INTRODUCTION TO REPRESENTATION THEORY. 6/12/04. LECTURE 8

1. Simple finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-modules

Our next goal is to describe all finite dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$. This will be done in two steps.

Step 1. An explicit construction of all irreducible finite dimensional representations.

Step 2. To prove that any finite dimensional representation is completely reducible.

The base field is $\mathbb{C}$; We denote by $\mathbb{N}$ the set of non-negative integers.

1.1. Recall that $\mathfrak{sl}_2 \subseteq \mathfrak{gl}_2 = \langle E_{11}, E_{12}, E_{21}, E_{22}\rangle$ where E_{ij} denotes the matrix whose only non-zero entry is 1 in the (ij) position.

Note that in this notation $e = E_{12},\ f = E_{21},\ h = E_{11} - E_{22}$.

Consider the polynomial algebra $\mathbb{C}[x,y]$ and define the action of $\mathfrak{gl}_2$ on it by the formulas

$$E_{11}(p) = xp'_x, \quad E_{22}(p) = yp'_y, \quad E_{12}(p) = xp'_y, \quad E_{21}(p) = yp'_x. \tag{1}$$

Lemma. *The formulas (1) define a $\mathfrak{gl}_2$-module structure on $\mathbb{C}[x,y]$.*

The set of homogeneous polynomials of a degree n is, obviously, a $\mathfrak{gl}_2$-submodule and an $\mathfrak{sl}_2$-submodule. Denote this $\mathfrak{sl}_2$-submodule by $V(n)$. Let us show that $V(n)$ is a simple $\mathfrak{sl}_2$-module.

1.1.1. *Module V(n).* Fix n. Consider the following basis of $V(n)$:

$$v_0 := x^n, v_1 := nx^{n-1}y, v_2 := n(n-1)x^{n-2}y^2, \dots,$$
$$v_k := n!/(n-k)!x^{n-k}y^k, \dots, v_n := n!y^n.$$

One has

$$Fv_k = E_{2,1}v_k = v_{k+1}, \quad Ev_k = E_{1,2}v_k = k(n+1-k)v_{k-1}, \tag{2}$$
$$Hv_k = (E_{1,1} - E_{2,2})v_k = (n-2k)v_k.$$

We see that H acts diagonally on the basis and all eigenvalues are distinct. Recall that any submodule $W \subseteq V(n)$ is spanned by the elements of our basis belonging to W. In particular, any non-zero submodule contains v_k for some k; the relations (2) imply that such a submodule contains all basis elements. Hence $V(n)$ is simple.

1.2. **We have got all of them...** Now we will prove there are no finite dimensional irreducible representations of $\mathfrak{sl}_2$ except for the $V(n)$ described above.

1.2.1. *Definitions.* A vector v of $\mathfrak{sl}_2$-module is called *a weight vector* if $Hv \in \mathbb{C}v$.

A vector v of $\mathfrak{sl}_2$-module is called *of weight c* ($c \in \mathbb{C}$) if $Hv = cv$.

A vector v of $\mathfrak{sl}_2$-module is called *primitive* if $Ev = 0$.

1.2.2. The set of vectors of weight λ in V is denoted V_λ.

Let $v \in V_\lambda$. We claim that $Ev \in V_{\lambda+2}$ and $Fv \in V_{\lambda-2}$. In fact,

$$HEv = EHv + [H,E]v = \lambda Ev + 2Ev = (\lambda+2)Ev$$

and similarly for Fv.

1.2.3. Let V be a finite dimensional $\mathfrak{sl}_2$-module. We claim that V has a primitive weight vector.

In fact, $H : V \to V$ is an endomorphism of a finite dimensional vector space. Therefore, H admits an eigenvector $v \in V$. Let $v \in V_\lambda$. Then $E^k v \in V_{\lambda+2k}$. Since V is finite dimensional, this proves that $E^k v = 0$ for k big enough. Thus, if $n = \max\{k | E^k v \neq 0\}$, the element $E^n v$ is a primitive weight vector.

1.2.4. Let V be a finite dimensional $\mathfrak{sl}_2$-module and let v_0 be a primitive vector of weight λ.

Put $v_n = F^n v_0$. One has $Fv_n = v_{n+1}$ and $Hv_n = (\lambda - 2n)v_n$. It turns out there is an very nice formula for Ev_n.

In Lemma 1.2.4 below we will prove the following identity.

$$EF^k = F^k E + kF^{k-1}(H - (k-1)). \tag{3}$$

The formula (3) implies that

$$Ev_n = EF^n v_0 = F^n E v_0 + nF^{n-1}(H - n + 1)v_0 = n(\lambda - n + 1)v_{n-1}.$$

Let us rewrite once more these formulas

$$Fv_n = v_{n+1},\ Hv_n = (\lambda - 2n)v_n,\ Ev_n = n(\lambda - n + 1)v_{n-1}. \tag{4}$$

Lemma. *The identity (3) is valid for any $n \geq 1$ for any representation of $\mathfrak{sl}_2$.*

Proof. Induction on k. For $k = 1$ it says that $EF = FE + H$ which is obvious. Suppose it has already been proven for $k = n$ and let $k = n + 1$. We have

$$EF^{n+1} = EF^n F = (F^n E + nF^{n-1}(H - n + 1))F = F^n EF + nF^{n-1}(H - n + 1)F = \\ F^n(FE + H) + nF^{n-1}F(H - n + 1) - nF^{n-1}(2F) = F^{n+1}E + (n + 1)F^n(H - n).$$

□

1.2.5. We have made a substantial progress. In fact, we already know that any finite dimensional $\mathfrak{sl}_2$ module V contains a primitive weight vector v_0; The collection of $v_n = F^n v_0$ is a submodule. This implies that only finite number of v_i is nonzero.

This has very unexpected consequences. In fact, suppose $n = \max\{i | v_i \neq 0\}$. Then

$$0 = Ev_{n+1} = (n + 1)(\lambda - n)v_n$$

and this implies that $\lambda = n$.

We have (easily!) proven the following

1.2.6. **Theorem.** *Let V be a finite dimensional representation and let v_0 be a weight primitive vector of weight λ. Then $\lambda \in \mathbb{N}$. The submodule of V generated by v_0 is $\langle v_0, v_1, \ldots, v_\lambda \rangle$. It has dimension $\lambda + 1$ and its module structure is given by the formulas (4).*

2. The structure of completely reducible modules (general stuff).

2.1. Jordan-Hölder theorem: any finite dimensional module N has (a non-unique) Jordan-Hölder series:

$$0 \subset N_1 \subset \ldots \subset N_{r-1} \subset N_r = N.$$

such that all N_i are submodules of N and all quotients N_i/N_{i-1} are simple. The set of subquotients $\{N_i/N_{i-1}\}_{i=1}^r$ is uniquely determined by N.

2.1.1. Example: ρ is a representation of one-dim. Lie algebra $L = kx$ where N has a basis $\{e_1, e_2, e_3\}$ and

$$\rho(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then

$$span\{e_1\} \subset span\{e_1, e_2\} \subset span\{e_1, e_2, e_3\} = N$$

is a Jordan-Hölder series; the set of subquotients contains 3 simple (one-dimensional) modules: two trivial modules and the one-dimensional given by $\rho'(x) = 2\,\mathrm{id}$.

2.2. **Theorem.** *The following conditions are equivalent*

(i) A module N is completely reducible.

(ii) A module N is a sum of simple submodules.

(iii) Any submodule N' of N is a direct summand that is there exists a submodule N'' of N such that $N = N' \oplus N''$.

2.2.1. **Corollary.** *All submodules and quotient modules of a completely reducible module are completely reducible.*

2.2.2. Let N be a module and N' be its submodule. A projection $\iota : N \to N'$ is a homomorphism whose restriction to N' is the identity map. The existence of a projection $\iota : N \to N'$ is equivalent to the condition that N' is a direct summand of N.

2.2.3. Let N be a direct sum of simple modules $\{N_i\}_{i\in I}$ and V be a simple module. *The isotypical component of V in N* is the sum of all simple submodules of N which are isomorphic to V.

2.2.4. **Lemma.** *Let N be a direct sum of simple modules $\{N_i\}_{i\in I}$. Any simple submodule of N is isomorphic to some N_i.*

Proof. Let V be a simple submodule of N. Theorem 2.2 (iii) implies the existence of a projection $\phi : N \to V$. Since $\phi \neq 0$, the restriction of ϕ to some N_i is non-zero. This restriction is a non-zero homomorphism from a simple module N_i to a simple module V. Therefore it is an isomorphism that is $N_i \cong V$ as required. $\square$

2.2.5. **Lemma.** *Let N be a direct sum of simple modules $\{N_i\}_{i\in I}$ and V be a simple module. The isotypical component N_V of V in N takes form*

$$N_V = \oplus_{i\in X:N_i\cong V} N_i.$$

Proof. It sufficies to show that any submodule V' of N which is isomorphic to V lies in

$$N' := \oplus_{i\in X:N_i\cong V} N_i.$$

Set $X' := i \in X : N_i \cong V$. One has $N = N' \oplus N''$ where

$$N'' :== \oplus_{i\in X\setminus X'} N_i.$$

Let ι be a projection of N to N'' with respect to the decomposition $N = N' \oplus N''$ (this means that the kernel of ι is N' and the restriction of ι to N'' is the identity map). The condition $V' \subset N'$ is equivalent to the condition $\iota(V') = 0$. Suppose that $\iota(V') \neq 0$. Since V' is simple, ι is injective and so N'' contains a submodule isomorphic to $V \cong V'$. This contradicts to Lemma 2.2.4. The lemma is proven. $\square$

2.2.6. Now we are ready to describe all submodules of completely reducible module. We will assume for simplicity that the base field is algebraically closed and that our module is finite dimensional (eventhough a similar description exists). Let $\mathfrak{g}$ be a Lie algebra and let N be a direct sum of simple $\mathfrak{g}$-modules $\{N_i\}_{i\in I}$. Any submodule M of N is a direct sum of its isotypical components M_V (where V runs through the set of simple modules). For a simple module V, M_V is a submodule of N_V. The submodules of N_V are in one-to-one correspondance with a subspaces of the vector space $\mathrm{Hom}_{\mathfrak{g}}(V, N)$: if M is a submodule then $\mathrm{Hom}_{\mathfrak{g}}(V, M)$ is a subspace of $\mathrm{Hom}_{\mathfrak{g}}(V, N)$ and if L is subspace of $\mathrm{Hom}_{\mathfrak{g}}(V, N)$ then the corresponding submodule is given by $M := \sum_{f\in L} f(V)$.

It remains to desribe the vector space $\mathrm{Hom}_{\mathfrak{g}}(V, N)$.

Fix a simple module V and set $X' := i \in X : N_i \cong V$. For each $i \in X'$ choose a homomorphism $\theta_i : V \to N$ whose image coincides with N_i (by Schur's lemma, this homomorphism is unique up to a scalar). We claim that $\{\theta_i\}_{i\in X'}$ form a basis of $\mathrm{Hom}_{\mathfrak{g}}(V, N)$. Indeed, the linear independence of $\{\theta_i\}_{i\in X'}$ follows immediately from the fact that the sum $\sum_{i\in X'} N_i$ is direct. Take $\theta \in \mathrm{Hom}_{\mathfrak{g}}(V, N)$ and let $p_i : N \to N$ be a homomorphism with the kernel $\oplus_{j\in X: j\neq i} N_j$ whose restriction to N_i is the identity map. The composition $p_i \circ \theta$ is a homomorphism from V to N whose image lies in N_i. Thus $p_i \circ \theta$ is zero if $i \notin X'$ and is proportional to θ_i if $i \in X'$. Hence for each $i \in X$ there exists a scalar a_i such that $p_i \circ \theta = a_i\theta_i$. Using the equality $\sum_{i\in X} p_i = \mathrm{id}$, one obtains $\theta = \sum_i a_i\theta_i$ as required.

2.2.7. **Definition.** If N is completely reducible module and V is a simple module, the dimension of $\mathrm{Hom}_{\mathfrak{g}}(V, N)$ is called *the multiplicity of V in N*.

We showed that for any decomposition of N into a direct sum of simple modules, the number of summands which are isomorphic to V is equal to the multiplicity of V in N.

3. Complete reducibility of a finite dimensional $\mathfrak{sl}(2, \mathbb{C})$ module

3.0.8. *The action of Casimir operator.* Let N be a finite dimensional $\mathfrak{sl}(2)$-module. Recall that

$$Q := H^2 + 2FE + 2EF = H^2 + 2H + 4FE \in \mathrm{End}_{\mathfrak{sl}(2)}(N).$$

If $v_0 \in N$ is a weight primitive vector of weight λ, one has $Qv_0 = (\lambda^2 + 2\lambda)v_0$. In particular, if $N = V(n)$ one has $Q = (n^2 + 2n)\,\mathrm{id}$ that is $Q + \mathrm{id} = (n + 1)^2\,\mathrm{id}$. If

$$N = V(m_1)^{\oplus r_1} \oplus \ldots \oplus V(m_s)^{\oplus r_s}$$

where m_i are distinct one has

$$V(m_i)^{\oplus r_i} := \{v \in N |\ (Q + \mathrm{id})(v) = (m_i + 1)^2\}.$$

Using this formula, one can prove

Theorem. *Any finite dimensional $\mathfrak{sl}(2)$-module is completely reducible.*

Complete reducibility of finite dimensional representations holds for all complex finite dimenionsal simple Lie algebras (H.Weyl theorem).

3.0.9. **Corollary.** *The element H acts semisimply on any finite dimensional $\mathfrak{sl}(2,\mathbb{C})$ module.*

3.1. Let N be a finite dimensional $\mathfrak{sl}(2,\mathbb{C})$ module. Since N is completely reducible, its structure is completely determined by the multiplicity of each $V(m)$ in N.

It turns out that these multiplicities can be easily expressed through the dimensions of weight spaces of N. Recall that H acts semisimply on N. Moreover all weights are integer because all weights of any simple finite dimensional module are integer. For any $c \in \mathbb{C}$ set

$$N_c := \{v \in N \mid Hv = cv\}.$$

Observe that $N_c = 0$ for all but finite number of integer values of c and that $\dim N_m = \dim N_{-m}$.

3.1.1. **Proposition.** *For $m \in \mathbb{N}$ the multiplicity of each $V(m)$ in N is equal to* $\dim N_m - \dim N_{m+2}$.

Proof. This follows from the fact that $\dim V(m)_c = 1$ for $c \in \{m, m-2, m-4, \dots, -m\}$ and that $\dim V(m)_c = 0$ for other values of c.

Indeed, for each $n \in \mathbb{N}$ let $ml(n)$ be the multiplicity of $V(n)$ in N. Then for $c \in \mathbb{N}$ one has

$$\dim N_c = \sum_{k \geq c; k-c \text{ is even}} ml(k)$$

and

$$\dim N_{c+2} = \sum_{k \geq c+2; k-(c+2) \text{ is even}} ml(k) = \sum_{k > c; k-c \text{ is even}} ml(k)$$

and so $\dim N_c - \dim N_{c+2} = ml(n)$ as required. ☐

3.1.2. *Example.* Let us compute $V(1) \otimes V(1)$. Let v_1, v_{-1} be a weight basis of $V(1)$: $hv_i = iv_i$. One has

$$h(v_i \otimes v_j) = (i+j)(v_i \otimes v_j).$$

That is $N := V(1) \otimes V(1)$ takes form

$$N = N_2 \oplus N_0 \oplus N_2$$

where $N_2 = span(v_1 \otimes v_1)$, $N_0 = span(v_1 \otimes v_{-1}, v_{-1} \otimes v_1)$, $N_{-2} = span(v_{-1} \otimes v_{-1})$. Hence $N = V(2) \oplus V(0)$.

Notice that $\Lambda^2(V(1))$ is one dimensional and $V(1) \otimes V(1) = \Lambda^2(V(1)) \oplus S^2(V(1))$. Hence $\Lambda^2(V(1)) \cong V(0)$, $S^2(V(1)) \cong V(2)$ that is

$$V(0) = span\{v_1 \otimes v_{-1} - v_{-1} \otimes v_1\}$$

and

$$V_2 = span\{v_1 \otimes v_1,\ v_1 \otimes v_{-1} + v_{-1} \otimes v_1,\ v_{-1} \otimes v_{-1}\}.$$