K-Pile Wythoff Games

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Abstract

An important aspect of the classic Wythoff game is that its P positions form a disjoint cover of the positive integers by two sequences. Though generalizations of Wythoff to $K > 2$ piles abound, we believe that the generalization presented here is the first where the P positions form a disjoint cover of the positive integers by K sequences. To achieve this we add a novel ingredient - we allow pile sizes to increase. This leads, inter alia, to games with infinitely many positions, yet every such game ends with no remaining tokens, due to a lexicographic order $\prec$ imposed on the moves. We refer to this **L**exicographic W**ythoff** game as Lythoff(K). We introduce Lythoff(K) in section 2. In section 3 we construct its P positions and prove they form a disjoint cover. In section 4 we extend the moves of Lythoff(K) to depend on a function f that depends on the position we move to, called Lythoff(K_f). In section 5 we study the special case Lythoff(2_f).

1 Introduction

There are many generalizations of Wythoff games. For example, see most of the 150 bibliographic items in 'Wythoff Wisdom' [1]. However, this paper constitutes an unexpected new twist. The positions are K vectors, common enough. The revolutionary aspect of our present contribution is in their *moves* and in the *tiling* of the positive integers by their P-positions. Given a $K \geq 2$ vector of the pile sizes $[a_1, a_2, ..., a_K]$ in nondecreasing order, $0 \leq a_1 \leq a_2 \leq ... \leq a_K$, choose any set S of $K-1$ integers and then for each pile i, either leave the pile untouched, or choose $x \in S$ with $x \leq a_i$, and remove x tokens from the pile. We may have to rearrange the resulting position so as to be nondecreasing. To be legal, the resulting position **B** must satisfy $\mathbf{B} \prec \mathbf{A}$ in *lexicographic* order. For $K = 2$ these rules reproduce the classic Wythoff game. We refer to our **L**exicographic W**ythoff** game as Lythoff(K). We discuss this in more detail, together with clarifying examples, in section 2.

This definition produces a number of unexpected results: $x \in S$ may be negative, in which case the 'remove' becomes *add*. That is, pile sizes may *grow* in size rather than diminish. A consequence is that in Lythoff(K) with $K > 2$ any position T with at least two non-empty piles has an infinite number of sub-positions and the number of moves till the game ends is unbounded from above, yet Lythoff(K) always ends after a finite number of moves. All of this is proved in Theorem 1 below.

We also define and discuss a *variable* Lythoff(K) whose rule set contains a wider set of moves, depending on a function f of the position we move to. We call this generalization Lythoff(K_f). Lythoff(2_f) is discussed in section 5. Notice that Lythoff(2_f) with f a constant, is [5], which itself is a generalization of Wythoff obtained by weakening the constraint of taking equal numbers from both piles.

The generalization of Wythoff to more than two heaps was a long sought-after problem. Some of the more successful generalizations are cited below. For a wider view see [1], section 4.

In [7], the P-positions for a K-Pile game, $K \geq 2$, are constructed using triangular numbers, and the resulting strategy is polynomial-time, whereas most games are either Pspace-complete or Exp-complete. However, the P-positions do not tile the positive integers. There is some resemblance between the proofs of [7] and the present paper.

A quite different generalization: Moores Nim$_k$, [8], is a variation of Nim in which up to k piles can be reduced. Thus Nim$_1$ is Nim. A tractable strategy can be given by expressing the pile sizes in binary as in Nim, but XOR-ing them to the base $k+1$. If this sum (without carries) is 0, we have a P-position. Otherwise, it is an N-position, and a move to 0 wins. No polynomial strategy seems to be known for this game. Another generalization: In Fraenkel, [2].

In [6] it is shown that a natural generalization of Nim to the case of $K > 2$ heaps of sizes $[a_1, a_2, ..., a_K]$ is to either remove any positive number of tokens from a single heap, or remove x_i tokens from each heap simultaneously, subject to the conditions: (i) $x_i > 0$ for some i, (ii) $x_i \leq a_i$ for all i, (iii) $x_1 \oplus x_2 \oplus ... \oplus x_K = 0$, where $\oplus$ denotes Nim-sum, (also known as addition over GF(2), or XOR). The player making the last move wins and the opponent loses. See also [1] section 4. What are the P-positions of this game?

Two conjectures about the P-positions of multiple pile game are formulated in [9] and [10]. See also [4].

2 K-Pile Wythoff Games

Notation: We will use $\mathbf{A}$ to denote a vector of integers $[a_1, a_2, \ldots, a_K]$. When comparing two such vectors we will use the usual lexicographic order. Specifically $\mathbf{B} \prec \mathbf{A}$ if $b_i < a_i$ for the first index i at which they differ.

Definition 1. *A two-player* Lythoff(K) *game is played on $K \geq 2$ piles of tokens. A position in the game is specified by giving the sizes of the piles in nondecreasing order $0 \leq a_1 \leq a_2 \leq \ldots \leq a_K$. A legal move from $\mathbf{A} = [a_1, a_2, \ldots, a_K]$ is to choose any set of $K-1$ integers, S, and then for each pile i, either leave the pile untouched, or choose $x \in S$ with $x \leq a_i$, and remove x tokens from the pile. We may have to rearrange the resulting position so as to be nondecreasing. To be legal, the resulting position $\mathbf{B}$ must satisfy $\mathbf{B} \prec \mathbf{A}$.*

Note 1: Reducing a pile to size 0 does not change the nature of the game. It is still a K-Pile game, not a (K-1)-Pile game. So the set S is still of size $K-1$, not $K-2$ and therefore the move directly to $\mathbf{0}$ is now legal (unless we are already at $\mathbf{0}$).

Note 2: We explicitly allow S to contain negative integers (which implies adding tokens to a pile). The restriction $\mathbf{B} \prec \mathbf{A}$ ensures that each move decreases the lexicographic order. Theorem 1 below shows that each game ends in a finite number of moves since all pile sizes get reduced to zero.

2.1 Examples

We give some examples of positions and moves for $K = 3$

- $[0, 0, 0]$ has no legal moves, since it is first in lexicographical order.
- $[0, 10, 30]$ has a legal move to $[0, 0, 0]$ as follows: For the set S choose $S = \{10, 30\}$. Then leave the first pile untouched, remove $10 \in S$ from the second pile and $30 \in S$ from the third pile.
- $[0, 10, 30]$ has a legal move to $[0, 9, 50]$ as follows: For the set S choose $S = \{1, -20\}$. Then leave the first pile untouched, remove $1 \in S$ from the second pile and remove $-20 \in S$ from the third pile (thus adding

+20). Since $[0,9,50] < [0,10,30]$ in lexicographical order, the move is legal.

- $[10,10,20]$ has a legal move to $[0,0,0]$ as follows: Choose $S = \{10,20\}$. Remove $10 \in S$ from the first and second piles and $20 \in S$ from the third pile.
- $[1,2,3]$ has no legal move to $[0,0,0]$ since such a move would require removing 1 from the first pile, two from the second and three from the third and therefore would require a *three* element set S.

Note: Though not immediately obvious, Lythoff(2) is just the classical Wythoff game. This is because the 1-element set S must contain a positive integer if the lexicographical order is to decrease and therefore we can only leave a pile untouched or remove the same positive number of tokens from both piles.

One might wonder if the un-natural permission to increase pile size actually "adds" anything to the game. Might these moves be reversible? To show that this is not so, below are the first 13 P positions of Lythoff(3) versus the alternate version that disallows moves that increase the size of a pile. Only the last row differs. The position $[23,39,58]$ is a P position in the alternate game, but in our version it has a legal move to the previous P position $[20,37,53]$ via $S = \{-14,38\}$. We remove -14 tokens from the first and second piles and 38 tokens from the third pile.

Lythoff(3)

a_1	a_2	a_3
0	0	0
1	2	3
4	7	10
5	9	13
6	11	16
8	15	22
12	21	30
14	25	36
17	29	41
18	31	44
19	34	49
20	37	53
23	42	61

Alternate definition

a_1	a_2	a_3
0	0	0
1	2	3
4	7	10
5	9	13
6	11	16
8	15	22
12	21	30
14	25	36
17	29	41
18	31	44
19	34	49
20	37	53
23	39	58

We note that in Lythoff(K) with $K > 2$ any position T with at least two non-empty piles has an infinite number of sub-positions and there is no upper bound on the number of moves till the game ends. This is true because if the two piles have sizes $0 < a \leq b$ then we can choose the two integers 1 and $-n < 0$ to be elements of S so the move $a \to a - 1$, $b \to b + n$ is legal for all $n > 0$. We therefore need the following theorem:

Theorem 1. *Lythoff(K) always ends after a finite number of moves.*

Proof. Consider Lythoff(K) for some fixed value K. We will prove by induction on n that when both players limit their moves to a fixed set of n piles, leaving the other $K - n$ piles untouched, those n piles will all have size zero after a finite number of moves. The case $n = K$ then proves the theorem.

For $n = 1$ it is obvious that since all other piles are unchanged, each move must decrease the size of the single pile, which can only happen a finite number of times. Given $n + 1$ piles, let s be the smallest pile size and let i be a pile whose size is s. If we leave pile i untouched then we are making moves on a fixed set of n piles. By the induction hypothesis after a finite number of moves all their sizes will be zero. At that point we will have to reduce the i'th pile. Whether we do this only at the end or at some intermediate step in either case the size of the smallest pile among the $n + 1$ has been reduced. Since this can only happen at most s times, all $n + 1$ piles will have size zero in a finite number of steps, thus proving the induction. □

One might wonder why we consider Lythoff(K) a natural generalization of Wythoff. We have already shown that Lythoff(2) is indeed the usual Wythoff game, but in view of the fact that for $K > 2$ the K-Pile game isn't even a short game (a short game may only have a finite number of sub-positions) and thus isn't even in the same class as the Wythoff game, in what sense is it a natural extension? Isn't the K-Pile game defined in section 5, Problem 11 of [3] more natural? The reason we feel our current definition best retains the flavor of the original Wythoff game is the fact that, as in the original game, the pile sizes of its P-Positions form a disjoint cover of the natural numbers. We prove this statement in the next section. To the best of our knowledge this does not occur for any other generalizations to K piles.

3 P-Positions of K-Pile Wythoff Games

Definition 2. *For any set of numbers S and a number x we define the shifted set, $x + S$, to be $\{ x + s \mid s \in S \}$. For two sets S and T we define $S + T = \{ s + t \mid s \in S, t \in T \}$.*

Definition 3. *For a non-decreasing sequence of numbers $\mathbf{A} = [a_1, a_2, \ldots, a_K]$ we define the difference set, $D(\mathbf{A})$, to be $\{ a_j - a_i \mid 1 \leq i < j \leq K \}$. Note that $D(\mathbf{A})$ contains only non-negative numbers.*

Notation:

1. The set of components $a_1, a_2, ..., a_K$ of the vector $\mathbf{A} = [a_1, a_2, \ldots, a_K]$ are denoted *set*($\mathbf{A}$).

2. To enhance readability we use the following conventions in the notation below: Subscripts i, j, k, l denote the index of a pile in a position, while superscripts n, m denote different positions.

Lemma 1. *If $\mathbf{B} \prec \mathbf{A}$ and $\mathbf{A} \cap \mathbf{B} \neq \varnothing$ then there exists a legal move $\mathbf{A} \to \mathbf{B}$.*

Proof. If $\mathbf{A} \cap \mathbf{B} \neq \varnothing$ there exist i, j such that $a_i = b_j$. We can then leave pile i untouched and choose a set of $K - 1$ integers to move the other piles from $\mathbf{A}$ to $\mathbf{B}$. So there exists a legal move from $\mathbf{A} \to \mathbf{B}$. □

Lemma 2. *If $\mathbf{B} \prec \mathbf{A}$ and $\mathbf{A} \cap \mathbf{B} = \varnothing$ then there exists a legal move $\mathbf{A} \to \mathbf{B}$ if and only if there exist i, j such that $a_j - a_i \in D(\mathbf{B})$.*

Proof. Since $\mathbf{B} \cap \mathbf{A} = \varnothing$ no pile in $\mathbf{A}$ is in $\mathbf{B}$. Since S only has $K - 1$ elements a legal move exists if and only if a single number $s \in S$ can be used to move two piles of $\mathbf{A}$ to two piles of $\mathbf{B}$. This can happen if and only if there exist indices i, j, k, l such that $a_i - b_k = a_j - b_l$, which happens if and only if $a_j - a_i = b_l - b_k$. By switching indices $i \leftrightarrow j$ and $k \leftrightarrow l$ if necessary, we can assume $b_k < b_l$ so $b_l - b_k \in D(\mathbf{B})$ as required. □

Motivation: Due to Lemmas 1 and 2 we want to construct the P-positions recursively so that the n'th P-position has no piles of the same size as in any of the previous P-positions and that the differences in the sizes of its piles don't match any previous differences. This is achieved in the following construction.

Construction 1. *We recursively construct a set of* candidate *P-positions,* $\mathbf{P^n}$ *and the sets* X^n*,* D^n *as follows:*

$$\mathbf{P^0} = [0, 0, \ldots, 0]$$

and then for all $n > 0$

$$X^n = \bigcup_{0 \le m < n} \text{set}(\mathbf{P^m})$$

$$D^n = \bigcup_{0 \le m < n} D(\mathbf{P^m})$$

and then, for a given n*, define*

$$Q_1^n = X^n$$

and then recursively for each $1 \le i \le K$

$$p_i^n = \text{mex}\{Q_i^n\}$$

$$Q_{i+1}^n = Q_i^n \cup (p_i^n + D^n) = X^n \bigcup_{j \le i} (p_j^n + D^n).$$

Lemma 3. *For all* $n > 0$ *we have* $p_1^n > p_1^{n-1}$ *and for all* $1 \le i < K$ *we have* $p_{i+1}^n > p_i^n$*. In particular,* $\mathbf{P^n}$ *lists the pile sizes in the correct order.*

Proof. Since $X^n \supset X^{n-1}$ we have $p_1^n \ge p_1^{n-1}$. Since $p_i^{n-1} \in \mathbf{P^{n-1}} \subset X^n$ we have $p_1^n \ne p_1^{n-1}$. So $p_1^n > p_1^{n-1}$.

Since $Q_{i+1}^n \supset Q_i^n$ we have $p_{i+1}^n \ge p_i^n$. Since $0 \in D^1 \subset D^n$ for all $n > 0$, we have $p_i^n = p_i^n + 0 \in Q_{i+1}^n$ and therefore $p_{i+1}^n \ne p_i^n$. So $p_{i+1}^n > p_i^n$. □

Lemma 4. *The sequences* $\{p_1^n\}, \{p_2^n\}, \ldots, \{p_K^n\}$ *for* $n > 0$ *form a disjoint cover of the positive integers.*

Proof. By the definition of $p_1^n = \text{mex}\{X^n\}$ it is clear that $\{X^n\}_{n>0}$ is a covering. For all i, $Q_i^n \supset Q_1^n = X^n$ so $p_i^n \notin X^n$. By Lemma 3 $p_{i+1}^n > p_i^n$. So the cover is disjoint. □

Lemma 5. *For all* $n > m$ *there is no legal move from* $\mathbf{P^n}$ *to* $\mathbf{P^m}$*.*

Proof. By Lemma 4 no pile in $\mathbf{P^n}$ has the same number of tokens as a pile in $\mathbf{P^m}$. So by Lemma 2 a legal move exists only if there exist i, j with $p_i^n - p_j^n \in D(\mathbf{P^m})$. But then from Lemma 3 we have $i > j$. So $p_i^n \in (p_j^n + D^n) \subset Q_i^n$ contradicting $p_i^n = \text{mex}\{Q_i^n\}$. □

Definition 4. *The set of candidate P-positions, P, is $\{\mathbf{P^n}\}_{n\geq 0}$. The set of candidate N-positions, N, is the complement of P.*

Lemma 6. *For every position in N there exists a legal move to some position in P.*

Proof. Let $\mathbf{V} \in N$. By Lemma 4 $v_1 = p_k^n$ for some n, k. If $k > 1$ then by Lemma 3 $v_1 = p_k^n > p_1^n$ so $\mathbf{P^n} \prec \mathbf{V}$. So the move $\mathbf{V} \to \mathbf{P^n}$ leaving v_1 unchanged is legal.

If $k = 1$ then $v_1 = p_1^n$ and there exists a first i such that $v_j = p_j^n$ for all $j < i$ and $v_i \neq p_i^n$ (since $\mathbf{V} \neq \mathbf{P^n}$). If $p_i^n < v_i$ then again the move $\mathbf{V} \to \mathbf{P^n}$ is legal.

If $v_i < p_i^n$ then, since $p_i^n = \operatorname{mex}\{Q_i^n\}$,

$$v_i \in Q_i^n = X^n \bigcup_{j<i} (p_j^n + D^n) = X^n \bigcup_{j<i} (v_j + D^n).$$

If $v_i \in X^n$ then v_i equals some p_l^m and since $v_1 = p_1^n > p_1^m$, by Lemma 3 the move $\mathbf{V} \to \mathbf{P^m}$ leaving v_i untouched is legal.

Finally, if

$$v_i \in \bigcup_{j<i} (v_j + D^n),$$

then $v_i - v_j \in D(\mathbf{P^m})$ for some $m < n$. Since $v_1 = p_1^n > p_1^m$, by lemma (2) the move $\mathbf{V} \to \mathbf{P^m}$ is legal. □

Theorem 2. *The P-positions other than* $\mathbf{0}$ *of Lythoff(K) are given recursively by construction (1). The corresponding sequences $\{p_1^n\}, \{p_2^n\}, \ldots, \{p_K^n\}$ for $n > 0$ form a disjoint cover of the positive integers.*

Proof. By Lemmas 5 and 6 P and N are the P-positions and N-positions of the game. By Lemma 4 the P-positions other than $\mathbf{0}$ form a disjoint cover of the positive integers. □

4 Variable K-Pile Wythoff Games

We now extend the above definitions and theorems to a wider class of games which we call *Variable* Lythoff(K) games. As motivation for the extension we first reformulate the definition of a legal move in a Lythoff(K).

Definition 5. *A move from* $\mathbf{A}$ *to* $\mathbf{B}$ *in Lythoff(K) is legal if* $\mathbf{B} \prec \mathbf{A}$ *and either* $\mathbf{A} \cap \mathbf{B} \neq \varnothing$ *or* $D(\mathbf{A}) \cap D(\mathbf{B}) \neq \varnothing$

By Lemmas 1 and 2 the new definition is equivalent to the old one given in Definition 1.

Before proceeding to the definition of a variable Lythoff(K) game we will also need the following

Definition 6. *We define the open interval* (i, j) *as the set of integers* $\{ k \mid i < k < j \}$. *We define simliarly the closed and half open intervals* $[i, j], [i, j), (i, j]$.

And finally

Definition 7. *A* Variable Lythoff(K) game *has the same positions as those of Lythoff(K). In addition there is given a function* $f : \mathbb{N}^{\mathbb{K}} \to \mathbb{N}^{+}$. *A move from* $\mathbf{A}$ *to* $\mathbf{B}$ *is legal if* $\mathbf{B} \prec \mathbf{A}$ *and either* $\mathbf{A} \cap \mathbf{B} \neq \varnothing$ *or* $D(\mathbf{A}) \cap \Big(D(\mathbf{B}) + \big[0, f(\mathbf{B})\big) \Big) \neq \varnothing$.

We denote the variable Lythoff(K) game defined by f as Lythoff(K_f).

An example Lythoff(K_f) game for $K = 3$ is given by $f(\mathbf{B}) \equiv f([B_1, B_2, B_3]) \equiv 1 + B_3 - B_2$. We list the first few positions $\mathbf{B}$ and the values of $D(\mathbf{B})$, $f(\mathbf{B})$ and $D(\mathbf{B}) + \big[0, f(\mathbf{B})\big)$. The *Legality* column specifies whether a move from $\mathbf{A} = [2, 4, 6]$ to $\mathbf{B}$ is legal and, if so, at least one reason why. Note that $D(\mathbf{A}) = \{2\}$. Since the number of possible sub-positions is infinite, the below table obviously doesn't contain *all* legal moves from $\mathbf{A}$.

Variable 3-Pile Wythoff						
B_1	B_2	B_3	$D(\mathbf{B})$	$f(\mathbf{B}) = 1 + B_3 - B_2$	$D(\mathbf{B}) + \big[0, f(\mathbf{B})\big)$	Legality
0	0	0	$\{0\}$	1	$\{0\}$	Illegal
0	0	1	$\{0,1\}$	2	$\{0,1,2\}$	D's intersect
0	0	2	$\{0,2\}$	3	$\{0,1,2,3,4\}$	D's intersect
0	1	1	$\{0,1\}$	1	$\{0,1\}$	Illegal
0	1	2	$\{1\}$	2	$\{1,2\}$	D's intersect
0	2	2	$\{0,2\}$	1	$\{0,2\}$	D's intersect
1	1	1	$\{0\}$	1	$\{0\}$	Illegal
1	1	2	$\{0,1\}$	2	$\{0,1,2\}$	D's intersect
1	2	2	$\{0,1\}$	1	$\{0,1\}$	A and B intersect
2	2	2	$\{0\}$	1	$\{0\}$	A and B intersect

Lemma 7. *If* $\mathbf{B} \prec \mathbf{A}$ *and* $\mathbf{A} \cap \mathbf{B} = \varnothing$ *then there exists a legal move* $\mathbf{A} \to \mathbf{B}$ *if and only if there exist* $i < j$ *such that* $a_j - a_i \in D(\mathbf{B}) + \big[0, f(B)\big)$.

Proof. Immediate from the definition of a legal move. □

Construction 2. *We duplicate construction (1) with a single change. We replace the definition of* D^n *with*

$$D^n = \bigcup_{0 \le m < n} D(\mathbf{P^m}) + \big[0, f(\mathbf{P^m})\big).$$

Theorem 3. *The P-positions other than* $\mathbf{0}$ *of Lythoff(*K_f*) are given recursively by construction (2) and form a disjoint cover of the positive integers.*

Proof. The corresponding proofs of Lemmas 3-6 and Theorem 2 remain unchanged after replacing Lemma 2 with Lemma 7. □

5 Variable 2-Pile Wythoff Games and Complementary Sequences

Theorem 4. *The P-positions of Lythoff(*2_f*) are given by* $p_1^n = \operatorname{mex}\{X^n\}$ *and* $p_2^n = p_1^n + \sum_{m=0}^{n-1} f(\mathbf{P^m})$.

Proof. $p_1^n = \operatorname{mex}\{X^n\}$ follows directly from construction 2 and Theorem 3. We prove $p_2^n = p_1^n + \sum_{m=0}^{n-1} f(\mathbf{P^m})$ by induction. Since, by convention, the empty sum is zero, we have $p_2^0 = p_1^0 + 0 = 0$ so $\mathbf{P^0} = [0, 0]$ as required. For $n > 0$ we have

$$\begin{aligned}
D^n &= \bigcup_{0 \le m < n} D(\mathbf{P^m}) + \big[0, f(\mathbf{P^m})\big) \\
&= \bigcup_{0 \le m < n} \sum_{s=0}^{m-1} f(\mathbf{P^s}) + \big[0, f(\mathbf{P^m})\big) \\
&= \bigcup_{0 \le m < n} \left[\sum_{s=0}^{m-1} f(\mathbf{P^s}), \sum_{s=0}^{m-1} f(\mathbf{P^s}) + f(\mathbf{P^m})\right) \\
&= \bigcup_{0 \le m < n} \left[\sum_{s=0}^{m-1} f(\mathbf{P^s}), \sum_{s=0}^{m} f(\mathbf{P^s})\right) \\
&= \left[0, \sum_{s=0}^{n-1} f(\mathbf{P^s})\right).
\end{aligned}$$

from Theorem 3, looking back at construction (1) we have

$$p_2^n = \text{mex}\{Q_2^n\} = \text{mex}\{X^n \cup (p_1^n + D^n)\}$$
$$= \text{mex}\left\{X^n \cup p_1^n + \left[0, \sum_{s=0}^{n-1} f(\mathbf{P^s})\right)\right\}$$
$$= \text{mex}\left\{X^n \cup \left[p_1^n\,,\, p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})\right)\right\}.$$

But X^n contains all the integers less than p_1^n and $\left[p_1^n\,,\, p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})\right)$ contains the rest of the integers up to $p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})$, so

$$p_2^n = p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})$$

(the last equality is true because, by induction, all elements of X^n are less than $p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})$). □

Note: If we choose f to be a constant function, $f(\mathbf{X}) = a$ for all $\mathbf{X}$ with $a > 0$, then the P positions of Lythoff(2_f) are

$$p_1^n = \text{mex}\left\{X^N\right\},\, p_2^n = p_1^n + na$$

which are the same as the P positions of the generalized Wythoff game introduced in [5].

Definition 8. *An ordered pair of sequences,* $(\{y^n\}_{n>0}, \{z^n\}_{n>0})$ *which form a disjoint cover of* $\mathbb{N}^+$ *is* monotonic *if* $y^1 < z^1$ *and for all* $n > 0$*,* $z^{n+1} - y^{n+1} > z^n - y^n$*.*

It is obvious that $y^1 = 1$ and $y^n < z^n$ for all n and therefore that $y^n = \text{mex}\{\{y^i\}_{i<n}, \{z^i\}_{i<n}\}$.

From Theorem 4 we know that the P positions, $(\{p_1^n\}_{n>0}, \{p_2^n\}_{n>0})$, other than $[0, 0]$ of Lythoff(2_f) form a monotonic disjoint cover of $\mathbb{N}^+$. The converse is also true:

Theorem 5. *Every monotonic disjoint cover of* $\mathbb{N}^+$*,* $(\{y^n\}_{n>0}, \{z^n\}_{n>0})$*, is the P positions other than* $[0, 0]$ *of some Lythoff(*2_f*).*

Proof. Define $f(0,0) = z^1 - y^1 = z^1 - 1 > 0$. For all $n \in \mathbb{N}^+$ either $n = y^i$ or $n = z^i$. If $n = y^i$ define $f(y^i, z^i) = (z^{i+1} - y^{i+1}) - (z^i - y^i)$. For all other (n, m) the value of $f(n, m)$ will turn out to be irrelevant, so define $f(n, m)$ to be an arbitrary positive integer (for example, 1). Then, since $(\{y^n\}, \{z^n\})$ is monotonic, f defines a function from $\mathbb{N}^2 \to \mathbb{N}^+$ and therefore defines a game Lythoff(2_f). If $[a^n, b^n]$ are the P-positions other than $[0, 0]$ of Lythoff(2_f) then we prove by induction that for all $n > 0$, $a^n = y^n$ and $b^n = z^n$.

For the case $n = 1$ we have $a^1 = 1 = y^1$ and $b^1 = a^1 + f(0) = a^1 + z^1 - y^1 = z^1$. Assume by induction that $a^m = y^m$ and $b^m = z^m$ for all $m < n$. Then

$$\begin{aligned} a^n &= \operatorname{mex}\{\{a^m\}_{m<n}, \{b^m\}_{m<n}\} \\ &= \operatorname{mex}\{\{y^m\}_{m<n}, \{z^m\}_{m<n}\} = y^n, \end{aligned}$$

and

$$\begin{aligned} b^n &= a^n + \sum_{m=0}^{n-1} f(a^m, b^m) \\ &= a^n - a^{n-1} + a^{n-1} + \sum_{m=0}^{n-2} f(a^m, b^m) + f(a^{n-1}, b^{n-1}) \\ &= a^n - a^{n-1} + b^{n-1} + f(a^{n-1}, b^{n-1}) \\ &= y^n - y^{n-1} + z^{n-1} + f(a^{n-1}, B^{n-1}) \\ &= y^n - y^{n-1} + z^{n-1} + ((z^n - y^n) - (z^{n-1} - y^{n-1})) \\ &= z^n. \end{aligned}$$

□

Corollary 1. *In particular, if two Beatty Sequences* $\lfloor np + \beta_p \rfloor_{n>0}$, $\lfloor nq + \beta_q \rfloor_{n>0}$ *form a disjoint cover of* $\mathbb{N}^+$ *with* $q \geq 3$ *then they are the* P *positions of some Lythoff(*2_f*).*

Proof. First we note that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$. Since $1/p + 1/q = 1$ we have $p < 2$. But then

$$\begin{aligned} \lfloor (n+1)q + \beta_q \rfloor - \lfloor (n+1)p + \beta_p \rfloor &= \lfloor nq + \beta_q + q \rfloor - \lfloor np + \beta_p + p \rfloor \\ &\geq \lfloor nq + \beta_q \rfloor + \lfloor q \rfloor - (\lfloor np + \beta_p \rfloor + \lfloor p \rfloor + 1) \\ &= \lfloor nq + \beta_q \rfloor + \lfloor np + \beta_p \rfloor + \lfloor q \rfloor - \lfloor p \rfloor - 1 \\ &\geq \lfloor nq + \beta_q \rfloor + \lfloor np + \beta_p \rfloor + 1. \end{aligned}$$

So the Beatty sequences form a monotonic disjoint cover of $\mathbb{N}^+$ and the result follows immediately from Theorem 5. □

6 Further Work

It would seem that many other token taking games that have been discussed in the literature would also be amenable to "Lexification" which might lead to interesting games in their own right. For classic Nim, where each move is restricted to taking tokens from a single pile, "Lexification" would add nothing. Similarly, we noticed that "Lexification" has no effect on the classic 2-Pile Wythoff game. But there are many games which allow taking from more than two piles. For example in [6] it is shown that a natural generalization of Nim to the case of $K > 2$ heaps of sizes $[a_1, a_2, ..., a_K]$ is to either remove any positive number of tokens from a single heap, or remove x_i tokens from each heap simultaneously, subject to the conditions: (i) $x_i > 0$ for some i, (ii) $x_i \leq a_i$ for all i, (iii) $x_1 \oplus x_2 \oplus ... \oplus x_K = 0$, where $\oplus$ denotes Nim-sum. This game has some interesting open conjectures regarding its P-positions. We can "Lexify" this game by allowing some of the x_i to be negative (thus adding tokens) and requiring that moves be to positions which are earlier in lexicographic order.

Additional directions of research specific to Lythoff games would be: To investigate Lythoff(2_f) for f linear, or, more generally, Lythoff(K_f) for suitable functions f that produce 'interesting' games; Misère play; Reducing the time complexity of calculating the P positions; Computing the Sprague-Grundy function to enable play of several games simultaneously.

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