The standard notion of approximating majority refers to distinguishing between $n$-bit strings that have at least $2n/3$ one-entries and $n$-bit strings that have at most $n/3$ one-entries. For sake of clarity and generality, we consider the following notion of approximating the Hamming weight of strings.

For $x = x_1 \cdots x_n \in \{0,1\}^n$, we let $\text{wt}(x) \overset{\text{def}}{=} |\{i \in [n]: x_i = 1\}|$ denote the Hamming weight of $x$. The relative weight of a string $x$ is $\overline{\text{wt}}(x) \overset{\text{def}}{=} \text{wt}(x)/|x|$. For fixed $\rho, \epsilon \in (0,1)$, we consider the promise problem in which the YES-instances are strings of relative weight at least $\rho$ and the NO-instances are strings of relative weight at most $\rho - \epsilon$.

We first observe that, for every $x \in \{0,1\}^n$, if we select a random $\ell$-subset, denoted $S \subset [n]$, where $\ell = \Theta(\log n)$, then, with probability at least $1 - n^{-3}$, it holds that $\overline{\text{wt}}(x_S) = \overline{\text{wt}}(x) \pm \epsilon/3$, where $x_S$ denotes the projection of $x$ on $S$.

Selecting $m = n^2$ such subsets $S_1, \ldots, S_m$, with probability at least $1 - 2^{-n}$, it holds that, for every $x \in \{0,1\}^n$, more than $m - n$ of the $S_i$’s satisfy $\overline{\text{wt}}(x_{S_i}) = \overline{\text{wt}}(x) \pm \epsilon/3$. This is the case because for every $x \in \{0,1\}^n$ we have

$$\Pr_{S_1, \ldots, S_m \in \binom{[n]}{\ell}} \left[ |\{i \in [m]: |\overline{\text{wt}}(x_{S_i}) - \overline{\text{wt}}(x)| > \epsilon/3\}| \geq n \right] = \left( \frac{m}{n} \right) \cdot \left( \Pr_{S \in \binom{[n]}{\ell}} \left[ |\overline{\text{wt}}(x_{S}) - \overline{\text{wt}}(x)| > \epsilon/3\right] \right)^n < m^n \cdot \left(1/n^3\right)^n = n^{-n}.
$$

Hence, there exists a sequence of $m$ sets, denoted $S_1, \ldots, S_m$, each of size $\ell$, such that for every $x \in \{0,1\}^n$ it holds that

$$|\{i \in [m]: |\overline{\text{wt}}(x_{S_i}) - \overline{\text{wt}}(x)| > \epsilon/3\}| < n.
$$

Fixing this sequence, and defining $F : \{0,1\}^{\ell} \rightarrow \{0,1\}$ such that $F(z) = 1$ if and only if $\overline{\text{wt}}(z) > \rho - (\epsilon/2)$, we consider the formula

$$\Phi(x) \overset{\text{def}}{=} \bigvee_{j \in [n]} \bigwedge_{k \in [n]} F(x_{S_{(j-1)n+k}}).
$$

Clearly, $F$ can be implemented by a poly($n$)-size CNF, and so the foregoing formula is in $\mathcal{AC}^0$. Furthermore, $\Phi$ is a monotone formula, because $F$ is a monotone function. We now show that this formula decides correctly on each input that satisfies the promise.

The case of YES-instances: If $\overline{\text{wt}}(x) \geq \rho$, then $|\{i \in [m]: \overline{\text{wt}}(x_{S_i}) \leq \rho - \epsilon/2\}| < n$.

It follows that there exists a $j \in [n]$ such that for every $k \in [n]$ it holds that $F(x_{S_{(j-1)n+k}}) = 1$.

Hence, $\Phi(x) = 1$.

The case of NO-instances: If $\overline{\text{wt}}(x) \leq \rho - \epsilon$, then $|\{i \in [m]: \overline{\text{wt}}(x_{S_i}) > \rho - \epsilon/2\}| < n$.

It follows that for every $j \in [n]$ there exists a $k \in [n]$ such that $F(x_{S_{(j-1)n+k}}) = 0$. Hence, $\Phi(x) = 0$.

**Perspective:** Recall that Majority is not in $\mathcal{AC}^0$. In fact, any constant depth (unbounded fan-in) circuit (with AND, OR, and NOT gates) that computes Majority must have sub-exponential size; specifically, computing $n$-way Majority in depth $d$ requires size $\exp(\Omega(n^{1/2d}))$. 
