

# On constructing expanders for any number of vertices

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**Abstract.** While typical constructions of explicit expanders work for certain sizes (i.e., number of vertices), one can obtain constructions of about the same complexity by manipulating the original expanders. One way of doing so is detailed and analyzed below.

For any  $m \in [0.5n, n]$  (equiv.,  $n \in [m, 2m]$ ), given an  $m$ -vertex expander,  $G_m$ , we construct an  $n$ -vertex expander by connecting each of the first  $n - m$  vertices of  $G_m$  to an (otherwise isolated) new vertex, and adding edges arbitrarily to regain regularity. The analysis of this construction uses the combinatorial definition of expansion.

A preliminary version of this memo was posted on the author's web-site in October 2019.<sup>1</sup> The current revision corrects various typos.

## 1 The story, which can be skipped

Expander graph have numerous applications in the theory of computation (see, e.g., [3]), which explains the extensive interest in constructing these objects. Actually, when talking about expander graphs, one typically refers to families of regular graphs of fixed degree for a varying number of vertices that are  $\Omega(1)$ -expanding, where the expansion factor is fixed for the entire family. That is, there exists a constant  $c > 0$  such that for every graph  $G = (V, E)$  in the family, and every  $S \subseteq V$  of size at most  $|V|/2$ , it holds that

$$|\{v \in V \setminus S : \exists u \in S \text{ s.t. } \{u, v\} \in E\}| \geq c \cdot |S|.$$

While designers of expanders focus on optimizing various parameters, their users tend to care most of having explicit expanders for any number of vertices (i.e., for any size). The most popular notions of being explicit are a *minimal* notion that requires that the graph be constructed in time that is polynomial in its length, and a *stronger* notion that requires that the neighbors of each vertex in each graph can be identified in time that is poly-logarithmic in the size of the graph (equiv., polynomial in the size of the description of the vertices, assuming a non-redundant representation).<sup>2</sup>

Unfortunately, typical constructions of explicit expanders work only for certain sizes (i.e., number of vertices). Yet, fortunately, one can obtain constructions

<sup>1</sup> See [http://www.wisdom.weizmann.ac.il/~oded/p\\_ex4all.html](http://www.wisdom.weizmann.ac.il/~oded/p_ex4all.html)

<sup>2</sup> See [3, Def. 2.3] or [2, Apdx. E.2.1.2].

of the same level of explicitness (or complexity) by manipulating the original expanders. One such construction was presented recently by Murtagh *et al.* [4]. It reminded me of a different construction, which I heard from Noga Alon many years ago. (In fact, checking something else in [2, Apdx. E.2], I noticed that I used Noga's construction there (see last paragraph in [2, Apdx. E.2.1.2])...)

The starting point in both cases is a construction for a “dense” set of sizes  $M$ ; that is, for every  $n \in \mathbb{N}$  there exists  $m \in M$  and an explicit  $m$ -vertex expander such that  $m \in [0.5n, n]$  (equiv.,  $n \in [m, 2m]$ ). The aim is to obtain an explicit  $n$ -vertex expander, for any given  $n \in \mathbb{N}$ .

The construction of Murtagh *et al.* [4] takes an  $m$ -vertex graph, where  $m \in M \cap [0.5n, n]$ , designates  $n - m$  pairs of vertices in it, joins each such pair to a single vertex (doubling the degree), and adds self-loops on the other  $m - (n - m)$  vertices to regain regularity. The analysis of this construction is conducted in terms of the algebraic definition of expansion (i.e., eigenvalues), and is presented in [4, Apdx. B]. Assuming that the  $m$ -vertex graph has a second eigenvalue smaller (in absolute value) than  $\beta < 1/3$ , the resulting  $n$ -vertex graph has a second eigenvalue smaller than  $(1 + 3\beta)/2$ .

Noga Alon's construction starts by picking  $m_1 \in M \cap [0.5n, n]$ . Discarding the fortunate case of  $m_1 = n$ , note that if  $m_1 = n/2$  we are done by connecting two copies of the  $m_1$ -vertex graph by a matching. The resulting  $n$ -vertex graph is shown to be an expander using the combinatorial definition of expansion (i.e., the expansion of vertex-sets). In general, we set  $r_1 = n - m_1 \in (0, 0.5n]$ , and proceed by picking  $m_2 \in M \cap [0.5r_1, r_1]$ , setting  $r_2 = r_1 - m_2$ , and so on; that is, in iteration  $i$  we pick  $m_i \in M \cap [0.5r_{i-1}, r_{i-1}]$  and set  $r_i = r_{i-1} - m_i$ , till we get to  $r_t = O(1)$ . At this point we connect the vertices of the  $t - 1$  smaller graphs to  $\sum_{i=2}^t m_i$  vertices of the  $m_1$ -vertex graph by using a matching (and add self-loops to maintain regularity).

The analysis of Noga's construction is less trivial than it seems. The source of trouble is that, when analyzing the expansion of sets, one needs to consider sets of size at most  $n/2$  and such sets may have more than  $m_1/2$  vertices in the large ( $m_1$ -vertex) expander. This difficulty can be resolved by using a definition that guarantees expansion also for larger sets (actually, it suffices to guarantee expansion for sets that have density at most  $3/4$ ). Furthermore, the standard definition of expansion does imply expansion also for larger sets (as needed above).

Thinking a little more about Noga's suggestion, I realized that, if one does not care about the specific expansion parameters, then the smaller expanders play no real role. Hence, the added small expanders can be replaced by isolated vertices; that is, wishing to have an  $n$ -vertex expander and given an  $m$ -vertex expander such that  $m \in [0.5n, n]$ , we connect  $n - m$  auxiliary vertices (which are otherwise isolated) to  $n - m$  vertices of the original expander (and then add edges arbitrarily to recover regularity). The analysis works via the combinatorial definition of expansion, with the aforementioned caveat.

## 2 The actual construction and its analysis

While typical constructions of explicit expanders work for certain sizes (i.e., number of vertices), one can obtain construction of about the same complexity by manipulating the original expanders. One way of doing so is detailed and analyzed below.

*The construction.* For  $m \in [0.5n, n]$  (equiv.,  $n \in [m, 2m]$ ), given an  $m$ -vertex expander,  $G_m$ , we construct an  $n$ -vertex expander by connecting each of the first  $n - m$  vertices of  $G_m$  to an (otherwise isolated) new vertex, and add edges arbitrarily to regain regularity. Hence, we obtain a construction of expanders for all sizes, provided we are given a construction of expanders for a sufficiently dense set of sizes (which is effectively accessible as assumed below).

**Construction 1** (padding and matching with isolated vertices): *For  $d \in \mathbb{N}$ , suppose that  $M \subseteq \mathbb{N}$  and  $\{G_m\}_{m \in M}$  is a set of  $d$ -regular graphs such that the following two conditions hold.*

1. *Given any  $m \in M$ , we can construct the  $m$ -vertex graph  $G_m = ([m], E_m)$ .*
2. *For every  $n \in \mathbb{N}$ , we can determine an  $m \in M$  such that  $m \in [0.5n, n]$  (equiv.,  $n \in [m, 2m]$ ).*

*Then, we construct a  $d'$ -regular  $n$ -vertex graph  $G_n = ([n], E_n)$  by picking  $m \in M \cap [0.5n, n]$ , constructing  $G_m = ([m], E_m)$ , and letting*

$$E_n = E_m \cup \{\{i, m + i\} : i \in [n - m]\} \cup E_{m,n},$$

*where  $d' \in \{d + 1, d + 2\}$  and  $E_{m,n}$  is an arbitrary set of  $\frac{d' \cdot n - d \cdot m}{2} - (n - m)$  edges that is added so to make  $G_n$  be  $d'$ -regular. Specifically,  $d' = d + 1$  may be used if either  $n$  is even or  $d$  is odd, and  $d' = d + 2$  is used otherwise.*

We say that a graph  $G = (V, E)$  is  $(\rho, c)$ -expanding if for every  $S \subset V$  such that  $|S| \leq \rho \cdot |V|$  it holds that  $|\partial(S)| \geq c \cdot |S|$ , where  $\partial(S) = \{u \in V \setminus S : \exists v \in S \text{ s.t. } \{v, u\} \in E\}$  is the boundary of  $S$ . The standard definition of expansion corresponds to  $(0.5, \Omega(1))$ -expansion, but it implies  $(\rho, \Omega(1))$ -expansion for any constant  $\rho < 1$ .<sup>3</sup> Hence, when showing that  $G_n$  is an expander, we may assume that  $G_m$  is  $(0.75, \Omega(1))$ -expanding, rather than  $(0.5, \Omega(1))$ -expanding.

**Theorem 2** (analysis of Construction 1): *If  $G_m$  is  $(0.75, c)$ -expanding, then  $G_n$  is  $(0.5, c/2)$ -expanding.*

<sup>3</sup> Assume that the graph is  $(0.5, c)$ -expanding, and let  $S \subset V$  be an arbitrary set such that  $0.5 \cdot |V| < |S| \leq \rho \cdot |V|$ . Then,  $R \stackrel{\text{def}}{=} V \setminus (S \cup \partial(S))$  has cardinality smaller than  $0.5 \cdot |V|$ , and it follows that  $|\partial(R)| \geq c \cdot |R|$ . On the other hand,  $\partial(R) \subseteq \partial(S)$ , and so  $|\partial(S)| \geq c \cdot |R| = c \cdot (|V| - |S| - |\partial(S)|)$ . Hence,  $|\partial(S)| \geq \frac{c}{1+c} \cdot (|V| - |S|) \geq \frac{c}{1+c} \cdot \frac{1-\rho}{\rho} \cdot |S|$ , and it follows that the graph is  $(\rho, c')$ -expanding for  $c' = \frac{c \cdot (1-\rho)}{(1+c) \cdot \rho}$ .

The proof does not use the edges in  $E_{m,n}$ , which makes sense given their arbitrary choice. Yet, it is quite likely that a more careful analysis of other aspects will yield a stronger result. In particular, assuming that  $G_m$  is  $(0.5, c)$ -expanding, we only conclude that  $G_n$  is  $(0.5, c/12)$ -expanding<sup>4</sup> (so the real challenge is to establish a higher expansion bound for  $G_n$ , when assuming that  $G_m$  is  $(0.5, c)$ -expanding).

**Proof:** Recall that  $0 \leq n - m \leq m$ . For an arbitrary set  $S \subset [n]$  of size at most  $0.5n$ , we consider the following four disjoint subsets of  $S$ :

$$\begin{aligned} S' &\stackrel{\text{def}}{=} \{i \in [n - m] : i \in S \ \& \ m + i \in S\} \\ S'' &\stackrel{\text{def}}{=} \{i \in ([m] \setminus [n - m]) : i \in S\} \\ S''' &\stackrel{\text{def}}{=} \{i \in [n - m] : i \in S \ \& \ m + i \notin S\} \\ R &\stackrel{\text{def}}{=} \{m + i \in S : i \notin S\} \end{aligned}$$

Note that  $(S', S'', S''')$  is a partition of  $S \cap [m]$  whereas  $(m + S', R)$  is a partition of  $S \setminus [m]$ . We may assume, without loss of generality, that  $S''' = \emptyset$ , because moving  $i \in S'''$  to  $m + i$  (i.e., replacing  $S$  by  $(S \setminus \{i\}) \cup \{m + i\}$ ) can only decrease the  $\partial(\cdot)$ -value.<sup>5</sup>

Next, we show that  $|S| \leq n/2$  implies  $|S' \cup S''| \leq 0.75 \cdot m$ . This holds because  $|S''| \leq m - (n - m) = 2m - n$ , which implies

$$\begin{aligned} |S'| + |S''| &\leq \frac{|S| - |S''|}{2} + |S''| \\ &\leq \max_{s \leq 2m - n} \left\{ \frac{|S| - s}{2} + s \right\} \\ &= \frac{|S| + 2m - n}{2} \\ &\leq \frac{2m - 0.5n}{2} \\ &\leq 0.75 \cdot m, \end{aligned}$$

where the third (resp., last) inequality is due to  $|S| \leq n/2$  (resp.,  $m \leq n$ ). Having established  $|S' \cup S''| \leq 0.75 \cdot m$  and using the  $(0.75, c)$ -expansion of  $G_m$ , we get  $|\partial(S' \cup S'')| \geq c \cdot (|S'| + |S''|)$ . Turning to  $R$ , and using the matching edges (i.e., the set  $\{i, m + i\} : i \in [n - m]\}$ ), we have  $|\partial(R)| = |R| \geq c \cdot |R|$ , since  $c \leq 1/3$ . Note that  $\partial(S' \cup S'') \cap R = \emptyset$  and  $\partial(R) \cap (S' \cup S'') = \emptyset$ , since the vertices in  $R$  are matched to vertices in  $[m] \setminus (S' \cup S'')$ . Hence,  $|\partial(S' \cup S'' \cup R)| \geq$

<sup>4</sup> We first infer that  $G_m$  is  $(0.75, c')$ -expanding for  $c' = \frac{0.25 \cdot c}{0.75 \cdot (1+c)}$  (see Footnote 3). Hence,  $G_n$  is  $(0.5, c'')$ -expanding for  $c'' = \frac{c}{6(1+c)}$ . Using  $c \leq 1$ , we get  $c'' \geq c/12$ .

<sup>5</sup> Suppose that  $i \in [m] \cap S$  and  $m + i \in [n] \setminus S$ , and let  $T = (S \setminus \{i\}) \cup \{m + i\}$ . Then,  $i \notin \partial(S)$  and  $m + i \in \partial(S)$ , whereas  $i \in \partial(T)$  and  $m + i \notin \partial(T)$ , which means that  $|\partial(T) \cap \{i, m + i\}| = 1 = |\partial(S) \cap \{i, m + i\}|$ . However,  $\partial(T) \setminus \{i, m + i\} \subseteq \partial(S) \setminus \{i, m + i\}$ , since the move may only eliminate a contribution of  $i$  to  $\partial(S) \setminus \{i, m + i\}$  (whereas  $m + i$  does not contribute to  $\partial(T) \setminus \{i, m + i\}$ ).

$c \cdot (|S'| + |S''| + |R|)/2$ , since each vertex may contribute at most twice to the sum  $|\partial(S' \cup S'')| + |\partial(R)|$ . Noting that  $|S'| + |S''| + |R| \geq |S|/2$ , we infer that  $|\partial(S)| \geq c \cdot |S|/4$ .

Using a more careful analysis, we note that  $|\partial(S' \cup S'' \cup R)| \geq |\partial(S' \cup S'')| + |\partial(R)| \geq c \cdot (|S'| + |S''|) + 0.5 \cdot |R|$ , since the double contribution may occur only on elements of  $\partial(R)$  and  $|\partial(R)| = |R|$ . Using  $|S| = 2 \cdot |S'| + |S''| + |R|$ , we get

$$\begin{aligned} |\partial(S' \cup S'' \cup R)| &\geq c \cdot (|S'| + |S''|) + 0.5 \cdot |R| \\ &= c \cdot \left( \frac{|S| - (|S''| + |R|)}{2} + |S''| \right) + 0.5 \cdot |R| \\ &= c \cdot \frac{|S| + |S''|}{2} + \frac{1-c}{2} \cdot |R|, \end{aligned}$$

and the claim follows (since  $c \leq 1/3$ ). ■

### 3 Postscript

As mentioned in Section 1, it turns out that I did mention Noga Alon’s construction in [2, Apdx. E.2.1.2] (but forgot of this). Also, it seems that Noga has mentioned the construction (and/or variants of it) in some old papers of his. Asking him about this in October 2019, he suggested a few alternative constructions, which are aimed at better expansion parameters. My favorite one, starts with an  $m$ -vertex  $d$ -regular graph,  $G_m$ , for  $n \in [m, m + o(m)]$ , and obtains an  $n$ -regular  $d'$ -regular graph by connecting each of the  $n - m$  new vertices to  $d'$  different old vertices.

A combinatorial analysis of the resulting graph,  $G_n$ , maintains much of the expansion features of  $G_m$ . Specifically, assume that, for some monotone non-decreasing function  $\mathbf{X} : [m] \rightarrow [m]$  (e.g.,  $\mathbf{X}(s) = \Omega(d \cdot s)$  for  $s < m/2d$ ), every  $s$ -subset of vertices of  $G_m$  has at least  $\mathbf{X}(s)$  neighbors (in  $G_m$ ) that are outside it. Consider an arbitrary set  $S \subset [n]$  of vertices in  $G_n$ , and let  $S' \stackrel{\text{def}}{=} S \cap [m]$  and  $S'' = S \setminus S'$ . If  $|S''| > \mathbf{X}(|S|)/2d'$ , then  $|\partial(S)| \geq |\partial(S'') \setminus S'| \geq d' \cdot |S''| - |S| > 0.5 \cdot \mathbf{X}(|S|) - |S|$ , since  $|\partial(S'')| = d' \cdot |S''|$ . Otherwise,  $|\partial(S)| \geq |\partial(S') \cap [m]| \geq \mathbf{X}(|S'|) \geq \mathbf{X}(|S| - (\mathbf{X}(|S|)/2d')) \geq \mathbf{X}(|S|/2)$ , since  $\mathbf{X}(|S|) \leq d \cdot |S|$ .

Noga is currently writing a paper with a spectral analysis of some of these alternative construction [1]. The tentative abstract reads as follows.

An  $(n, d, \lambda)$ -graph is a  $d$ -regular graph on  $n$  vertices in which the absolute value of any nontrivial eigenvalue is at most  $\lambda$ .

- For any constant  $d \geq 3$  and  $\epsilon > 0$ , and all sufficiently large  $n$  we show that there is a deterministic  $\text{poly}(n)$ -time algorithm that outputs an  $(n, d, \lambda)$ -graph (on exactly  $n$  vertices) with  $\lambda \leq 2\sqrt{d-1} + \epsilon$ .
- For any  $d = p + 2$  with  $p \equiv 1 \pmod 4$  prime and all sufficiently large  $n$ , we describe a strongly explicit construction of an  $(n, d, \lambda)$ -graph (on exactly  $n$  vertices) with  $\lambda \leq \sqrt{2(d-1)} + \sqrt{d-2} + o(1)$  ( $< (1 + \sqrt{2})\sqrt{d-1} + o(1)$ ), with the  $o(1)$  term tending to 0 as  $n$  tends to infinity.

- For every  $\epsilon > 0$ ,  $d > d_0(\epsilon)$  and  $n > n_0(d, \epsilon)$  we show a strongly explicit construction of an  $(m, d, \lambda)$ -graph with  $\lambda < (2 + \epsilon)\sqrt{d}$  and  $m = n + o(n)$ .

All constructions are obtained by starting with known ones of Ramanujan or nearly Ramanujan graphs, modifying or packing them in an appropriate way. The spectral analysis relies on the delocalization of eigenvectors of regular graphs in cycle-free neighborhoods.

## References

1. N. Alon. Explicit expanders of every degree and size. In preparation, 2019.
2. O. Goldreich. *Computational Complexity: A Conceptual Perspective*. Cambridge University Press, 2008.
3. S. Hoory, N. Linial, and A. Wigderson. Expander Graphs and Their Applications. *Bulletin (New Series) of the American Mathematical Society*, Vol. 43 (4), pages 439–561, 2006.
4. J. Murtagh, O. Reingold, A. Sidford, and S. Vadhan. Deterministic Approximation of Random Walks in Small Space. In *23rd RANDOM, LIPIcs 145*, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019.