$. The two sequences split the nonnegative integers, and $\mathcal{L}R_2(a_n) = R_2(b_n)$. Is there again a "digital" characterization?…Yes, it appears that $\mathcal{P}$ of MARK's misère play is the set of all dopey numbers except that all dopey powers of 2 are swapped with all vile powers of 2; and $\mathcal{N}$ is the set of all vile numbers except that all vile powers of 2 are swapped with all dopey powers of 2. This provides a linear algorithm in the input size $O(\log k)$ of k for deciding whether $k \in \mathcal{P}$ or $k \in \mathcal{N}$ for misère play of MARK.

WIL: These two sequence are not yet in the NESLOPEDIA, but, as you point out, the $\{a_n\}_{n\ge1}$-sequence is the same as A003159 except for the interchange of vile by dopey powers of 2, and $\{b_n\}_{n\ge1}$ is the same as A036554 except for the interchange of dopey by vile powers of 2…We could compute the g-function for this game, but it would only enable us to play sums of MARK's game, where the end position of each component game is 1. For the sum of MARK's misère plays, the end position of every component game is 0, except for the one played last, whose end position is 1. What do you think?

GERT: I agree, and we better leave the misère sum analysis to the famous misère gurus Thane Plambeck and Aaron Siegel, though the P-positions of normal and misère play agree except that the powers of 2 are swapped. This value-swapping occurs sometimes in swapping between normal and misère play. For example, the renowned mathematician Vladimir Gurvich has recently observed and proved a similar behavior in swapping between normal and misère play in a certain generalization of Wythoff's game…We might now wish to examine the game UPMARK, which is the same as MARK, except that when halving, we round up rather than down. For example, the only follower of 3 is 2.

WIL: …In normal play, where the player making the last move to 0 wins, the position 1 has followers 0 and 1 in UPMARK. The game is thus loopy, and it is the generalized Sprague–Grundy function γ that's needed. In fact, $\gamma(1) = \infty(0)$, and the sum of two UPMARK games with position $(1, 1)$ is clearly a draw.

GERT: Alright, so for the time being, let us consider the version where 1 is the end-position: the player first reaching 1 wins…My PC produced the following table for UPMARK, where $N_n = a_n$ for $n \ge 1$ and $P_n = b_n$ for $n \ge 0$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	2	4	5	6	8	10	12	13	14	16	17	18	20	21	22	24	26	28	29	30	32	34	36
b_n	1	3	7	9	11	15	19	23	25	27	31	33	35	39	41	43	47	51	55	57	59	63	67	71

WIL: It appears that $a_n = \text{mex}\{a_i, b_i : 0 \le i < n\}$ for $n \ge 0$, $b_0 = 1$ and $b_n = 2a_n - 1$ for $n \ge 1$, and the two sequences split the nonnegative integers…It further seems that the P-positions are all odd numbers that are alternately odious and evil, *greedily chosen*, beginning with 1, which is odious. Define the *spite* of k to be evil if k is evil; odious if k is odious. Then "greedily chosen" means the following: $1 \in \mathcal{P}$, and if we have already shown that $2k - 1 \in \mathcal{P}$, then $2k + 1 \in \mathcal{P}$ if $2k - 1$ and $2k + 1$ are of opposite spite. Otherwise $2k + 3 \in \mathcal{P}$. Also odious and evil numbers alternate…Incidentally, neither the a_n- nor the b_n-sequence is in the NESLOPEDIA.

GERT: The set of odd *excluded* numbers from $\{b_n\}$ is

$$S_0 = 5, 13, 17, 21, 29, 37, 45, 49, 53, 61, 65, 69, \ldots,$$

which is not in the NESLOPEDIA either. But the set of b_n-numbers just *before* the excluded ones, namely

$$S_- = 3, 11, 15, 19, 27, 35, 43, 47, 51, 59, 63, 67, \ldots$$

is A131323 in the NESLOPEDIA—all odd numbers whose binary representation ends in an even number of 1's. Evil and odious numbers alternate. On the other hand, the set of b_n-numbers just *after* the excluded ones, namely

$$S_+ = 7, 15, 19, 23, 31, 39, 47, 51, 55, 63, 67, 71 \ldots$$

is not in the NESLOPEDIA. They appear to be all odd numbers whose binary representations end in 11, such that odious and evil numbers alternate. Thus $S_0 = S_- + 2 = \text{A131323} + 2 = S_+ - 2$.

WIL: Is there a concise characterization of $\mathcal{P}$ for UPMARK, I wonder?…Yes! Given a positive odd integer k in binary expansion, we examine its spite. If k and $k - 2$ have opposite spite-parity, then $k \in \mathcal{P}$. Otherwise, $k - 2 \in \mathcal{P}$ and $k \in \mathcal{N}$. This constitutes a linear algorithm in the input size $O(\log k)$ of k for determining whether or not $k \in \mathcal{P}$.

GERT: Notice that S_0 appears to consist of precisely all alternately evil and odious numbers whose binary representation ends in $1^k0^{2n+1}1$, where $n \ge 0$, $k \ge 1$, i.e., a prefix of an odd number of 0s followed by 1. And $\{a_n\}_{n\ge0}$ consists of all even positive integers together with S_0, so $R_2(a_n)$ either ends in 0 or in $0^{2n+1}1$ for every $n \ge 0$. Since $\{a_n\}_{n\ge0}$ and $\{b_n\}_{n\ge0}$ split the nonnegative integers, $R_2(b_n)$ consists of precisely all numbers whose binary representation ends in 1^k, $k > 1$, or in $0^{2k}1$ for every $k \ge 0$. Evil and odious numbers alternate in $\{b_n\}_{n\ge0}$. This provides another linear algorithm.

WIL: Nice. I'd like to break out of the binary numeration system…Let us consider the ternary system first…Define MARK-3 by taking 1 or 2 from n or moving to $\lfloor n/3 \rfloor$. Again the game ends when the integer reaches 0, and the player making the last move wins. Will the ternary system provide a winning strategy?

GERT: Well, here is what my PC produced. As usual we seem to have $a_n = \text{mex}\{a_i, b_i : 0 \le i < n\}$, where now $b_n = 3a_n$, which looks promising. We have $P_n = b_n$ for $n \ge 0$, and $N_n = a_n$ for $n \ge 1$, the sequences split the positive integers for $n \ge 1$…Bravo, the sequence $\{a_n\}_{n\ge1}$ seems to consist of all vile numbers in the ternary numeration system, and therefore $\{b_n\}_{n\ge1}$ seems to consist of all dopey numbers in the ternary numeration system. The latter part follows from $\mathcal{L}R_3(a_n) = R_3(b_n)$. So for every $k \ge 2$, define MARK-k as the game of removing one of $1, 2, \ldots, k - 1$ from n, or moving n to

$\lfloor n/k \rfloor$ (where MARK-2 = MARK). Then the conjecture is that the P-positions and N-positions of MARK-k are the dopey and vile numbers respectively in the numeration system to base k, and $\mathcal{L}R_k(a_n) = R_k(b_n)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	1	2	4	5	7	8	9	10	11	13	14	16	17	18	19	20	22	23	25	26	28	29	31
b_n	0	3	6	12	15	21	24	27	30	33	39	42	48	51	54	57	60	66	69	75	78	84	87	93

WIL: I see that the sequence $\{b_n\}_{n\geq1}$ of MARK-3 is A145204 in the Neslopedia, and $\{a_n\}_{n\geq1}$ is A007417…I have just verified our conjecture for $k = 4$, as can be seen from the next table. In fact, $a_n = \text{mex}\{a_i, b_i : 0 \leq i < n\}$, $b_n = 4a_n$ for all $n \geq 0$. These two sequences, which are not in the Neslopedia, split the positive integers $(n \geq 1)$, $R_4(b_n)$ is dopey $(n \geq 1)$, $R_4(a_n)$ is vile $(n \geq 1)$ in the quaternary numeration system, and $\mathcal{L}R_4(a_n) = R_4(b_n)$. Incidentally, MARK-$k$ resembles SUBTRACTION GAMES, which are take-away games with a finite set of positive integers that can be taken away from the game positions. The main result on subtraction games is that their g-function is periodic. Alas, this does not seem to be the case for MARK-k for any $k \geq 2$…Now we might change the move $\lfloor n/k \rfloor$ to $\lceil n/k \rceil$ as we did for $k = 2$, examine the g-function for MARK-k, look at misère play…But I'd like to consider the variation…

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	1	2	3	5	6	7	9	10	11	13	14	15	16	17	18	19	21	22	23	25	26	27	29
b_n	0	4	8	12	20	24	28	36	40	44	52	56	60	64	68	72	76	84	88	92	100	104	108	116

GERT: I admit that we really had fun, however, I'm getting rather famished…

WIL: But what about proofs?

GERT: Well, we did *experimental* math, a topic in which the eminent guru Doron Zeilberger excels. We might as well leave proofs to the younger generation…for example to Aviezri. I was born October 28, 1929, way before the end of 1929, so it stands to reason that he was born much later in 1929!

WIL: So we'll have to wait a zillion years until we see proofs!

GERT: Don't be a smarty. You know that I meant an exclamation mark, not a factorial—Let us now go to have dinner.

WIL: All I meant is that it will take that guy a zillion years to write up the proofs…Let me provide a short sample proof before we go to dinner. Denote by $C(n) \in \mathcal{P}, \mathcal{N}$ the character of a given $n \in \mathbb{Z}_{\geq0}$, that is its membership $n \in \mathcal{P}$ or $n \in \mathcal{N}$.

Proposition 1. *The game* MARK *is aperiodic.*

Proof. Suppose that there are constants $r, n_0 \in \mathbb{Z}_{\geq1}$ such that $C(n) = C(n + r)$ for all $n \geq n_0$. For $t \in \mathbb{Z}_{\geq1}$, we may assume that $2rt \in \mathcal{P}$, since if $2rt \in \mathcal{N}$, then $4rt \in \mathcal{P}$, so we replace t by $2t$. For t large enough, we have, in addition, $C(2rt) = C(2rt + 2(t + 1)r) = C(4rt + 2r)$ by the assumed periodicity. Now one of the followers of $4rt + 2r$ is half of this number, namely $2rt + r$. Then $C(2rt) = C(2rt + r) = C(4rt + 2r)$. Thus both $4rt + 2r$ and its follower $2rt + r$ are in $\mathcal{P}$, a contradiction. □

GERT: Nice. Since the P-positions are aperiodic, so is the g-function, a fortiori…Thanks for the very nice time. Now we are indeed ready for dinner.