

# LIM IS NOT SLIM

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ABSTRACT. In this paper LIM, a recently proposed impartial combinatorial ruleset, is analyzed. A formula to describe the  $\mathcal{G}$ -values of LIM positions is given, by way of analyzing an equivalent combinatorial ruleset LIM', closely related to the classical NIM. Also, an enumeration of  $\mathcal{P}$ -positions of LIM with  $n$  stones, and its relation to the Ulam-Warburton cellular automaton, is presented.

KEYWORDS: Combinatorial game theory, impartial games, NIM, Sprague-Grundy theory, Ulam-Warburton cellular automaton.

## 1. INTRODUCTION

Combinatorial game theory studies perfect information games in which there are no chance devices (e.g. dice) and two players take turns moving alternately. Here we are concerned with games under normal play, where the last player to move wins. This paper is self contained; see [1, 2, 5] for background and [6] for a survey. Readers fluent in combinatorial game theory may wish to proceed to the subsection 1.2.

### 1.1. BACKGROUND ON RELEVANT COMBINATORIAL GAME THEORY

The *options* of a game are all those positions which can be reached in one move. Using the standard notation for combinatorial game theory of [2] where Left and Right are the players, games can be expressed recursively as  $G = \{\mathcal{G}^L | \mathcal{G}^R\}$  where  $\mathcal{G}^L$  are the Left options and  $\mathcal{G}^R$  are the Right options of  $G$ . We distinguish between multiple meanings of the word *game* by using the words *ruleset* and *game*. The word *ruleset* has a concrete meaning related to some particular set of rules (what is called a “game” informally). The word *game*, by contrast, has the abstract mathematical meaning defined by Conway [2, 5]. When we speak of the *value* of a game, we are

emphasizing that it is being considered in this latter sense, as an algebraic object which can be compared for equality with, or added to, other games.

An example of a combinatorial ruleset is the classic game of NIM, first studied by C. Bouton [3]. NIM is played with piles of stones. On his turn, each player can remove any number of stones from any pile. The winner is the player who takes the last stone. NIM is an example of an *impartial* ruleset: Left options and Right options are the same for the game and all its followers. The values involved in NIM are called numbers (or stars):

$$*k = \{0, *, \dots, *(k-1) \mid 0, *, \dots, *(k-1)\}$$

It is a surprising fact that all impartial rulesets take only numbers as values (Sprague-Grundy Theory, see [7, 11]).

The *minimum excluded value* of a set  $S$  is the least nonnegative integer which is not included in  $S$  and is denoted  $\text{mex}(S)$ . The *nim-value* of an impartial game  $G$ , denoted by  $\mathcal{G}(G)$ , is given by

$$\mathcal{G}(G) = \text{mex}\{\mathcal{G}(H) : H \text{ is an option of } G\}.$$

The value of an impartial game  $G$  is the number  $*\mathcal{G}(G)$ . The game  $G$  is a previous player win, i.e. the next player has no good move, if and only if  $\mathcal{G}(G) = 0$ . If a game is a previous player win, we say it is a  $\mathcal{P}$ -position. If a game is a next player win, we say it is a  $\mathcal{N}$ -position. The set of  $\mathcal{P}$ -positions is noted  $\mathcal{P}$  and the set of  $\mathcal{N}$ -positions is noted  $\mathcal{N}$ .

The *nim-sum* of two nonnegative integers is the exclusive or (XOR), written  $\oplus$ , of their binary representations. It can also be described as adding the numbers in binary without carrying. The disjunctive sum of games  $H$  and  $K$  is written  $G = H + K$ . In this game, the player to move must choose one of  $H$  and  $K$  and make a legal move in that game. One important result about impartial games is the following: if  $G = H + K$ , then  $\mathcal{G}(G) = \mathcal{G}(H) \oplus \mathcal{G}(K)$  (see [1, 2, 5]).

## 1.2. THE GAME OF LIM

This paper studies the impartial ruleset LIM with very simple rules proposed by Jorge Nuno Silva <sup>1</sup>. In [9, 10], there is a correct conjecture about the  $\mathcal{P}$ -positions, but no proof is presented.

There are 3 piles. A player takes the same number  $N$  of stones from 2 piles and adds  $N$  stones to the third. The last player wins (i.e. we consider the normal play version). There is a board game implementation of LIM has a board game implementation. The players move a pile of checkers diagonally in one of the three directions depicted. If a player moves Southwest or Northeast, he can move a number of cells smaller or equal to the number of checkers of the pile and, when the move is finished, the player removes that number of checkers from the pile. If a player moves Northwest, when the move is finished, the player adds the number of checkers that is equal to the number of traveled cells. See Figure 1.

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<sup>1</sup>Etymological note: the name LIM, aside from rhyming with NIM, is an acronym for **L**aura e **M**anuel, the names of Silva's children (the Portuguese word *e* "and" is pronounced /i/).

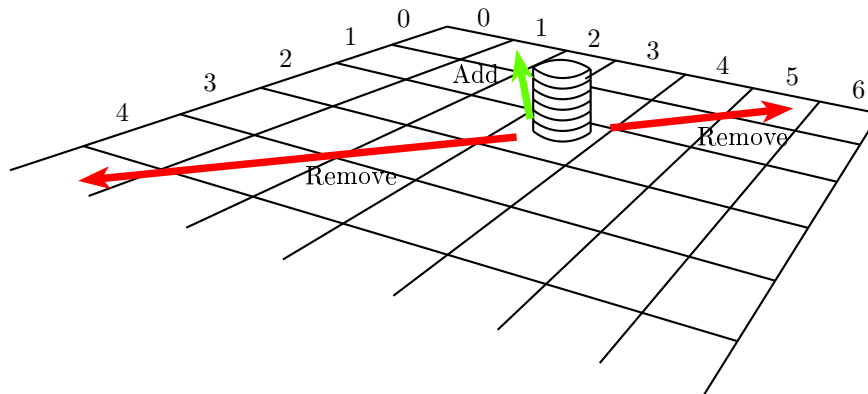


FIGURE 1. Moves in the board game implementation of LIM.

In this paper a complete analysis of LIM is presented. Theorem 4.1 gives a formula for the nim-values of LIM. This formula intermixes the ordinary sum and the nim-sum, and thus our analysis necessitates establishing some lemmas on how these two notions of sum interact, which we do in section 2. Moreover, it is very convenient for the analysis of LIM to introduce an equivalent ruleset LIM', a sort of coordinate transform of LIM: it turns out that we can use NIM moves to find good moves in LIM'. This is the subject of Section 3.

We also enumerate the  $\mathcal{P}$ -positions of LIM and connect it to the Ulam-Warburton cellular automaton in Section 5, exhibiting a bijection between  $\mathcal{P}$ -positions of LIM with  $n$  stones and cells born in a region of this automaton on tick  $n$ .

## 2. NIM, SUMS, AND NIM SUMS

In this section we prove some useful general results relating the usual sum and subtraction to the nim sum, culminating with Lemma 2.4 on NIM.

**Lemma 2.1.** *Let  $a$ ,  $b$  and  $c$  be nonnegative integers. Then,*

- (1)  $a + b = a \oplus b + 2(a \odot b)$
- (2)  $a + b + c = a \oplus b \oplus c + 2(a \odot b + a \odot c + b \odot c) - 4(a \odot b \odot c)$   
*where  $\odot$  is the bitwise product (i.e. AND).*

*Proof.* For the first item we just observe that nim sum cancels repeating bits in binary expansions of  $a$  and  $b$ . So, in order to obtain the usual sum from nim sum, we have to add the repeating bits twice.

Again, the nim sum for three summands cancels repeating bits in binary expansions of  $a$ ,  $b$  and  $c$ . It is easy to see that a bit of the binary expansion of  $(a \odot b + a \odot c + b \odot c) - 2(a \odot b \odot c)$  is 1 if and only if  $a$ ,  $b$  and  $c$ , for that bit, have two or three 1s (there are repetitions). So, in order to obtain the usual sum from nim sum, we have to add  $(a \odot b + a \odot c + b \odot c) - 2(a \odot b \odot c)$  twice.  $\square$

**Lemma 2.2.** *Let  $a, b$  be nonnegative integers. Then,*

$$a - b \leq a \oplus b \leq a + b$$

*Proof.*  $a \oplus b \leq a + b$  is trivial, as we can argue with the previous lemma. For the second inequality,

$$b + (b \oplus a) = b \oplus (b \oplus a) + 2(b \odot (a \oplus b)) = a + 2(b \odot (a \oplus b)) \geq a$$

So,  $a - b \leq a \oplus b$ . □

The next lemma is our most technical. Those not reading in depth may wish to skip ahead to Lemma 2.4, where it is used.

**Lemma 2.3.** *Let  $d$  be a positive integer, and suppose given four expressions for  $d$  as a signed sum of distinct powers of two: that is, let  $\varepsilon_{i,\ell} \in \{-1, 0, 1\}$  satisfy*

$$d = \sum_{i \geq 0} \varepsilon_{i,\ell} 2^i$$

for each  $\ell = 1, \dots, 4$ . Suppose that there is no  $i \geq 0$  such that  $\varepsilon_{i,1}\varepsilon_{i,2}\varepsilon_{i,3}\varepsilon_{i,4} = 1$ . Let  $i_0$  be the maximal index such that  $\varepsilon_{i_0,1} + \varepsilon_{i_0,2} + \varepsilon_{i_0,3} + \varepsilon_{i_0,4}$  is odd. Then, at least one of the  $\varepsilon_{i_0,\ell}$  equals 1.

*Proof.* We first compare two such expansions of  $d$ , say with coefficient sequences  $\{\varepsilon_{i,1}\}$  and  $\{\varepsilon_{i,2}\}$ . Let  $i$  be the greatest index at which  $\varepsilon_{i,1}$  differs from  $\varepsilon_{i,2}$ . Without loss of generality we may take  $\varepsilon_{i,1} > \varepsilon_{i,2}$ . The difference  $\varepsilon_{i,1} - \varepsilon_{i,2}$  must equal 1; if it were greater, then since  $\sum_{j=0}^i \varepsilon_{j,1} 2^j = \sum_{j=0}^i \varepsilon_{j,2} 2^j = d$ , subtracting the disagreeing terms  $\varepsilon_{i,\ell}$  would yield

$$\sum_{j=0}^{i-1} (\varepsilon_{j,1} - \varepsilon_{j,2}) 2^j \leq -2 \cdot 2^i,$$

which is impossible since,

$$\sum_{j=0}^{i-1} (\varepsilon_{j,1} - \varepsilon_{j,2}) 2^j \geq \sum_{j=0}^{i-1} -2 \cdot 2^j = -2(2^i - 1).$$

Moving on to the  $(i-1)$ th terms, an argument of the same type shows that  $\varepsilon_{i-1,1} - \varepsilon_{i-1,2} \leq -1$ : if instead this difference were  $\geq 0$ , the remaining terms would have to make up a discrepancy of size  $2 \cdot 2^{i-1}$ , and could not. Likewise, if  $\varepsilon_{i-1,1} - \varepsilon_{i-1,2} = -1$ , then another similar argument shows  $\varepsilon_{i-2,1} - \varepsilon_{i-2,2} \leq -1$ . And one can continue iteratively, concluding that if  $\varepsilon_{k,1} - \varepsilon_{k,2} = -1$  for all  $k = i-1, i-2, \dots, j$ , then  $\varepsilon_{j-1,1} - \varepsilon_{j-1,2} \leq -1$ .

Also, we can not have  $\varepsilon_{k,1} - \varepsilon_{k,2} = -1$  for all  $k = i-1, i-2, \dots, 0$  because  $\sum_{j=0}^{i-1} 2^j < 2^i$  so, for some  $k < j$ , we have  $\varepsilon_{k,1} - \varepsilon_{k,2} = -2$ .

Taking up the situation of interest with all four expansions  $\varepsilon_{i,\ell}$ , let  $i$  be the maximal index such that any  $\varepsilon_{i,\ell}$  is nonzero. Since  $d$  is positive,  $\varepsilon_{i,\ell} \in \{0, 1\}$  for each  $\ell$ . The number of  $\ell$  such that  $\varepsilon_{i,\ell}$  equals 1 cannot be 4 by hypothesis, and if it is 1 or 3 then our conclusion is immediate with  $i_0 = i$ . So we may assume there are two such  $\ell$ , without loss of generality that  $\varepsilon_{i,1} = \varepsilon_{i,2} = 1$  and  $\varepsilon_{i,3} = \varepsilon_{i,4} = 0$ .

Now let us examine the remaining coefficients  $\varepsilon_{k,\ell}$  with  $k < i$ . By the above, consider  $i_0$  the first bit such that  $\varepsilon_{i_0,1} - \varepsilon_{i_0,3} = -2$  or  $\varepsilon_{i_0,2} - \varepsilon_{i_0,4} = -2$ . Say that  $\varepsilon_{i_0,1} - \varepsilon_{i_0,3} = -2$ . Therefore,  $\varepsilon_{i_0,1} = -1$  and  $\varepsilon_{i_0,3} = 1$ . Also, we can not have  $\varepsilon_{i_0,2} - \varepsilon_{i_0,4} = -2$  because, by the assumptions of the theorem,  $\varepsilon_{i,1}\varepsilon_{i,2}\varepsilon_{i,3}\varepsilon_{i,4} \neq 1$ . So, it is mandatory that either  $\varepsilon_{i_0,2} = -1$  and  $\varepsilon_{i_0,4} = 0$  or  $\varepsilon_{i_0,2} = 0$  and  $\varepsilon_{i_0,4} = 1$ . This completes the proof because  $i_0$  is indeed the maximal bit sought and at least one of the  $\varepsilon_{i_0,\ell}$  equals 1.  $\square$

**Lemma 2.4.** *Consider  $a, b, c$  nonnegative integers and  $0 < d \leq a \oplus b \oplus c$ . Then, the NIM position  $(a + d, b + d, c + d)$  has a move to a NIM position of Grundy value  $(a \oplus b \oplus c) - d$ .*

*Proof.* It is enough to prove that we can not have simultaneously the following three inequalities:

$$\left\{ \begin{array}{l} ((a \oplus b \oplus c) - d) \oplus (b + d) \oplus (c + d) \geq a + d \\ ((a \oplus b \oplus c) - d) \oplus (a + d) \oplus (c + d) \geq b + d \\ ((a \oplus b \oplus c) - d) \oplus (a + d) \oplus (b + d) \geq c + d. \end{array} \right.$$

To justify this fact, say that  $((a \oplus b \oplus c) - d) \oplus (b + d) \oplus (c + d) < a + d$ . If so, we have a NIM move from  $(a + d, b + d, c + d)$  to  $((a \oplus b \oplus c) - d) \oplus (b + d) \oplus (c + d), b + d, c + d)$  with Grundy value  $(a \oplus b \oplus c) - d$ . So, if we prove that at least one of the inequalities fails, the lemma is proved.

Let  $\text{bit}_i(x)$  denote the  $i$ th bit of the binary expansion of an integer  $x$ . Define the integers  $\varepsilon_{i,\ell} \in \{-1, 0, 1\}$  by

$$\begin{aligned} \varepsilon_{i,1} &= \text{bit}_i(a + d) - \text{bit}_i(a) \\ \varepsilon_{i,2} &= \text{bit}_i(b + d) - \text{bit}_i(b) \\ \varepsilon_{i,3} &= \text{bit}_i(c + d) - \text{bit}_i(c) \\ \varepsilon_{i,4} &= \text{bit}_i(a \oplus b \oplus c) - \text{bit}_i((a \oplus b \oplus c) - d) \end{aligned}$$

Of course,  $\sum_i \varepsilon_{i,\ell} 2^i = d$  for  $\ell = 1, \dots, 4$ .

There is no  $i$  such that  $\varepsilon_{i,1}\varepsilon_{i,2}\varepsilon_{i,3}\varepsilon_{i,4} = 1$ , because that would imply that an odd number of  $\text{bit}_i(a)$ ,  $\text{bit}_i(b)$ ,  $\text{bit}_i(c)$ , and  $\text{bit}_i(a \oplus b \oplus c)$  were 1. Therefore Lemma 2.3 applies to the  $\varepsilon_{i,\ell}$ .

Each  $\varepsilon_{i,\ell}$  is odd if and only if the nim-sum of the two bits subtracted in its definition is odd, so the  $i_0$  of the Lemma 2.3 equals the index of the leading 1 bit in

$$(a + d) \oplus a \oplus (b + d) \oplus b \oplus (c + d) \oplus c \oplus (a \oplus b \oplus c) \oplus ((a \oplus b \oplus c) - d),$$

which therefore equals the leading 1 bit in

$$(a + d) \oplus (b + d) \oplus (c + d) \oplus ((a \oplus b \oplus c) - d).$$

At last, suppose none of  $a + d$ ,  $b + d$ , or  $c + d$  have their  $i_0$ th bit equal to 1. Then  $\varepsilon_{i,\ell} \neq 1$  for  $\ell = 1, 2, 3$ , so  $\varepsilon_{i,4} = 1$ , implying that  $\text{bit}_{i_0}(a \oplus b \oplus c) = 1$ . But then an odd number of  $\text{bit}_{i_0}(a)$  and  $\text{bit}_{i_0}(b)$  and  $\text{bit}_{i_0}(c)$  equal 1, so an odd number of  $\varepsilon_{i,1}$  and  $\varepsilon_{i,2}$  and  $\varepsilon_{i,3}$  are odd, contradicting the definition of  $i_0$ .

In conclusion, in  $i_0$ , the leading 1 bit of

$$(a + d) \oplus (b + d) \oplus (c + d) \oplus ((a \oplus b \oplus c) - d),$$

at least one of the  $i_0$ th bits of  $a + d$ ,  $b + d$ , and  $c + d$  must be 1.

This suffices to argue that one of the initial three inequalities must fail. Say that we have the  $i_0$ th bits of  $a + d$ ,  $b + d$ ,  $c + d$ , and  $((a \oplus b \oplus c) - d)$  equaling 1, 1, 0, and 1. In that case, both the first and second inequalities fail. And a similar argument for the other cases leads to, at least, one failed inequality. This completes the proof.  $\square$

**Example 2.5.** Consider the triple  $(41, 30, 23)$  and  $d = 15$ . Because  $41 \oplus 30 \oplus 23 = 32 > 15 > 0$ , the hypotheses of Lemma 2.4 are satisfied. So, it is possible to find a NIM move from  $(41 + 15, 30 + 15, 23 + 15) = (56, 45, 38)$  to a NIM position with Grundy value  $32 - 15 = 17$ . To find such a move we can perform the following calculations:

- (1)  $((41 \oplus 30 \oplus 23) - 15) \oplus (30 + 15) \oplus (23 + 15) = 17 \oplus 45 \oplus 38 = 26$
- (2)  $((41 \oplus 30 \oplus 23) - 15) \oplus (41 + 15) \oplus (23 + 15) = 17 \oplus 56 \oplus 38 = 15$
- (3)  $((41 \oplus 30 \oplus 23) - 15) \oplus (41 + 15) \oplus (30 + 15) = 17 \oplus 56 \oplus 45 = 4.$

In this example, all the three inequalities  $56 > 26$ ,  $45 > 15$  and  $38 > 4$  hold (all the three inequalities of the proof failed). So, there are three good possibilities:  $(56, 45, 28) \rightarrow (26, 45, 28)$ ,  $(56, 45, 28) \rightarrow (56, 15, 28)$  or  $(56, 45, 28) \rightarrow (56, 45, 4)$ .  $\diamond$

### 3. THE GAME OF LIM'

In this section we introduce and analyze the ruleset LIM' which is equivalent to LIM. LIM' is played on triangles  $(A, B, C)$  (that is  $A + B \geq C$ ,  $A + C \geq B$  and  $B + C \geq A$ ) such that  $A + B + C \equiv 0 \pmod{2}$  and a move consists of subtracting from a single side length an even number less or equal than this side length. There are several classical variations of NIM; *subtraction games* are played like NIM but the players can only remove a number of stones from a pile if it is an element of a given *subtraction set*  $\{s_1, \dots, s_k\}$  ([1, 2, 5], for more information). LIM' is the subtraction game played on triangles defined as above whose subtraction set is the set of positive even numbers.

**Lemma 3.1.** LIM and LIM' are equivalent combinatorial rulesets.

*Proof.* Consider the digraphs  $(\mathbb{P}, E)$  and  $(\mathbb{P}', E')$ , where  $\mathbb{P}$  and  $\mathbb{P}'$  are the sets of vertices representing the positions of LIM and LIM' and  $E$  and  $E'$  are the edges corresponding to the moves of each ruleset. We want to prove that the digraphs are isomorphic.

Consider  $\psi : \mathbb{P} \rightarrow \mathbb{P}'$  such that  $\psi(a, b, c) = (A, B, C)$  where  $A = b + c$ ,  $B = a + c$ ,  $C = a + b$ . It is easy to see that  $(A, B, C)$  is a triangle such that  $A + B + C \equiv 0 \pmod{2}$ , so that the image of  $\psi$  indeed lies in the set of LIM' positions.

First, we observe that  $\psi$  is a bijection. To wit,  $\psi(a, b, c) = \psi(a', b', c') \Rightarrow (b + c, a + c, a + b) = (b' + c', a' + c', a' + b')$  and elementary linear algebra shows that  $b + c = b' + c'$ ,  $a + c = a' + c'$  and  $a + b = a' + b'$  implies  $a = a'$ ,  $b = b'$

and  $c = c'$ . Also, consider an arbitrary  $(A, B, C) \in \mathbb{P}'$ . Let  $a = (-A + B + C)/2$ ,  $b = (A - B + C)/2$ , and  $c = (A + B - C)/2$ . Because all the fractions are nonnegative integers ( $(A, B, C)$  is a triangle and  $A + B + C \equiv 0 \pmod{2}$ ), we have  $(a, b, c) \in \mathbb{P}$  and  $\psi(a, b, c) = (A, B, C)$ .

As for the edges of our digraphs, consider the LIM move  $(a, b, c) \longrightarrow (a - k, b - k, c + k)$ . Of course, we have  $k \leq a$ ,  $k \leq b$ ,  $2k \leq a + b = C$ , and  $A + B + C - 2k \equiv 0 \pmod{2}$ . Also,  $(A, B, C - 2k)$  is a triangle because  $C - 2k + A = a + b - 2k + b + c = B + 2b - 2k \geq B$  and the same argument for  $C - 2k + B \geq A$ . So,  $(A, B, C) \longrightarrow (A, B, C - 2k)$  is a legal LIM' move. Conversely, consider the LIM' move  $(A, B, C) \longrightarrow (A, B, C - 2k)$ . It is mandatory that  $(A, B, C - 2k)$  is a triangle. So,  $C - 2k + B \geq A \Rightarrow k \leq (-A + B + C)/2 = a$  (and similarly  $k \leq b$ ). Therefore,  $(a, b, c) \longrightarrow (a - k, b - k, c + k)$  is a legal LIM move. The conclusion is that an edge is in  $E$  if and only if the correspondent edge is in  $E'$ . We have a graph isomorphism and this completes the proof.  $\square$

To be explicit, here again are the correspondences from moves in LIM to moves in LIM':

- (1)  $(a, b, c) \longrightarrow (a - k, b - k, c + k)$  is identified with  $(A, B, C) \longrightarrow (A, B, C - 2k)$
- (2)  $(a, b, c) \longrightarrow (a - k, b + k, c - k)$  is identified with  $(A, B, C) \longrightarrow (A, B - 2k, C)$
- (3)  $(a, b, c) \longrightarrow (a + k, b - k, c - k)$  is identified with  $(A, B, C) \longrightarrow (A - 2k, B, C)$ .

So, a complete analysis of LIM' provides a complete analysis of LIM. The following proposition is such a complete analysis.

**Proposition 3.2.** *Let  $(A, B, C)$  be a LIM' position. If  $(A - g) \oplus (B - g) \oplus (C - g) = g$  then  $(A, B, C)$  has  $\mathcal{G}$ -value  $g$ .*

*Proof.* We will prove the theorem following the usual induction in  $A + B + C$ . The base case  $A = B = C = 0$  is trivial. We will show that a position we have asserted to have  $\mathcal{G}$ -value  $g$  has no option of the same  $\mathcal{G}$ -value, and has an option of  $\mathcal{G}$ -value  $h$  for each  $0 \leq h < g$ .

The former is clear: if  $\text{LIM}'(A, B, C)$  and  $\text{LIM}'(A - k, B, C)$  had the same  $\mathcal{G}$ -value  $g$ , so would the NIM positions  $(A - g, B - g, C - g)$  and  $(A - k - g, B - g, C - g)$ , which they cannot.

Let  $0 \leq h < g$ . Consider  $(A - g, B - g, C - g)$  and  $d = g - h$ . We have  $0 < d \leq (A - g) \oplus (B - g) \oplus (C - g) = g$ . So, we are in the conditions of Lemma 2.4. It is possible to find a NIM move from  $(A - g + d, B - g + d, C - g + d) = (A - h, B - h, C - h)$  to a position with  $\mathcal{G}$ -value  $g - d = h$ . There exists  $k$  such that the nim sum of  $(A - k - h, B - h, C - h)$  is  $h$ .

Now, let us see that  $(A - k, B, C)$  is a LIM' position and  $k$  is a nonnegative even number. To begin,  $B + C \geq A - k$  because  $B + C \geq A$ . Also, because  $A - k - h =$

$h \oplus (B - h) \oplus (C - h)$  implying  $A - k = h + (h \oplus (B - h) \oplus (C - h))$ ,

$$\begin{aligned}
& (A - k) + B \\
&= (h + (h \oplus (B - h) \oplus (C - h))) + B \\
&\geq (h \oplus (h \oplus (B - h) \oplus (C - h))) + B \\
&= ((B - h) \oplus (C - h)) + B \\
&\geq ((C - h) - (B - h)) + B \\
&= C.
\end{aligned}$$

We have used Lemma 2.2 in the previous manipulations. Similarly  $(A - k) + C \geq B$  so,  $(A - k, B, C)$  is a triangle.

Also, the nim sum of  $(A - k - h, B - h, C - h)$  is  $h$ . A simple parity analysis allows us to observe that  $(A - h) + (B - h) + (C - h) \equiv h \pmod{2}$ , and therefore  $k$  must be even.

Finally, because  $k$  is even,  $(A - k) + B + C \equiv 0 \pmod{2}$  and  $(A - k, B, C)$  is a LIM' position. Now, using induction, the  $\mathcal{G}$ -value of this position is  $h$  because the nim sum of  $(A - k - h, B - h, C - h)$  is  $h$ .

□

#### 4. $\mathcal{G}$ -VALUES OF LIM POSITIONS

**Theorem 4.1.** *Let  $(a, b, c)$  be a LIM position. Then,*

$$\mathcal{G}(a, b, c) = \frac{1}{2}((a + b + c) - (a \oplus b \oplus c)).$$

This theorem solves LIM completely; Example 4.2 below illustrates this in practice. We observe that the  $\mathcal{P}$ -positions of LIM are all those positions  $(a, b, c)$  such that the usual sum coincides with nim sum. This observation was first published in [9] as a conjecture, without proof.

*Proof.* As seen in Lemma 2.1,  $\frac{1}{2}((a + b + c) - (a \oplus b \oplus c)) = (a \odot b + a \odot c + b \odot c) - 2(a \odot b \odot c)$  and the  $i$ th bit of its binary expansion is 1 if and only if the binary expansions of  $a$ ,  $b$ , and  $c$ , for that bit, have two or three 1s (there are repetitions).

Let  $g = \frac{1}{2}((a + b + c) - (a \oplus b \oplus c))$ . Exhaustively we can observe that

$$\text{bit}_i(g) = \text{bit}_i(a + b - g) \oplus \text{bit}_i(a + c - g) \oplus \text{bit}_i(b + c - g).$$

In fact,

- (1) If  $\text{bit}_i(a) = 0$ ,  $\text{bit}_i(b) = 0$ , and  $\text{bit}_i(c) = 0$ , then  $\text{bit}_i(g) = 0$  and  $\text{bit}_i(a + b - g) \oplus \text{bit}_i(a + c - g) \oplus \text{bit}_i(b + c - g) = 0 \oplus 0 \oplus 0 = 0$ ;
- (2) If  $\text{bit}_i(a) = 1$ ,  $\text{bit}_i(b) = 0$ , and  $\text{bit}_i(c) = 0$ , then  $\text{bit}_i(g) = 0$  and  $\text{bit}_i(a + b - g) \oplus \text{bit}_i(a + c - g) \oplus \text{bit}_i(b + c - g) = 1 \oplus 1 \oplus 0 = 0$ ;
- (3) If  $\text{bit}_i(a) = 1$ ,  $\text{bit}_i(b) = 1$ , and  $\text{bit}_i(c) = 0$ , then  $\text{bit}_i(g) = 1$  and  $\text{bit}_i(a + b - g) \oplus \text{bit}_i(a + c - g) \oplus \text{bit}_i(b + c - g) = 1 \oplus 0 \oplus 0 = 1$ ;
- (4) If  $\text{bit}_i(a) = 1$ ,  $\text{bit}_i(b) = 1$ , and  $\text{bit}_i(c) = 1$ , then  $\text{bit}_i(g) = 1$  and  $\text{bit}_i(a + b - g) \oplus \text{bit}_i(a + c - g) \oplus \text{bit}_i(b + c - g) = 1 \oplus 1 \oplus 1 = 1$ .



But  $\text{bit}_i(g) = \text{bit}_i(a+b-g) \oplus \text{bit}_i(a+c-g) \oplus \text{bit}_i(b+c-g)$  implies naturally that  $(a+b-g) \oplus (a+c-g) \oplus (b+c-g) = g$ . By Proposition 3.2, this implies that  $\mathcal{G}(A, B, C) = g$  where  $A = b+c$ ,  $B = a+c$ ,  $C = a+b$  and  $(A, B, C)$  is the related LIM' position. This finishes the proof.  $\square$

**Example 4.2** (An example of playing LIM). Consider the game  $\text{LIM}(22, 33, 40) + *17$ . By Theorem 4.1,

$$\mathcal{G}(22, 33, 40) = \frac{(22 + 33 + 40) - (22 \oplus 33 \oplus 40)}{2} = 32.$$

Therefore, in order to win the game, we must find a move in the LIM component to  $*17$ .

The related LIM' position is  $(33 + 40, 22 + 40, 22 + 33) = (73, 62, 55)$ . Let  $h = 17$ , in order to use Proposition 3.2, we must find a NIM move in  $(73 - h, 62 - h, 55 - h) = (56, 45, 38)$  to  $*17$ .

The NIM move  $(56, 45, 38) \mapsto (26, 45, 38)$  is such a move. This was obtained subtracting  $k = 30$  from the first pile 56 so, in the original LIM position, we will add 15 to the first pile.

Returning again to LIM, we must add 15 to the first component of the LIM position. Thus  $\text{LIM}(22, 33, 40) \mapsto \text{LIM}(37, 18, 25)$  obtains the desired  $*17$ .  $\diamond$

## 5. ENUMERATION OF $\mathcal{P}$ -POSITIONS OF LIM

**Theorem 5.1.** *Consider  $\mathcal{P}_n = \{(N_1, N_2, N_3) \in \mathcal{P} : N_1 + N_2 + N_3 = n\}$  the set of  $\mathcal{P}$ -positions of LIM with  $n$  stones. Then, for  $n = 0$ ,  $|\mathcal{P}_n| = 1$ , and for  $n > 0$ ,  $|\mathcal{P}_n| = (3^{w(n)-1} + 1)/2$ , where  $w(n)$ , the binary weight of  $n$ , is the number of 1s in the binary expansion of  $n$ .*

*Proof.* If  $n = 0$ , the result is trivial.

Consider  $n$ , a positive integer. To count the  $\mathcal{P}$ -positions such that  $N_1 + N_2 + N_3 = n$ , we must distribute the bits of the binary expansion of  $n$  among the three numbers  $N_1, N_2, N_3$ . There are  $3^{w(n)}$  ways to accomplish this.

This generates 6 repetitions for positions with at most one zero and 3 repetitions for positions with two zeros. Because there are exactly 3 positions with two zeros, avoiding repetitions brings our count to

$$\frac{3^{w(n)} - 3}{6} + \frac{3}{3} = \frac{3^{w(n)-1} + 1}{2}.$$

$\square$

The sequence  $(3^{w(n)-1} + 1)/2$ , the number of  $\mathcal{P}$ -positions of LIM with  $n$  stones (including the sinks  $(0, 0, n)$ ), is the sequence A079318 in the Online Encyclopedia of Integer Sequences [8], closely related to Ulam's cellular automaton. We proceed to explain this relation.

**Ulam-Warburton Cellular Automaton.** The cells are the squares in an infinite square grid, and the neighbors of each cell are defined to be the four squares which share an edge with it. At stage 0, a single cell is turned ON. Thereafter, a cell is changed from OFF to ON at stage  $n$  if and only if exactly one of its four neighbors was ON at stage  $n - 1$ . Once a cell is ON it stays ON. Figure 2 gives the first several stages.

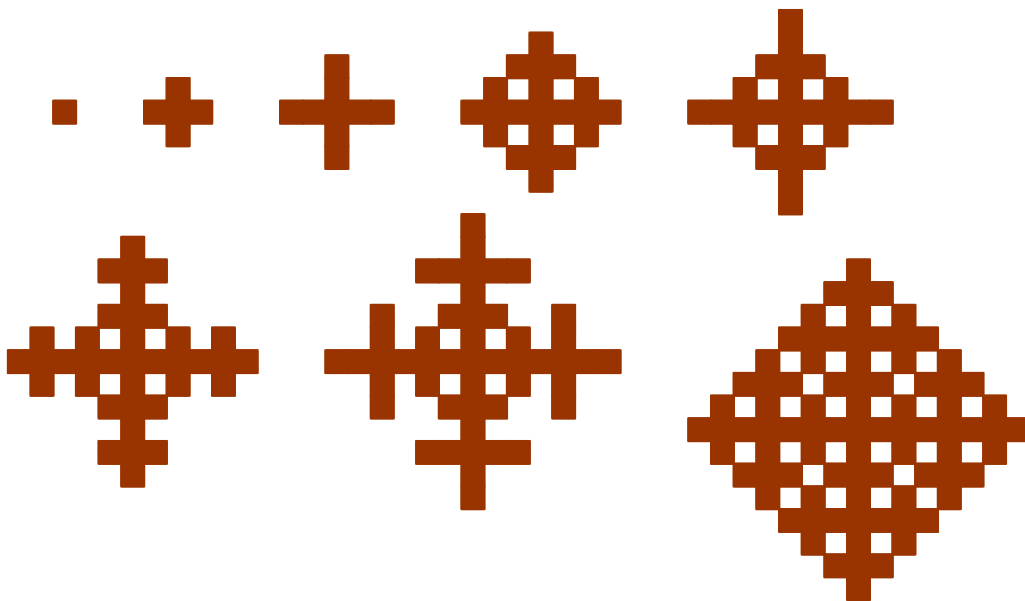


FIGURE 2. Stages 0 through 7 of the evolution of the Ulam-Warburton structure.

In [12], Richard P. Stanley proposed the following problem. Let  $\mathcal{L}$  be the integer lattice in  $\mathbb{R}^d$ , i.e.,  $\mathcal{L}$  is the set of points  $(x_1, x_2, \dots, x_d)$  with all  $x_j \in \mathbb{Z}$ . Consider a graph  $\mathbf{L}$  with vertex set  $\mathcal{L}$  by declaring two lattice points to be adjacent if the distance between them is 1. Define a sequence  $S_0, S_1, \dots$  of subsets of  $\mathcal{L}$  inductively as follows:  $S_0 = \{(0, 0, \dots, 0)\}$  and  $S_n = \{P \in \mathcal{L} \setminus \bigcup_{0 < k < n} S_k : P \text{ is adjacent to exactly one element of } \bigcup_{0 < k < n} S_k\}$ . Let  $\mathbf{S}$  be the full subgraph of  $\mathbf{L}$  whose vertices are  $S = \bigcup S_n$ . Thus,  $P \in S$  is adjacent in  $\mathbf{S}$  to  $P' \in S$  if and only if the distance between  $P$  and  $P'$  is 1.

- (a) Characterize  $S_n$ .
- (b) How many elements are in  $S_n$ ?

Later, in [4], Robin J. Chapman gave some answers. We concentrate on the special case  $d = 2$  which is studied in the present work.

- (a) Consider the binary expansion  $n = \sum_{j=1}^k 2^{r_j}$ ,  $r_1 > r_2 > \dots > r_k \geq 0$ . Consider  $v_1 = (1, 0)$ ,  $v_2 = (0, 1)$ ,  $v_3 = (-1, 0)$ , and  $v_4 = (0, -1)$ .

- $S_n$  is exactly the set of lattice points  $P$  that can be represented as  $P = \sum_{j=1}^k 2^{r_j} v_j$  such that  $v_j \neq -v_{j-1}$  for  $j > 1$ . For example,  $(3, 2) = 4(1, 0) + 2(0, 1) + 1(-1, 0)$ , so  $(3, 2) \in S_7$ .
- (b)  $|S_n| = 4 \cdot 3^{w(n)-1}$ .

The last expression and Theorem 5.1 let us guess a relation between the Ulam-Warburton structure and  $\mathcal{P}_n$ . In fact, it is possible to construct a one-to-one correspondence between  $\mathcal{P}_n$  and an octant of the UW cellular automaton.

Consider  $n = \sum_{j=1}^k 2^{r_j}$  and  $P \in S_n$  with the representation  $\sum_{j=1}^k 2^{r_j} v_j$ . Let

- (1)  $a = \sum_{j=1}^k \alpha_j 2^{r_j}$  where  
 $\alpha_j = 1$  if  $v_j = (1, 0)$  or  $(v_{j-1} = (-1, 0)$  and  $v_j = (0, -1))$ ;  $\alpha_j = 0$  otherwise
- (2)  $b = \sum_{j=1}^k \beta_j 2^{r_j}$  where  
 $\beta_j = 1$  if  $v_j = (0, 1)$  or  $(v_{j-1} = (0, -1)$  and  $v_j = (-1, 0))$ ;  $\beta_j = 0$  otherwise
- (3)  $c = \sum_{j=1}^k \gamma_j 2^{r_j}$  where  
 $\gamma_j = 1$  if  $(v_j = (-1, 0)$  and  $v_{j-1} \neq (0, -1))$  or  $(v_j = (0, -1)$  and  $v_{j-1} \neq (-1, 0))$ ;  $\gamma_j = 0$  otherwise

We have  $(a, b, c) \in \mathcal{P}_n$  and  $a \geq b \geq c$ .

Conversely, we have the following inverse association:

Consider  $(a, b, c) \in \mathcal{P}_n$ ,  $a \geq b \geq c$ , and  $n = a + b + c = \sum_{j=1}^k 2^{r_j}$ .

- (1) If  $2^{r_j}$  is a bit of the binary expansion of  $a$  then  
 (a) if the already attached  $v_{j-1} = (-1, 0)$ ,  $v_j = (0, -1)$   
 (b)  $v_j = (1, 0)$  otherwise
- (2) If  $2^{r_j}$  is a bit of the binary expansion of  $b$  then  
 (a) if the already attached  $v_{j-1} = (0, -1)$ ,  $v_j = (-1, 0)$   
 (b)  $v_j = (0, 1)$  otherwise
- (3) If  $2^{r_j}$  is a bit of the binary expansion of  $c$  then  
 (a) if the already attached  $v_{j-1} = (1, 0)$  or  $v_{j-1} = (0, -1)$ ,  $v_j = (0, -1)$   
 (b) if the already attached  $v_{j-1} = (0, 1)$  or  $v_{j-1} = (-1, 0)$ ,  $v_j = (-1, 0)$

We get the related  $P = \sum_{j=1}^k 2^{r_j} v_j$  in the octant of the UW cellular automaton.

For example,  $(6, 3) \in S_{15}$  because  $15 = 8 + 4 + 2 + 1$  and  $(6, 3) = 8(1, 0) + 4(0, 1) + 2(-1, 0) + 1(0, -1)$ . Following the one-to-one correspondence, this cell is related to  $(8 + 1, 4, 2) = (9, 4, 2) \in \mathcal{P}_{15}$ .

We observe that, as expected,  $|S_n| = 4 \cdot 3^{w(n)-1}$  is consistent with Theorem 5.1. In fact, first we take out four cells belonging simultaneously to two octants, after we divide the result by eight and, finally, we add one cell:

$$\frac{4 \cdot 3^{w(n)-1} - 4}{8} + 1.$$

Simplifying, this is the result obtained in Theorem 5.1.

## 6. FINAL REMARKS ON ENUMERATING POSITIONS

We have seen in the previous section that the  $\mathcal{P}$ -positions of LIM are enumerated by sequence A079318 in OEIS, closely related to Ulam's cellular automaton.

The rules of LIM only allow for positions to have three piles. An enumeration of the  $\mathcal{P}$ -positions of NIM with  $2n$  stones, allowing for an arbitrary number of piles, is quite difficult. However, in view of the relationship between LIM and three-pile NIM, it is not unexpected that Theorem 5.1 relates to enumeration of three-pile  $\mathcal{P}$ -positions in NIM. Sequence A128975 in OEIS nearly enumerates these positions but it insists on three *positive* piles, excluding the positions of the form  $(n, n, 0)$ : that is, it counts the unordered triples of *positive* integers  $(A, B, C)$  with  $A + B + C = n$ , whose nim sum is zero. If  $n > 0$  then  $(A, B, C)$  is a  $\mathcal{P}$ -position of NIM with  $2n$  stones and at most 3 piles if and only if  $(A, B, C)$  is a  $\mathcal{P}$ -position of LIM' with  $A + B + C = 2n$  (we are including the cases  $(n, n, 0)$ ). In fact, if  $(A, B, C)$  is a  $\mathcal{P}$ -position of LIM' then, by Proposition 3.2,  $A \oplus B \oplus C = 0$  and, conversely, if  $(A, B, C)$  is a  $\mathcal{P}$ -position of NIM, it is necessary that  $A + B + C \equiv 0 \pmod{2}$  and  $(A, B, C)$  is a triangle (if not, there would be a bit spoiling the fact that  $A \oplus B \oplus C = 0$ ).

To count the  $\mathcal{P}$ -positions of NIM with  $2n$  stones and at most 3 piles, we just need to count the  $\mathcal{P}$ -positions of LIM' with  $A + B + C = 2n$ . Because of the equivalence of LIM and LIM', this is the same as counting the  $\mathcal{P}$ -positions of LIM with  $a + b + c = n$ . Therefore, Theorem 5.1 provides the solution; for  $n > 0$ ,  $(3^{w(n)-1} + 1)/2$  is the number of different  $\mathcal{P}$ -positions of NIM with  $2n$  stones and 2 or 3 piles, and the number of these  $\mathcal{P}$ -positions is given by OEIS sequence A079318.

Along these lines, here is a problem for future work: is there a generalization of LIM, allowing arbitrarily many piles, which would aid in solving the open problem of describing the number of  $\mathcal{P}$ -positions of NIM with  $2n$  stones and any number of piles?

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