Take-away games on Beatty's Theorem and the notion of k-invariance

Aviezri S. Fraenkel,
Dept. of Computer Science and Applied Mathematics,
Weizmann Institute of Science,
Rehovot 76100, Israel; fraenkel@wisdom.weizmann.ac.il
Urban Larsson,
University of Goteborg and Chalmers University of Technology;
urban.larsson@yahoo.se

January 26, 2015

Abstract

We formulate three reasonably short game rules for three two-pile take-away games, which share one and the same set of P-positions. This set is comprised of a pair of complementary homogeneous Beatty sequences together with $(0, 0)$. We relate the succinctness of the game rules with the complexity of the P-positions by means of a notion dubbed k-invariance.

1 Introduction

Let us recall the rules of d-Wythoff [10], d a fixed positive integer. The available positions are (x, y), x and y non-negative integers. The legal moves are

(I) Nim type: $(x, y) \to (x - t, y)$, if $x - t \geq 0$ and $(x, y) \to (x, y - t)$, if $y - t \geq 0$; $t > 0$.

(II) Extended diagonal type: $(x, y) \to (x - s, y - t)$ if $|t - s| < d$ and $x - s \geq 0,\ y - t \geq 0$; $s > 0$, $t > 0$.

This game is a so-called impartial take-away game [2], vol. 1. We restrict attention to *normal* play, that is, the player first unable to move loses. For our games it means that the player called upon to move from $(0, 0)$ loses.

Rules (I) and (II) imply that d-Wythoff is a so-called *invariant* [5, 16] (take-away) game, that is, each available move is legal from any position as long as the resulting position has non-negative coordinates. Every move in any invariant game is an *invariant move*. In this note we study another type of take-away game, where certain positions have some local restrictions on the set of otherwise invariant moves. Such games are called *variant* [5, 16]. We define these notions in Section 4.

Central to our investigation is Beatty's Theorem [1] (predated by Lord Rayleigh [19]): Let $\beta > 2$ be an irrational number and define its *complement*, $\hat{\beta}$, by $\hat{\beta}^{-1} + \beta^{-1} = 1$ so that $\hat{\beta} = \beta/(\beta - 1)$. This clearly implies $1 < \hat{\beta} < 2 < \beta$. Let $A_n = \lfloor n\hat{\beta} \rfloor$, $B_n = \lfloor n\beta \rfloor$, $A = \cup_{n\geq 1}\{A_n\}$, $B = \cup_{n\geq 1}\{B_n\}$. Beatty's Theorem then asserts that A and B are *complementary* sets, that is, $A \cup B = \mathbb{Z}_{\geq 1}$, $A \cap B = \emptyset$. Since $\beta > \hat{\beta} > 1$, the (homogeneous) *Beatty sequences* (A_n) and (B_n) are strictly increasing.

1.1 Three Games

We formulate three game rules. Let $\beta > 2$ be a fixed irrational and let $d = \lfloor \beta \rfloor$. Fix a pair of non-negative integers (x, y). Recall that $B_n = \lfloor n\beta \rfloor$ for all n:

(G1) The moves are as in Nim on two piles (I), except that, if $B \cap \{x, y\} = \emptyset$, then in addition to the Nim-type move a player may also take away $s \in \{0, \dots, d\}$ from the other pile in the same move. This game is denoted by β*-Nim*.

(G2) The moves are as in d-Wythoff, subject to (I) and (II), except that if $B \cap \{x, y\} \neq \emptyset$, then only Nim-type moves (I) are permitted. This game is denoted by β*-Wynim*.

(G3) The moves are as in d-Wythoff, subject to (I) and (II), except that if $B \cap \{x, y\} \neq \emptyset$, then the pair (s, t), with s and t as in (II), cannot belong to the pair of β*-triangles* defined by

$$\{(x, y), (y, x) \mid (x, y) \in \{(1, \lfloor \beta \rfloor), (2, \lfloor \beta \rfloor), (2, \lfloor \beta \rfloor + 1)\}\}.$$

This game is denoted by β^{T}*-Wynim*.

The name Wynim derives from Wythoff-Nim; in (G3) the T in β^{T} stands for Triangles. The main result of this note is as follows:

Theorem 1. *The set of P-positions of β-Nim, β-Wynim and β^{T}-Wynim is the same. It is*

$$\mathcal{P} := \bigcup_{n\geq 0}\{(A_n, B_n)\} \bigcup_{n\geq 0}\{(B_n, A_n)\},$$

where $A_n = \lfloor n\hat{\beta}\rfloor$, $B_n = \lfloor n\beta\rfloor$.

We prove this result in Section 3. In Section 4 we develop the distinction between invariant and variant games and relate our findings to certain complexity issues. In the section to come we give some examples.

2 Examples and tables of P-Positions

In the proof of the main result and in the examples of sets of P-positions to come, we use the following illustrative notation:

Notation 1. For every $n \geq 0$,
(i) $\Delta A_n := A_{n+1} - A_n$, $\Delta B_n := B_{n+1} - B_n$ are the *gaps*.
(ii) $\Delta_n := B_n - A_n$.
(iii) $\Delta_n^2 := \Delta_{n+1} - \Delta_n$.

For some (invariant) take-away games on two heaps where short formulas for both the rules and the P-positions are known, such as [10, 12, 14], the coordinates of the P-positions are defined via certain algebraic numbers together with the floor function. Our first example rather uses a well known transcendental number.

Example 1. In the game of π-Wynim, a player may move as in Nim on two piles (I), or, if the position does not contain a coordinate of the form $\lfloor \pi n\rfloor$, deviate at most $\lfloor\pi\rfloor - 1 = 2$ positions from the 'main diagonal' as given by the game d-Wythoff, that is use (II) with $d = 3$. The result of this note implies that the P-positions of this game are the set

$$\cup_{n\geq 0}\{(\lfloor\hat{\pi}n\rfloor, \lfloor\pi n\rfloor), (\lfloor\pi n\rfloor, \lfloor\hat{\pi}n\rfloor)\},$$

the first few of which are displayed in Table 1.

Example 2. Example 1 illustrates Theorem 1 for a member of our second game family, β-Wynim. A further example: Let $d = 2$ in the formula $\beta = (2 + d + \sqrt{d^2 + 4})/2$ (see d-Wythoff and paper [10]) and with $\hat{\beta} = \beta - d$. Then $\beta = \sqrt{2} + 2$, $\hat{\beta} = \beta - 2$; note that $\lfloor\beta\rfloor = 3$ as in Example 1. The first few P-positions are shown in Table 2. Since $\beta - \hat{\beta} = d = 2$, we have $\Delta_n = dn = 2n$, so $\Delta_n^2 = d = 2$ for all $n \geq 0$, and the β-triangles, as in (G3), for both these games, will be $\{(1, 3), (2, 3), (2, 4)\} \cup \{(3, 1), (3, 2), (4, 2)\}$.

Table 1: The first few P-positions (A_n, B_n) for β-Nim, β-Wynim and β^{T}-Wynim, $\beta = \pi = 3.14159\ldots$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
A_n	0	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	24	26	27	29	30	32	33	35	36	38	39	41	42
B_n	0	3	6	9	12	15	18	21	25	28	31	34	37	40	43	47	50	53	56	59	62	65	69	72	75	78	81	84	87	91
Δ_n	0	2	4	5	7	8	10	11	14	15	17	18	20	21	23	25	27	29	30	32	33	35	37	39	40	42	43	45	46	49
Δ_n^2	2	2	1	2	1	2	1	3	1	2	1	2	1	2	2	2	2	1	2	1	2	2	2	1	2	1	2	1	3	1

Remark 1. It is remarkable that, for $\beta = (2 + d + \sqrt{d^2 + 4})/2$, d-Wythoff has the same set of P-positions as our three games. In particular, for $d = 1$, 1-Wythoff is the classical Wythoff game [2]. For $d = 2$, the first few P-positions of the games are displayed in Table 2. In [4] it was shown that from the classical Wythoff game no move can be deleted while preserving the set of P-positions of the classical Wythoff game. In the present note, Wythoff moves were deleted, and the P-positions are still preserved. The difference is that in [4] only *invariant* moves were permitted. See section 4 for more on the latter topic.

3 Proof of the Main Result

We preface the proof of Theorem 1 by collecting some facts on the sets $\{A_n\}$ and $\{B_n\}$.

Proposition 1. *For every* $n \geq 0$,
(i) *The only possible gap pairs are*

$$(\Delta A_n, \Delta B_n) \in \{(1, \lfloor\beta\rfloor),\ (1, \lfloor\beta\rfloor + 1),\ (2, \lfloor\beta\rfloor),\ (2, \lfloor\beta\rfloor + 1)\}.$$

(ii) $\Delta_n^2 = \Delta B_n - \Delta A_n$.
(iii) $\Delta_n^2 \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$.

Table 2: The first few P-positions (A_n, B_n) for 2-Wythoff, β-Nim β-Wynim and β^{T}-Wynim; β as in Example 2.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
A_n	0	1	2	4	5	7	8	9	11	12	14	15	16	18	19	21	22	24	25	26	28	29	31	32	33	35	36	38
B_n	0	3	6	10	13	17	20	23	27	30	34	37	40	44	47	51	54	58	61	64	68	71	75	78	81	85	88	92

Proof. (i) This is a well known result.

(ii) $\Delta_n^2 = (B_{n+1} - A_{n+1}) - (B_n - A_n) = (B_{n+1} - B_n) - (A_{n+1} - A_n) = \Delta B_n - \Delta A_n$.

(iii) Follows directly from (i) and (ii). ■

Example 3. Notice that in Example 1, Table 1, Δ_n^2 assumes all three possible values $\{1, 2, 3\} = \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$. In Example 2, Δ_n^2 assumes only the value $2 = \lfloor\beta\rfloor - 1$.

Proof of Theorem 1. Since our games are acyclic, it suffices to demonstrate the following two properties for each game:

P $\to$ N: Every move from any position of the form

$$(A_n, B_n) \text{ or } (B_n, A_n) \tag{1}$$

results in a position outside (1).

N $\to$ P: Given any position outside (1), there exists a move into (1).

For the direction P $\to$ N we use the same argument for the games (I) β-Nim and (II) β-Wynim, namely: Suppose that we play from a position of the form (1). The game rules imply that only Nim type moves (I) are permitted so that by complementarity, there is no move to a position of the same form.

For game (III) β^{T}-Wynim, we have to show that both (i) $(A_n, B_n) \to (A_m, B_m)$ and (ii) *cross moves* $(A_n, B_n) \to (B_m, A_m)$ are blocked for every $0 \le m < n$.

(i) By Proposition 1, $(B_n - B_m) - (A_n - A_m) = \Delta_n - \Delta_m \ge \Delta_n - \Delta_{n-1} = \Delta^2_{n-1} \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$ where the $\ge$ follows from the fact that $\beta > \hat{\beta}$, which implies that Δ_i is a non-decreasing function of i. Therefore the move $(A_n, B_n) \to (A_m, B_m)$ is either blocked by the triangle move restriction of β^{T}-Wynim (if $\Delta^2_{n-1} \le \lfloor\beta\rfloor$), or by the $\lfloor\beta\rfloor$-Wythoff constraint (if $\Delta^2_{n-1} \ge \lfloor\beta\rfloor$).

(ii) Notice that this move is possible only if $A_n > B_m$. Now $(B_n - A_m) - (A_n - B_m) = \Delta_n + \Delta_m$. Similarly to (i), if $\Delta_n + \Delta_m \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$, this forces $m = 0$ and $n = 1$ so that the move is blocked by the β-triangle move restriction; otherwise by the $\lfloor\beta\rfloor$-Wythoff constraint.

For the direction N $\to$ P, let (x, y), $0 \le x \le y$ be a position not of the form (1). We assume first that this position has a coordinate of the form B_n, so for each game it suffices to show that a Nim type (I) move suffices for moving into (1). If $x = B_n$ then move $y \to A_n$. If $y = B_n$ and $x > A_n$ then move $x \to A_n$. If $y = B_n$ and $x < A_n$, complementarity implies that there exists $m < n$ such that either $x = A_m$ so the move $y \to B_m$ restores (1); or else $x = B_m$, so the move $y \to A_m$ does it.

Hence we may assume that both x and y are in A, say $x = A_m \le A_n = y$. If $y > B_m$, then the Nim type (I) move $y \to B_m$ suffices for each game. We may therefore assume that

$$x = A_m \le A_n = y < B_m. \tag{2}$$

Since each of (A_i) and (B_i) is strictly increasing, a Nim type move to a position (1) does not exist, so we have to find a (II) extended diagonal type move for the games β-Wynim and β^{T}-Wynim. Observe that for both these games, this type of moves is now unrestricted with $k = \lfloor\beta\rfloor$.

Let $d := y - x$. Then $d = A_n - A_m < B_m - A_m = \Delta_m$. By Proposition 1, Δ_i grows from 0 to Δ_m as i grows from 0 to m, in steps $\Delta^2_i = \Delta B_i - \Delta A_i \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1, \lfloor\beta\rfloor\}$, bounded above by $\lfloor\beta\rfloor$. Hence there exists j such that $0 \le d - \Delta_j < \lfloor\beta\rfloor$. Then move $(x, y) \to (A_j, B_j)$. We need to show three things: (i) $j < m$, (ii) $y > B_j$, (iii) $|(y - B_j) - (A_m - A_j)| < \lfloor\beta\rfloor$.

(i) $\Delta_j \le d = y - A_m < B_m - A_m = \Delta_m$. Since Δ_i is an increasing function of i, we have $j < m$.

(ii) $y = A_m + d > A_j + d \geq A_j + \Delta_j = B_j$.

(iii) $|(y - B_j) - (A_m - A_j)| = |(y - A_m) - (B_j - A_j)| = |d - \Delta_j| < \lfloor \beta \rfloor$.

On the other hand, for the game β-Nim and a position of the form in (2), by Proposition 1 (i) a nearest lower P-position is attainable by an extended 'horizontal' Nim-type move. Precisely, since $\Delta B_n \in \{\lfloor \beta \rfloor, \lfloor \beta \rfloor + 1\}$ we can lower $y = A_n$ to B_i, where $i \geq 0$ is such that $B_i < A_n < B_{i+1}$, and $x = A_m$ to A_i, that is move $(A_m, A_n) \to (A_i, B_i)$. By the (I) Nim type move we have to show that $A_i < A_m$. But the definition of i together with (2) give $B_i < A_n < B_m$ which, by $i < m$, implies $A_i < A_m$.

Thus the set $\mathcal{P}$ is indeed the set of P-positions for our three games. ■

4 The notion of k-invariance and game complexity

Let us continue our brief discussion of variant versus invariant games from the introduction and Remark 1, and relate it to the complexity of numbers and games. We think of an integer as the simplest number, followed by the rationals, algebraic numbers and transcendental numbers, the most complex numbers.

Our three games are, in fact, 'minimally variant' in the sense that all their positions can be partitioned into precisely two sets, namely,

$$\{(A_i, A_j) \mid i, j \in \mathbb{Z}_{>0}\} \text{ and } \{(B_i, A_j), (A_i, B_j), (B_i, B_j) \mid i, j \in \mathbb{Z}_{\geq 0}\},$$

such that, for each game, for each set, the possible moves are invariant. This observation motivates a weakening of the notion of invariance to *k-invariance*, $k \in \mathbb{Z}_{>0}$.

Definition 1. Let X be a subset of the set of positions (j-tuples of non-negative integers) of a game G on j heaps. Then m (also a j-tuple of non-negative integers, but not all 0) is an *invariant move* in X, if for all $x \in X$, $x - m$ is an option, provided $x - m$ is a position in G.

Definition 2. Let G be a game and X a subset of all positions in G. Then m is a *variant move* in X if there exist $x, y \in X$ such that both $x - m$ and $y - m$ are positions in G, $x - m$ is an option but $y - m$ is not.

Definition 3. Let $k \in \mathbb{Z}_{>0}$. A game G is *k-invariant* if

- its set of positions can be partitioned into k subsets, such that, within each subset X, each allowed move is invariant in X;
- for any partition $\sqcup X_i$ of G's positions into $< k$ subsets, there is an i and an m, such that m is a variant move in X_i.

If a game G is not k-invariant for any $k \geq 1$, then it is ∞-*invariant*. If $k = 1$, then G is *invariant* (the second item does not apply). If $k \neq 1$, then G is *variant*.

The games in this paper are all 2-invariant. The "mouse game" in [6] is 2-invariant. However it is known that the $\star$-operator for invariant subtraction games [16, 15] produces an invariant game, the "mouse trap" [13], with the same sets of P-positions as the mouse game. Note that the mouse game is still 2-invariant; the k-invariance of game rules have got nothing to do with the possibility of existence of l-invariant games for $l < k$, for a given set of P-positions (such definitions appear to need special attention). The game Mark [8, 9] is ∞-invariant. Of course, any ∞-invariant game is variant.

Let $\gamma = \beta - \hat{\beta}$. It appears that the complexity of γ, the simplicity of the game rules and the size of l are related. If $\gamma = l$ is an integer, there are simple game rules, (I) and (II), and the game is invariant (Example 2). For our three games, γ is not necessarily an integer, the game rules are longer and the 1-invariance is replaced by 2-invariance. To shed more light on these suggested relationships, it might be well to investigate whether γ rational, algebraic [5, 17], or transcendental has any effect on the length of the game rules and k-invariance.

We close this section with a conjecture which requires a little background.

The succinct input size of a given ordered pair of integers (x, y) is $\log(xy)$. The time complexity of deciding whether a given ordered pair (x, y) is of the form (A_n, B_n) is polynomial in $\log(xy)$, see [10], §3. In [16, Main Theorem] it is demonstrated that, given the set $\mathcal{P}$ in Theorem 1, there is an *invariant* game for which the time complexity of determining whether a given ordered pair (s, t) is a legal move is exponential in $\log(st)$. In [5, 17] polynomial time invariant game rules are determined for the set $\mathcal{P}$ when γ is restricted to some specific algebraic numbers of degree 2.

We make the following related conjecture.

Conjecture 1. *Let $\beta > 2$ be irrational. Let $\mathcal{Q} = \cup_{n\geq 0}\{(A_n, B_n)\} \cup_{n\geq 0} \{(B_n, A_n)\}$ where $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are complementary Beatty sequences and suppose that $\gamma = \beta - \hat{\beta}$ transcendental, $A_0 = B_0 = 0$. Let G be an invariant game, with* polynomial time complexity *in* $\log(xy)$*, for finding a* *move from each N-position (x, y) into a P-position. Then the set $\mathcal{Q}$, is not the set of P-positions of G.*

Perhaps this conjecture holds even if γ is algebraic, or even if γ is a non-integer rational number. Perhaps it holds even if the Beatty sequences A and B are not complementary [9]. By the results of this paper, we know that the conjecture is false if we replace invariance by 2-invariance.

The notion of k-invariance is also interesting in a somewhat different context. In [18] certain k-invariant 2-heap subtraction games with a finite number of (variant) moves are studied and it is showed that they embrace computational universality.

Many heap games in the literature have move-size dynamic rules (e. g. Fibonacci nim), blocking maneuvers (e. g. blocking Wythoff nim), depend on positions moved to, rather than position moved from, and so on, and new variations yet to come. The notion of k-invariance in this paper is only intended as a small guide for a larger classification in the future.

5 Discussion

We have formulated three reasonably short game rules for three 2-invariant games, which have identical sets of P-positions. Suppose that we fix β and then increase the density of the pairs of sequences from 1 to say an arbitrary number $\zeta > 1$ (or decreases to a density < 1) where α is defined via $1/\alpha + 1/\beta = \zeta$. (That is, for all β, $\alpha \neq \hat{\beta}$.) Given candidate P-positions as above, is there still a short/succinct but non-trivial way of formulating the game rules without disclosing both irrationals or/and the joint density of the sequences? It is unknown to us whether or not there exist invariant rules for such games, see [16], [9]. Is it possible to find 2-invariant rules in the sense of this note? As another remark, observe that neither $\hat{\beta}$ nor the density 1 is disclosed in the presentation of the rules of our games in this note. In [16] invariant game rules are given for candidate P-positions constructed from complementary Beatty sequences, but not in a single case have we found a succinct description. In this note we have chosen to remove the nice condition of invariance (and reverted to 2-invariance) from the game rules and, maybe even more notably, one of the coordinates of the candidate P-positions is disclosed within the game rules. This could be argued to be a severe drawback in a definition of the rules of a game. But, on the other hand, we were able to give a very succinct formulation, without a complete trivialization of game rules, for all complementary homogeneous Beatty sequences and these are uncountably many. For other investigations that relate Nim and Wythoff, see [3], [11].

Acknowledgment. We thank Eric Duchêne for some informative discussion on invariance vs k-invariance of games.

References

[1] S. Beatty, Problem 3173, *Amer. Math. Monthly* **33** (1926) 159, Solution [to Problem 3177, numbering in **33** was erroneous] in **34** (1927) 159–160.

[2] E. R. Berlekamp, J. H. Conway and R. K. Guy [2001-2004], *Winning Ways for your Mathematical Plays*, Vol. 1-4, A K Peters, Wellesley, MA, 2nd edition: vol. 1 (2001), vols. 2, 3 (2003), vol. 4 (2004).

[3] E. Duchêne, A. S. Fraenkel, S. Gravier and R. J. Nowakowski [2009], Another bridge between NIM and WYTHOFF, *Australas. J. Combin.* **44** (2009) 43–56.

[4] E. Duchêne, A. S. Fraenkel, R. J. Nowakowski and M. Rigo [2010], Extensions and restrictions of Wythoff's game preserving its P positions, *J. Combin. Theory Ser. A* **117**, 545–567.

[5] E. Duchêne and M. Rigo [2010], Invariant games, *Theoret. Comput. Sci.* **411**(34-36), 3169–3180.

[6] The Rat game and the Mouse game, to appear in *Games of no Chance 4*.

[7] A. S. Fraenkel, Aperiodic subtraction games, *Electr. J. Combin.* Vol. 18(2) (2011) P19 12pp.

[8] A.S. Fraenkel, The vile, dopey, evil and odious game players, *Discrete Math.* 312 (2012) 42–46, special volume in honor of the 80th birthday of Gert Sabidussi.

[9] A. S. Fraenkel, Beating Your Fractional Beatty Game Opponent and: What's the Question to Your Answer?, Advances in Combinatorics, in Memory of Herbert S. Wilf, Proc. Waterloo Workshop on Computer Algebra, I.S. Kotsireas and E.V. Zima, eds., May 2011, Springer (2013), 175–186.

[10] A. S. Fraenkel, How to beat your Wythoff games' opponent on three fronts, *Amer. Math. Monthly* **89** (1982) 353–361.

[11] A. S. Fraenkel and M. Lorberbom, Nimhoff games, *J. Combin. Theory* (Ser. A) **58** (1991) 1–25.

[12] U. Larsson, Restrictions of m-Wythoff Nim and p-complementary sequences, to appear in *Games of no Chance 4.*

[13] U. Larsson, "The $\star$-operator and invariant subtraction games", preprint (to be renamed to: Permutation games and the $\star$-operator) http://arxiv.org/1009.4220

[14] U. Larsson, Blocking Wythoff Nim, *The Electr. J. Combin.*, 18 (2011) P120.

[15] The $\star$-operator and invariant subtraction games, *Theoret. Comput. Sci.* Vol. 422, (2012) 52–58.

[16] U. Larsson, P. Hegarty and A. S. Fraenkel [2011], Invariant and dual subtraction games resolving the Duchêne-Rigo conjecture, *Theoret. Comput. Sci.* **412**, 729–735.

[17] U. Larsson, M. Weimerskirch, Impartial games whose rulesets produce given continued fractions, to appear in *Games of no Chance 5.*

[18] U. Larsson, J. Wästlund, From heaps of matches to the limits of computability, *The Electr. J. Combin.*, 20 (2013) P41.

[19] J. W. S. Rayleigh [1937], *The Theory of Sound*, MacMillan, London, 1st ed. 1877, 2nd ed. 1894, reprinted 1928,1929,1937. There is also a Dover 1945 publication, which is a reprint of the 2nd. The "sound" equivalent of Beatty's theorem, is stated, without proof, in sect. 92a, pp. 122-123, in the 1937 reprint of the 2nd ed., the only one we saw. Kevin O'Bryant told us that he checked the 1st edition and the statement is not there.