On the NP-Completeness of Certain Network Testing Problems

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Let $G(V, E)$ be an undirected graph which describes the structure of a communication network. During the maintenance period every line must be tested in each of the two possible directions. A line is tested by assigning one of its endpoints to be a transmitter, the other to be a receiver, and sending a message from the transmitter to the receiver through the line. We define several different models for communication networks, all subject to the two following axioms: a vertex cannot act as a transmitter and as a receiver simultaneously and a vertex cannot receive through two lines simultaneously. In each of the models, two problems arise: What is the maximum number of lines one can test simultaneously? and What is the minimum number of phases necessary for testing the entire network?, where, by "phase" we mean a period in which some tests are conducted simultaneously. We show that in most models, including the "natural" model of radio communication, both problems are NP-hard. In some models the problems can be solved by reducing them to either a maximum matching problem or an edge coloring problem for which polynomial algorithms are known. One model remains for which the complexity of the minimization problem is unknown.

I. INTRODUCTION

Let $G(V, E)$ be an undirected graph which describes the structure of a communication network. During the maintenance period every line must be tested in each of the two directions. A line is tested by assigning one of its endpoints to be a transmitter, the other to be a receiver, and sending a text message from the transmitter to the receiver through the line.

The first set of problems we shall examine are of the type What is the maximum number of lines one can test simultaneously? Different problems arise when the ground rules are changed. Let us call the set of tests which are performed simultaneously a *phase*.

The second set of problems we shall examine are of the type What is the minimum number of phases necessary for testing all the lines in both directions? Again, the problems differ by the ground rules to be assumed.

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We shall always assume that a vertex can be either a transmitter or a receiver during one phase, but not both. Also, a vertex can receive through one line only, during a certain phase.

However, we may or may not assume that a transmitter can transmit only on one line, during a certain phase, and different problems arise accordingly.

We may or may not assume that G is bipartite.

In certain applications *interference* is not allowed; i.e., if t_1 transmits to r_1 and t_2 transmits to r_2 and $t_1 \neq t_2$, then there is no line connecting r_1 and t_2 in the graph. This is a natural assumption in certain wireless communication networks. The assumption "interference allowed" or "interference not allowed" changes the problem.

Finally, we shall also examine the problems in which certain vertices are allowed to be transmitters only, if used at all in the test, or receivers only. This is a special subcase of the bipartite case; i.e., $G = (T, R, E)$, where T is the set of potential transmitters, R is the set of potential receivers, and E is the set of edges, where $E \subseteq T \times R$.

In order to be able to name the problem succinctly, we shall use the following notation:

MAX, if the problem is to maximize the number of lines tested in one phase.
MIN, if the problem is to minimize the number of phases.
G, for general graphs.
BG, for bipartite graphs.
MT, if a transmitter may transmit to any number of receivers during a phase.
ST, if a transmitter may transmit to one receiver only during a phase.
IA, if interference is allowed.
INA, if interference is not allowed.
PA, if vertices are preassigned as potential receivers or transmitters.
NPA, if vertices are not preassigned.

A problem is now defined by a quintuple. For example, (MAX, G, MT, IA, NPA) is the problem of maximizing the number of lines to be tested in one phase, of a general graph, if a vertex can transmit to any number of neighbors, interference is allowed, and there is no preassignment. Also, we shall refer to a problem by number. For example, the problem above is No. 1 since the corresponding quintuple is (0, 0, 0, 0, 1).

The most natural problems, which were suggested to us by "network people" are Nos. 17, 19, and 7. We shall prove that these problems, and many others, are NP-hard (see ref. 1, 2, or 3 for discussions of NP-completeness and NP-hardness).

Out of the possible 32 problems, 8 are of no interest. These are Nos. 0, 2, 4, 6, 16, 18, 20, and 22. These eight problems are all about general graphs, when the vertices are preassigned as potential receivers or transmitters. The lines between two vertices assigned to be transmitters are of no interest; they can not be tested and no interference can occur by them. Thus they can be deleted altogether. The same observation follows for lines between two receivers. Thus, the problems are actually stated for bipartite graphs and correspond to problems Nos. 8, 10, 12, 14, 24, 26, 28, 30.

Our state of knowledge is summarized in the following table:

		MT				ST			
		IA		INA		IA		INA	
		PA	NPA	PA	NPA	PA	NPA	PA	NPA
MAX	G	0	1 NPH Sec. III	2	3 NPH Sec. III	4	5 POLY Sec. II	6	7 NPH Sec. V
	BG	8 POLY (trivial) Sec. III	9 NPH Sec. III	10 NPH Sec. III	11 NPH Sec. III	12 POLY Sec. II	13 POLY Sec. II	14 NPH Sec. V	15 NPH Sec. V
MIN	G	16	17 NPH Sec. IV	18	19 NPH Sec. IV	20	21 ? between $2d$ & $2d+2$ Sec. II	22	23 NPH Sec. VI
	BG	24 POLY (trivial) Sec. IV	25 NPH Sec. IV	26 NPH Sec. IV	27 NPH Sec. IV	28 POLY Sec. II	29 POLY Sec. II	30 NPH Sec. VI	31 NPH Sec. VI

Some of the results have been achieved independently by L. J. Stockmeyer and V. V. Vazirani [9].

II. PROBLEMS EQUIVALENT TO MATCHING OR EDGE COLORING PROBLEMS

Let us consider problems 5, 12, 13, 21, 28, and 29. In these problems a transmitter can transmit to only one receiver and interference is allowed. Thus, the set of lines tested in one phase is nothing but a matching. Hence, problem 13 is equivalent to the maximum matching problem in bipartite graphs, which can be solved in polynomial time [4]. In fact, the preassignment of vertices does not change this observation at all. Thus, problem 12 is in fact the same as 13.

Problem 5 is just the maximum matching problem in general graphs, which is also polynomial [5].

Problems 29 and 28 are equivalent to the problem of minimum edge coloring in bipartite graphs; in case of problem 29 edges should be duplicated. The number of colors necessary and sufficient to color the edges of a bipartite graph is equal to the maximum degree of the vertices [6].

Problem 21 is equivalent to the following. Given a graph G, first duplicate the edges; i.e., for each edge in the graph add another one parallel to it. Then look for the minimum number of colors necessary to color the edges in the resulting graph. Clearly, if d is the maximum degree of the vertices of G, then $2d$ colors are necessary. By

Vizing's theorem [7], $2d+2$ colors are sufficient. But we do not know if the determination of the minimum number of colors is polynomial. We conjecture that the problem is NPH, since for general graphs (not having, necessarily, the edges duplicated) the problem has recently been shown to be NPH [8].

III. MAXIMIZATION PROBLEMS OF MT TYPE

Let us start with a demonstration that problem 10 (MAX, BG, MT, INA, PA) is NPH. The proof is by reduction of 3XC (3 exact cover) to problem 10. 3XC is defined as follows [1, 2, 3]:

Input: A family of sets $\{S_1, S_2, \ldots, S_m\}$ such that for every $1 \leqslant i \leqslant m$, $|S_i| = 3$.
Question: Is there a subset I of $\{1, 2, \ldots, m\}$ such that

$$\text{(i)} \quad \bigcup_{i \in I} S_i = \bigcup_{i=1}^{m} S_i,$$

$$\text{(ii)} \quad \text{if } \; i, j \in I \; \text{ then } \; S_i \cap S_j = \phi?$$

Let us denote the rhs. of (i) by U, and call it the *universal set;* also assume that

$$U = \{u_1, u_2, \ldots, u_{3n}\}.$$

3XC is known to be NPC (NP-complete) [1, 2, or 3].

Our aim is to display a polynomial reduction of 3XC to problem 10, thus proving that it is NPH.

Define the BG(X, Y, E) as follows:

$$X = \{1, 2, \ldots, m\}, \qquad Y = \{1, 2, \ldots, 3n\}, \qquad E = \{x - y \mid u_y \in S_x\}.$$

Preassign the elements of X to be transmitters and the elements of Y to be receivers. Recall that we want to maximize the number of lines tested in one phase, where each transmitter can transmit to many (adjacent) receivers and interference is not allowed. We claim that the answer to the 3XC problem is positive if and only if the maximum number of lines that can be tested in BG(X, Y, E) is $3n$.

First, assume that I yields a solution of the 3XC instance. Let the set of active transmitters be equal to I and let each active transmitter transmit to all its 3 adjacent receivers. Clearly, $3n$ lines are tested, and since the cover is exact, by condition (ii), no interference occurs.

Now assume that $3n$ lines can be tested. If $t \in X$ is an active transmitter then it must transmit on all 3 lines incident to it, since if $t - r$ is not tested then r cannot be an active receiver at all, or interference will occur. Thus, the set of active transmitters defines an exact cover.

This concludes our proof that problem 10 is NPH.

Next, let us show that problem 11 is NPH too. Now, there is no preassignment, but we can still show a reduction of 3XC to problem 11. In fact, we shall show a reduction of 3XC to the following NP version of problem 11:

Input: A bipartite graph (X, Y, E) and an integer N.
Question: Can one test simultaneously N lines of the graph, assuming MT, INA, and NPA?

Starting with an instance $\{S_1, S_2, \ldots, S_m\}$ of the 3XC, the reduction is defined as follows:

$$X = \{x_1, x_2, \ldots, x_{m+q}\}, \quad q = n + 1,$$
$$Y = \{y_0, y_1, \ldots, y_{3n}\},$$
$$E = \{x_i - y_0 \mid 1 \leqslant i \leqslant m + q\} \cup \{x_i - y_j \mid u_j \in S_i\},$$
$$N = q + 3n + (m - n) = 3n + m + 1.$$

We have to show that there exists an exact cover if and only if the NP version of problem 11 has a positive answer.

First assume I specifies an exact cover. Assign y_0 to be a transmitter and $x_{m+1}, \ldots, x_{m+q}$ to be receivers. This produces q line tests. Also let x_i be a transmitter if $i \in I$, and $y_1, y_2, \ldots, y_{3n}$ be receivers. This produces $3n$ line tests. Finally, let x_i, $1 \leqslant i \leqslant m$ and $i \notin I$, be a receiver (from y_0). Since $|I| = n$, this produces $m - n$ line tests. Thus, N line tests are achieved.

Next, assume N line tests are achievable in (X, Y, E). Observe that y_0 must be a transmitter, or the number of tests is at most $3n + m$ $(<N)$. Thus, we may assume that $x_{m+1}, \ldots, x_{m+q}$ are all receivers, yielding q line tests, and at least $3n + (m - n)$ more receivers must be found among $\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_{3n}\}$. Let the number of receivers among $\{x_1, \ldots, x_m\}$ be r. Clearly, $r \geqslant m - n$.

Denote by t the number of transmitters among $\{x_1, \ldots, x_m\}$. Clearly, $t + r \leqslant m$. Since each such transmitter can transmit to at most 3 vertices among $\{y_1, \ldots, y_{3n}\}$, the number of receivers in $\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_{3n}\}$ is $\leqslant m - t + 3t$. Thus, $t \geqslant n$, implying that $r \leqslant m - t \leqslant m - n$. We conclude that $r = m - n$, $t = n$, and $y_1, \ldots, y_{3n}$ are receivers.

Now define I to be the set of indices which correspond to elements of $\{x_1, \ldots, x_m\}$ which are transmitters. By the above $|I| = n$, and I defines an exact cover of U.

This concludes the proof that problem 11 is NPH. It immediately implies the NP-hardness of problem 3, since problem 11 is a subproblem of it.

Note that the reduction used to prove the NP-completeness of the NP version of problem 11 is insensitive to whether we assume IA or INA. Thus problems 9 and 1 are NPH too.

It is interesting to note that Stockmeyer and Vazirani [9] prove the NP-hardness of problem 1 by a trivial reduction from the Dominating Set problem (see [2] or [3]). In fact problem 1 is equivalent to the Dominating Set problem; minimizing the dominating set (transmitters) maximizes the dominated set (receivers). Thus, our proof that problem 9 is NP-hard implies the NP-completeness of the Dominating Set problem for bipartite graphs. The authors are not aware of any published proof of this fact.

Problem 8 is trivial. Every assigned receiver can receive from any of its adjacent assigned transmitters, and the maximum is immediately achieved.

IV. MINIMIZATION PROBLEMS OF MT TYPE

Problem 24 (MIN, BG, MT, IA, PA) is trivial; the highest degree of a vertex which is assigned to be a receiver is the number of phases which is necessary and also sufficient.

Consider now problem 17 (MIN, G, MT, IA, NPA). The corresponding decision problem can be stated as follows:

Given a graph $G(V, E)$ and an integer N, determine whether all its edges can be tested in both directions within N phases, when a vertex can transmit to several vertices during a certain phase, interference is allowed and there is no preassignment. Let us call this problem P_1.

Theorem 1. P_1 is NP-complete.

Proof. Clearly $P_1 \in NP$. To complete the proof we show that $4C \propto P_1$. ($A \propto B$ means that there is a polynomial-time reduction of problem A to problem B; see ref. 2 or 3).

The k-colorability (kC) problem was shown to be NP-complete by Stockmeyer (see [2] or [3]) for every $k \geqslant 3$. It consists of a graph $G(V, E)$. One is required to determine whether there exists a vertex coloring function f: $V \to \{i\}_{i=1}^{k}$ such that $f(v) = f(u)$ implies $v \not- u$ in G (i.e., there is no edge between vertices v and u).

Define a *coordinator* to be a subgraph which consists of 6 vertices denoted $c_1, c_2, \ldots, c_6$ and the following edge set (see Fig. 1):

$$\{c_i - c_{i+3} \mid 1 \leqslant i \leqslant 3\} \cup \{c_i - c_j \mid 1 \leqslant i < j \leqslant 3\}.$$

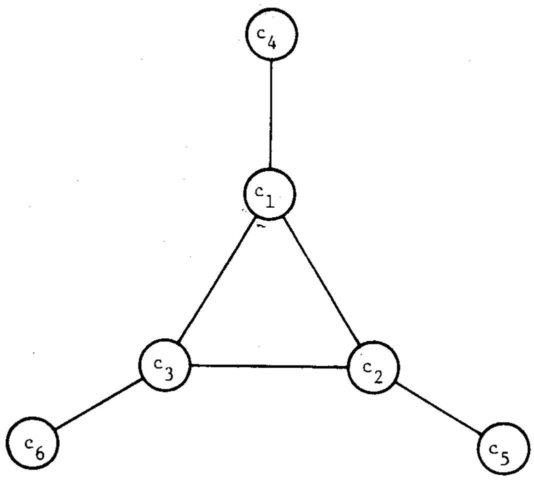
Let us show that a coordinator can be tested (i.e., all its edges can be tested in both directions) within 4 phases in a unique way up to the order of the phases.

We show first that a coordinator can be tested in 4 phases by specifying a way in which this could be done, to which we later refer as the *standard schedule* (see Fig. 1). For $i < 4$, transmit in phase number i from c_i to all its neighbors (denote this type T_i phase). In phase number 4 transmit from c_i to c_{i-3} for all $4 \leqslant i \leqslant 6$ (denote this type R phase).

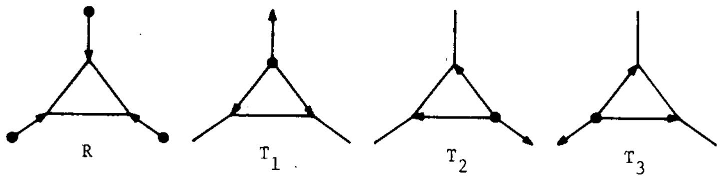
To show that this is the only way in which the coordinator can be tested in 4 phases, observe that any vertex of degree 3 should transmit to all its neighbors during one phase and thus exclude any other testing in the coordinator during that phase.

In the reduction $4C \propto P_1$, the given graph $G(V, E)$ will be reduced to $G'(V', E')$. The latter graph is constructed from subgraphs representing the vertices and edges of G.

Assume the degree of v, in G, is d. Define the subgraph $G_v(V_v, E_v)$, which represents v as a cascade of d coordinators denoted $C_1, C_2, \ldots, C_d$ and a set of $d - 1$ auxiliary edges denoted $\{a_i - b_i \mid 1 \leqslant i \leqslant d - 1\}$, linked as follows: For every $1 \leqslant i < d$,



A Coordinator



A Standard Schedule

FIG. 1.

vertex a_i plays the role of c_5 in C_i and the role of c_6 in C_{i+1} (see Fig. 2). The vertices denoted c_4 in the coordinators are called the *links* of G_v.

Edge e of G will be represented by a subgraph $G_e(V_e, E_e)$, which is a cascade of two coordinators and two auxiliary edges $d_1 - f_1$ and $d_2 - f_2$, linked as follows: d_1 (d_2) plays the role of c_6 (c_5) in the first (second) coordinator. The c_2 (c_5) vertex of the first coordinator is the same as c_6 (c_3) of the second, while the two edges merge (see Fig. 3). We call the vertices d_1, d_2 the *links* of G_e.

If v is an endvertex of e in G then G_v shares one of its links with one of the links of G_e. Each link of a G_v (G_e) is shared by one and only one link of some G_e (G_v).

To summarize, the 4C instance $G(V, E)$ is reduced to P_1 on $G'(V', E')$ and 4 phases, where

$$V' = \left(\bigcup_{v \in V} V_v\right) \cup \left(\bigcup_{e \in E} V_e\right),$$

$$E' = \left(\bigcup_{v \in V} E_v\right) \cup \left(\bigcup_{e \in E} E_e\right).$$

First, we show that if the 4C instance can be colored by 4 colors then the corre-

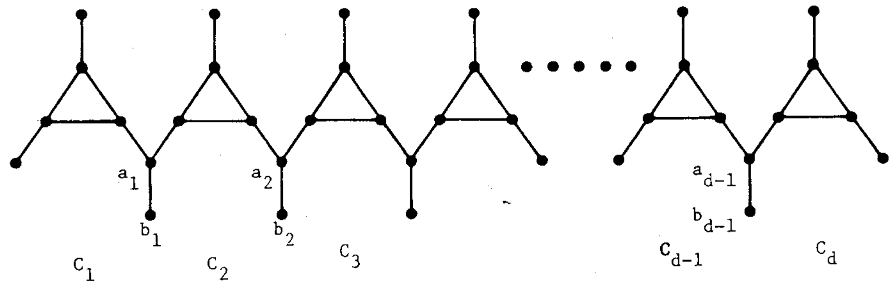


FIG. 2.

sponding P_1 instance can be tested within 4 phases. Let $\{i\}_{i=1}^{4}$ be the set of colors and $f\colon V \rightarrow \{i\}_{i=1}^{4}$ a coloring function that satisfies $f(v) = f(u) \Rightarrow v \not\leftrightarrow u$ in G. $G'(V',E')$ can be tested within 4 phases in the following way.

Testing the edges of a G_v is done by applying type R phase of the standard schedule to all the coordinators of G_v in phase $f(v)$ and applying T_i in phase $((f(v) + i - 1) \bmod 4) + 1$. The auxiliary edges of G_v are tested from the a_i in phase $f(v)$ and from the b_i in the phase in which T_1 is applied.

Specifying the way in which the edges of a G_e are tested requires a case study which depends on $f(v)$ and $f(u)$ where v, u are the endvertices of e. With no loss of generality, it is sufficient to check the case of $f(v) - f(u) \equiv 1$ or $2 \bmod 4$. The implementation of both cases is shown in Fig. 4, in which without loss of generality we assume that $f(v) = 1$.

To conclude, we show that if the P_1 instance can be tested within 4 phases then the source 4C instance can be colored by 4 colors.

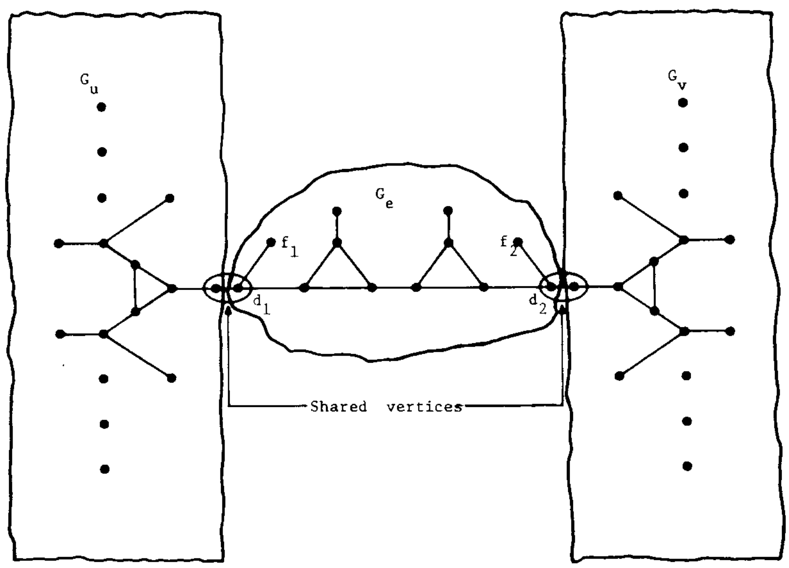


FIG. 3. $G_e(V_e, E_e)$ links $G_u(V_u, E_u)$ and $G_v(V_v, E_v)$.

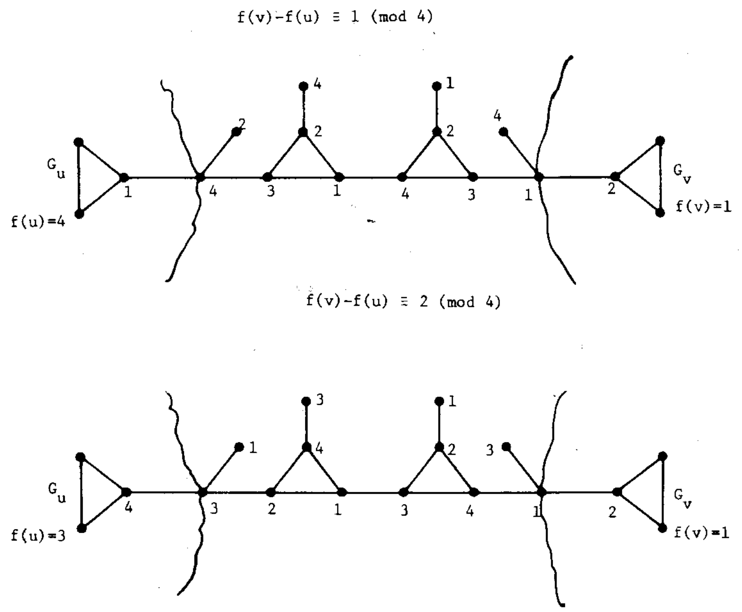


FIG. 4. Case study (the labeling of a vertex determines the phase in which it transmits).

Assume that there is a way in which $G'(V', E')$ can be tested within 4 phases; we refer to it as the test schedule. We claim that with regard to the test schedule the following holds: for every $v \in V$ all the links of G_v transmit once and in the same phase. [Note that all the links are of degree 3 in $G'(V', E')$ and therefore transmit in one phase only. Owing to the structure of G_v, the statement follows.]

Define the color of a vertex $v \in V, f(v)$, as the number of the phase in which the links of G_v act as transmitters.

Let us show that the coloring defined above satisfies the condition that for every $v, u \in V$ if $v \overset{e}{-} u$ in G then $f(v) \neq f(u)$.

Assume, on the contrary, that $f(u) = f(v) = \pi$. Consider the first coordinator of G_e. The assumption implies that type R phase is applied to it during phase π, and for similar reasons the same must hold for the second coordinator. This is a contradiction since it implies that the shared edge is tested in both directions simultaneously.

Thus we have proven that there is a 4 coloring of the source 4C instance. ∎

We turn now to problem 25 (MIN, BG, MT, IA, NPA). Let P_2 denote the problem of deciding whether a given bipartite graph can be tested in 4 phases. We show that $P_1 \propto P_2$.

Every edge $v \overset{e}{-} u$ of the P_1 instance is replaced by a simulation component which consists of vertices $s_1^e, s_2^e, \ldots, s_6^e$ and edges $v - s_1^e$, $u - s_2^e$, $s_1^e - s_3^e$, $s_1^e - s_4^e$, $s_2^e - s_3^e$, $s_2^e - s_4^e$, $s_3^e - s_5^e$, $s_4^e - s_6^e$ (see Fig. 5).

It is easy to see that testing the component in 4 phases requires that if s_1^e transmits to v in a certain phase then u transmits to s_2^e in the same phase. Notice that the graph constructed by the reduction is indeed bipartite. Thus problem 25 is NP-hard.

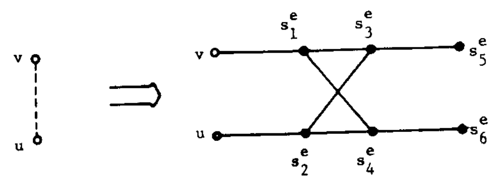


FIG. 5. Edge simulation component.

Notice that in the reduction used in the proof of Theorem 1, the graph defined has the property that the degrees of the vertices are either 1 or 3; thus the *restricted* P_1, for graphs with degrees 1 or 3 only, is also NP-complete. Furthermore, if one starts with a restricted P_1 instance, and applies the reduction to P_2, as above, the resulting graph still has only vertices of degree 1 or 3. Thus the restricted P_2 is also NP-complete.

In the testing schedule of restricted P_2 instances in 4 phases, every vertex is a transmitter only once, and during the phase in which it is a transmitter it tests all its incident edges outward. Thus interference never occurs. This proves that problem 27 (MIN, BG, MT, INA, NPA) is also NP-hard. Clearly, problem 19 (MIN, G, MT, INA, NPA) is NP-hard too.

The NP-completeness of the k-phase decision problem version of problem 17 ($k > 4$) is proved in a manner similar to that of Theorem 1; instead of using a triangle-based coordinator, use a $(k-1)$-clique-based coordinator. As to the NP-completeness of the k-phase decision problem version of problem 25, use in the reduction the simulation component shown in Fig. 6 instead of the one shown in Fig. 5.

Now, the graphs produced by the reductions to the k-phase versions of these problems have vertices of degree 1 or $k-1$. Thus the NP-completeness of the k-phase versions of problems 27 and 19 follows similarly.

Consider now problem 26 (MIN, BG, MT, INA, PA). We show that the problem of deciding whether all edges having a transmitter at one endpoint and a receiver at the other can be tested in k phases ($k \geqslant 3$) is NP-complete. This is done by a reduction from kC.

Each vertex, v, of the kC instance graph G is represented by a single transmitter T_v. Each edge, e, of G is represented by a k-star S_e whose center is a receiver and whose k leaves are transmitters; 2 of its leaves are called *links*.

If $v \overset{e}{-} u$ in G, then T_v is common with one of the links of S_e, while the other link of S_e is common with T_u.

Let us call the resulting graph G'.

Clearly, if G' can be tested in k phases, then in each k-star one and only one leaf is a transmitter in each of the k phases.

If G can be colored by k colors, then use T_v as a transmitter during the phase which corresponds to v's color. If $u \overset{e}{-} v$ in G then the phases during which the two links of S_e have been used are different. The other leaves of S_e can be used during the $k-2$ remaining phases in any order. Thus the demonstration that G' can be tested in k phases is complete.

If G' can be tested in k phases, define the color of v to be the phase during which

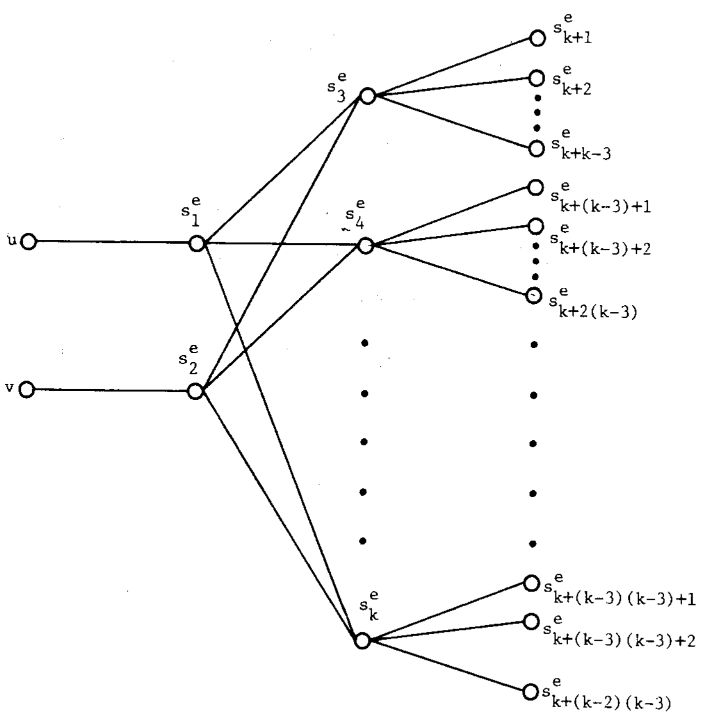


FIG. 6.

T_v is used. Note that this is well defined since T_v cannot be active in more than one phase owing to the INA condition. Clearly, this is a k-coloration of G.

V. MAXIMIZATION PROBLEMS OF ST, INA TYPE

Consider problem 14 (MAX, BG, ST, INA, PA). The set of edges that can be tested simultaneously under the ST condition forms a matching; every transmitter can transmit on one edge only, and every receiver can receive on one edge only. Such a matching, in which no interference occurs, is called *TR matching.*

Let P_3 be the decision problem, where the input is a bipartite graph $G(T, R, E)$; given an integer N, one is required to decide whether there is a TR-matching M such that $|M| \geqslant N$. Clearly, the TR-matching must be consistent with the preassignment.

We prove that P_3 is NP-complete by a polynomial reduction from 3XC.

Let the input of the 3XC consist of $\{S_1, S_2, \ldots, S_m\}$ where for every $1 \leqslant i \leqslant m$, $|S_i| = 3$, and the universal set U is defined by $U = \cup_{i=1}^{m} S_i = \{u_1, u_2, \ldots, u_{3n}\}$.

The graph $G(T, R, E)$ is defined as follows. The set U is a subset of R. Each set S_i is represented by a subgraph $G_i(T_i, R_i, E_i)$:

$$T_i = \{s_1^i, s_2^i, s_3^i, s_4^i\},$$

$$R_i = \{s_5^i\},$$

$$E_i = \{s_j^i - s_5^i \mid 1 \leqslant j \leqslant 3\} \cup \{s_4^i - s_5^i\}.$$

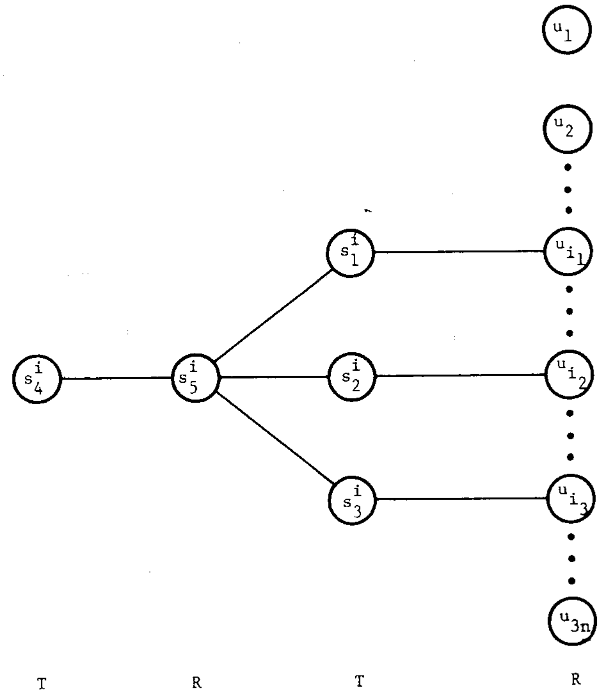


FIG. 7.

If $S_i = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ then define the edge set E_i^U by

$$E_i^U = \{s_j^i - u_{i_j} \mid 1 \leqslant j \leqslant 3\}.$$

The edges in E_i^U are called the *links* of G_i (see Fig. 7).

G and N are now defined as follows:

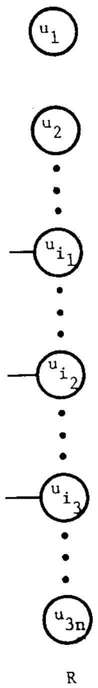
$$T = \bigcup_{i=1}^{m} T_i, \qquad R = \bigcup_{i=1}^{m} R_i \cup U,$$

$$E = \bigcup_{i=1}^{m} (E_i \cup E_i^U), \qquad N = 3n + (m - n).$$

First we prove that, if there is an exact cover of the 3XC instance, then the corresponding P_3 instance has a *TR*-matching consistent with the preassignment and of size N.

Let C be an exact cover of the 3XC instance. Define the following *TR*-matching:

$$\bigcup_{i=1}^{m} (\{s_4^i - s_5^i, \mid S_i \notin C\} \cup \{s_j^i - e_{i_j} \mid S_i \in C, 1 \leqslant j \leqslant 3\}).$$



n).

C instance, then the corre-
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:ollowing TR-matching:

$1 \leqslant j \leqslant 3\}$).

Note that this is indeed a TR-matching consistent with the preassignment, and that its size is exactly N.

We conclude by proving that if there is a TR-matching consistent with the preassignment and its size is N, then there is an exact cover of the 3XC instance.

Assume M is such a TR-matching. Note that if $M \cap E_i^U \neq \phi$ then $M \cap E_i = \phi$. (If s_j^i, $j = 1, 2, 3$, transmits to u_{i_j} then s_5^i cannot be used as a receiver, owing to interference, and thus none of the edges of G_i can be used.) Also, there is only one receiver in G_i, and therefore at most one edge of G_i can be used. Thus the only way to use 3 transmitters of G_i is to transmit on the 3 links.

Let l be the number of G_i for which links are used, and $m - l$ be the number of G_i for which no links are used. The maximum number of edges that can be tested in $3l + (m - l)$. Since $|M| \geqslant N$ we get $3l + (m - l) \geqslant 3n + (m - n)$, and therefore $l \geqslant n$. However, no two active links can be connected to the same u_i. Thus the number of edges in M is bounded by $3n + (m - l)$. This implies that $l \leqslant n$. It follows that there are exactly n G_i which use links, and each of them uses all its 3 links. This defines an exact cover of U.

Consider now problem 15 (MAX, BG, ST, INA, NPA). Let P_4 be the corresponding decision problem; i.e., given a bipartite graph $G'(X, Y, E')$ and an integer N', determine whether it has a TR-matching M' such that $|M'| \geqslant N'$.

We use a reduction $P_3 \propto P_4$, in order to prove P_4's NP-completeness.

Let the input of P_3 consist of $G(T, R, E)$ and N. Let $p = |T| + 1$ and define $G'(X, Y, E')$ and N' as follows:

$$X = T \cup \{a_i \mid 1 \leqslant i \leqslant p\},$$

$$Y = R \cup \{b_i \mid 1 \leqslant i \leqslant p\},$$

$$E' = E \cup \{t - b_i \mid t \in T, 1 \leqslant i \leqslant p\} \cup \{a_i - b_i \mid 1 \leqslant i \leqslant p\};$$

$$N' = p + N.$$

Assume, first, that there is a TR-matching M of $G(T, R, E)$ consistent with the preassignment and $|M| \geqslant N$. Let us show that, in G', there is a TR-matching M' such that $|M'| \geqslant N'$. Define M' as follows (the notation $u \to v$ means that u transmits to v):

$$M' = \{t \to r \mid t - r \text{ in } M\} \cup \{b_i \to a_i \mid 1 \leqslant i \leqslant p\}.$$

It is easy to see that M' is a TR-matching and $|M'| = |M| + p \geqslant N'$.

Next, we assume that there is a TR-matching M' in G' for which $|M'| \geqslant N'$, and show that this implies the existence of a TR-matching M in G consistent with the preassignment and $|M| \geqslant N$.

Every edge of $M' - E$ "consumes" one b_i. Thus $|M' - E| \leqslant p$. It follows that $|M' \cap E| \geqslant N$. If there are in M' two edges $t_1 \to r_1$ and $r_2 \to t_2$, where $t_1, t_2 \in T$ and $r_1, r_2 \in R$, then none of the b_i can be active, or interference will occur either through $b_i - t_1$ or $b_i - t_2$. In this case, the number of edges tested is $|M'| \leqslant |T| < p \leqslant N'$. We conclude that if $|M'| \geqslant N'$ then either all the active T-vertices are transmitters, or all are receivers. In P_4, if the tasks of transmitters and receivers of a TR-matching are exchanged then the result is also a *TR*-matching. We may, therefore, assume that all the active *T*-vertices of M' are transmitters. Thus, one concludes that $M' \cap E$ is a *TR*-matching which fulfills the requirements of P_3.

The NP-hardness of problem 15 implies the NP-hardness of problem 7.

VI. MINIMIZATION PROBLEMS OF ST, INA TYPE

Consider problem 31 (MIN, BG, ST, INA, NPA). Let us call the corresponding decision problem, for 6 phases, P_5. That is, one is given a bipartite graph and it is necessary to determine whether its edges can be tested in 6 phases. We shall show that this problem is NP-complete. In Theorem A1 at the Appendix we show that the corresponding decision problem for $k \leqslant 5$ phases can be solved in polynomial time.

In the sequel we shall use the notation "$x \to y$ is tested at phase i" to denote "at phase i, x transmits to y." We shall also denote a schedule in which x transmits to y at phase i and y transmits to x at phase j by the notation given in Fig. 8 below.

We need some definitions and lemmas before we give the reduction. Define an "outlet" to be the subgraph shown in Fig. 9.

Lemma 1. If an outlet is tested in 6 phases such that $x_1 \to y_1$ and $x_2 \to y_4$ are tested at phase 0 and $y_1 \to x_1$ and $y_4 \to x_2$ are tested at phase 1 (see Fig. 9), then $z \to y_2$ is tested at phase 0 and $y_2 \to z$ is tested at phase 1.

Proof. Note that edges $y_1 - y_2$ and $y_2 - y_3$ cannot be tested during phases 0 and 1 in either direction. Hence, $y_2 - z$ must be tested at phases 0 and 1. The lemma follows since y_2 cannot transmit at phase 0 and cannot receive at phase 1. ■

Let a "multiplier" be the subgraph shown in Fig. 10.

Lemma 2. If a multiplier is tested in 6 phases, then for $i = 1, \ldots, 4$, $\{x_i \to y_i\}$ are tested at the same phase and $\{y_i \to x_i\}$ are tested at the same phase.

Proof. In a 6-phase testing, nodes $y_1, \ldots, y_4$ must be active in every phase. If in a certain phase y_1 transmits to y_2, then y_3 must act as a transmitter and y_4 must act as a receiver, which means that $y_3 \to y_4$ is tested at the same phase. Similarly, $y_2 \to y_1$ and $y_4 \to y_3$ are tested simultaneously, as well as $y_1 \to y_3$ and $y_2 \to y_4$, $y_3 \to y_1$ and $y_4 \to y_2$. It follows that if y_1 transmits to x_1, y_3 must be active in testing edge $y_3 - x_3$. Moreover, y_3 must transmit to x_3, otherwise interference will occur. Similarly, if x_1 transmits to y_1, x_3 must transmit to y_3. The lemma follows. ■

A coordinator is a subgraph decomposed of two multipliers and one outlet, as shown in Fig. 11. Nodes x_1, x_2, x_3, x_4 are "link nodes," and edges $x_1 - x_2$, $x_3 - x_4$ are "link edges." Node z is an "out-node" and edge $y - z$ is an "out-edge."

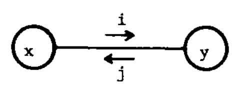

FIG. 8.

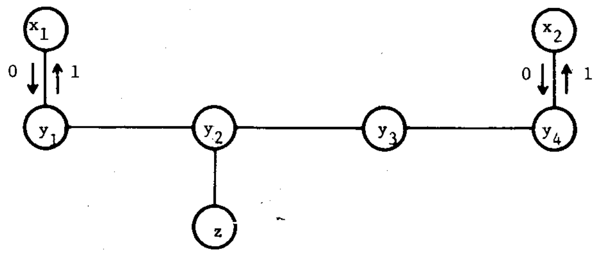

FIG. 9.

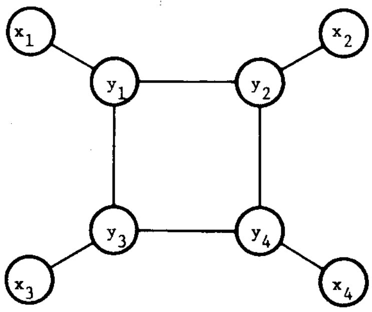

FIG. 10.

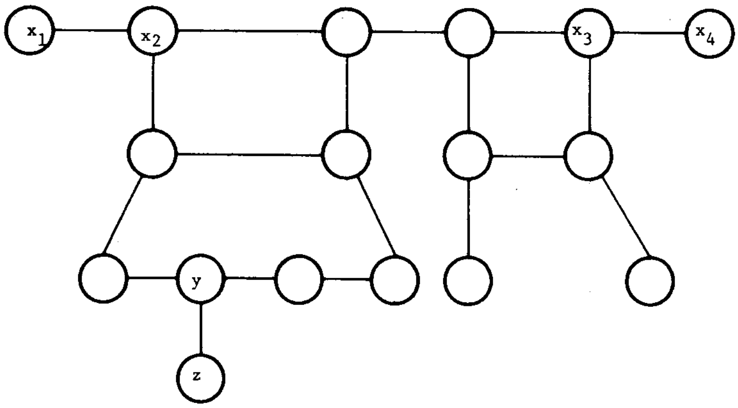

FIG. 11.

Lemma 3. If a coordinator is tested in 6 phases, then $x_1 \to x_2, x_3 \to x_4$, and $y \to z$ are tested at the same phase. Similarly, $x_2 \to x_1, x_4 \to x_3$, and $z \to y$ are tested at the same phase.

Proof. By repeated applications of Lemmas 1 and 2. ■

We need one more definition and lemma for the reduction. An edge-coordinator is the subgraph shown in Fig. 12 below; y_1 and y_2 are out-nodes, and $x_1 - y_1, x_2 - y_2$ are out-edges.

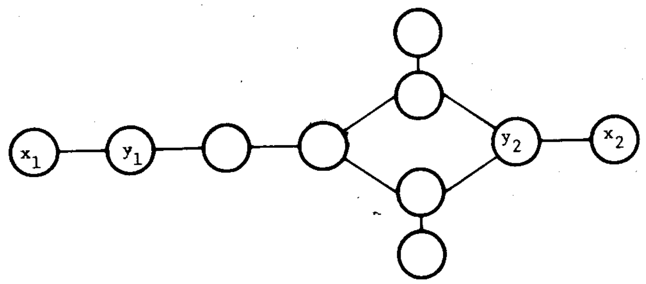

FIG. 12.

Lemma 4. If an edge-coordinator is tested in 6 phases, then $x_1 \to y_1$ ($y_1 \to x_1$) is not tested at the same phase in which $x_2 \to y_2$ ($y_2 \to x_2$) is tested.

Proof. By Lemma 2. ∎

We are now ready to describe the reduction of the 6C problem to P_5. Let $G(V, E)$ be an input to the 6C problem. Replace each $v \in V$ of degree d by a graph G_v which consists of a cascade of d coordinators connected by their link edges (see Fig. 13).

If $u - v$ is an edge in E, connect an out-node in G_u to an out-node in G_v by an edge coordinator as shown in Fig. 14.

Lemma 5. Let $G'(V', E')$ be the output of the reduction described above. Then, if G' can be tested in 6 phases, G is 6 colorable.

Proof. Suppose G' can be tested in 6 phases. Then by Lemma 3 for each subgraph G_v of G', all the out-nodes of the coordinators in G_v transmit along their out-edges at the same phase. Define this phase to be the color of node v. By Lemma 4, if v is connected to u, the out-nodes of G_v and G_u do not transmit along their out-edges at the same phase, and hence v and u are colored by different colors, which means that the described coloring is proper. ∎

In order to prove that if G is 6-colorable G' can be tested in 6 phases, we need a few more definitions.

A standard schedule of a coordinator is one of the following 4 schedules: *schedule A* is described in Fig. 15; *schedule B* is obtained by interchanging phases 2 and 4 in schedule A; *schedule C* interchanges phases 3 and 5 in schedule A; *schedule D* interchanges phases 2 and 4, and also 3 and 5, in schedule A.

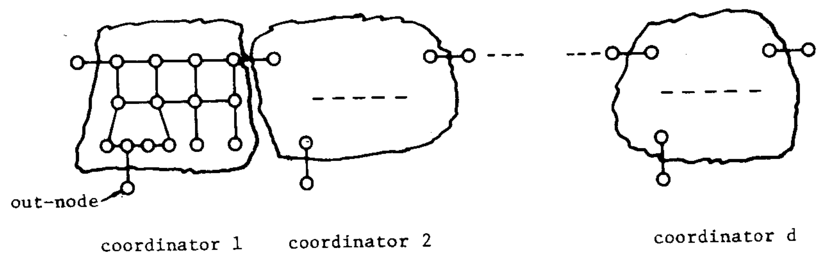

FIG. 13.

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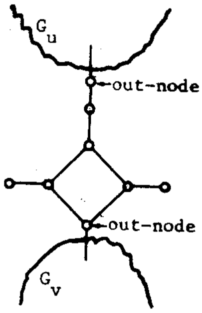

FIG. 14.

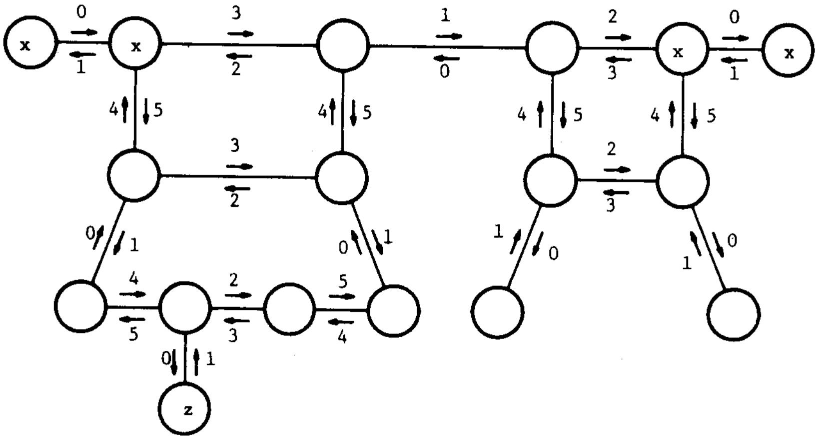

FIG. 15. Schedule A: link nodes are marked by x and the out-node by z.

Lemma 6. Let G_v be the subgraph corresponding to v described above. Then, by testing each of the coordinators in G_v by any one of the four standard schedules, chosen at random, we obtain a 6-phase testing of G_v.

Proof. The only possible problem may be caused by the link nodes, which may interfere with one another. However, this does not happen, since all of them transmit at phases 3, 5 and receive at phases 2, 4. ■

For an integer $k \in \{0, 1, \ldots, 5\}$, let schedule (A, k) denote schedule A, where each phase i is replaced by phase $i + k$ (mode 6). Schedules (B, k), (C, k), and (D, k) are defined similarly. We are now ready to prove the final lemma corresponding to the reduction of 6C to P_5.

Lemma 7. If G is 6-colorable, then G' can be tested in 6 phases.

Proof. Let $f: V \rightarrow \{0, 1, \ldots, 5\}$ be a 6-coloring of G. A 6-phase testing of G' is defined in the following way.

Let $u - v \in E, f(u) = i, f(v) = i + k$ (mod 6) $(0 \leqslant i \leqslant 5, 1 \leqslant k \leqslant 5)$. Let C_u and C_v be the coordinators in G_u and G_v which are connected by an edge coordinator G_{u-v}. The schedules (X, i) for G_u, $(Y, i+k \pmod 6)$ for G_v ($X, Y \in \{A, B, C, D\}$), and the schedule for the edges of G_{u-v} are determined by the value of k, as shown in Fig. 16 for $i = 0$. This figure shows the schedules of the edges of G_{u-v} and the relevant edges of C_u and C_v. The out-nodes of C_u (C_v) are marked by z_u (z_v).

It is straightforward to check that the above schedules give a proper 6-phase test of the edges of G_{u-v} and the corresponding edges in G_u and G_v, for all $u - v$ in E. Also, by Lemma 6, the above schedules give a proper 6-phase testing of the edges of G_u for all $u \in V$. The lemma follows. ∎

By Lemmas 5 and 7 we obtain

Theorem 2. Problem 31 (MIN, BG, ST, INA, NPA) is NP-hard.

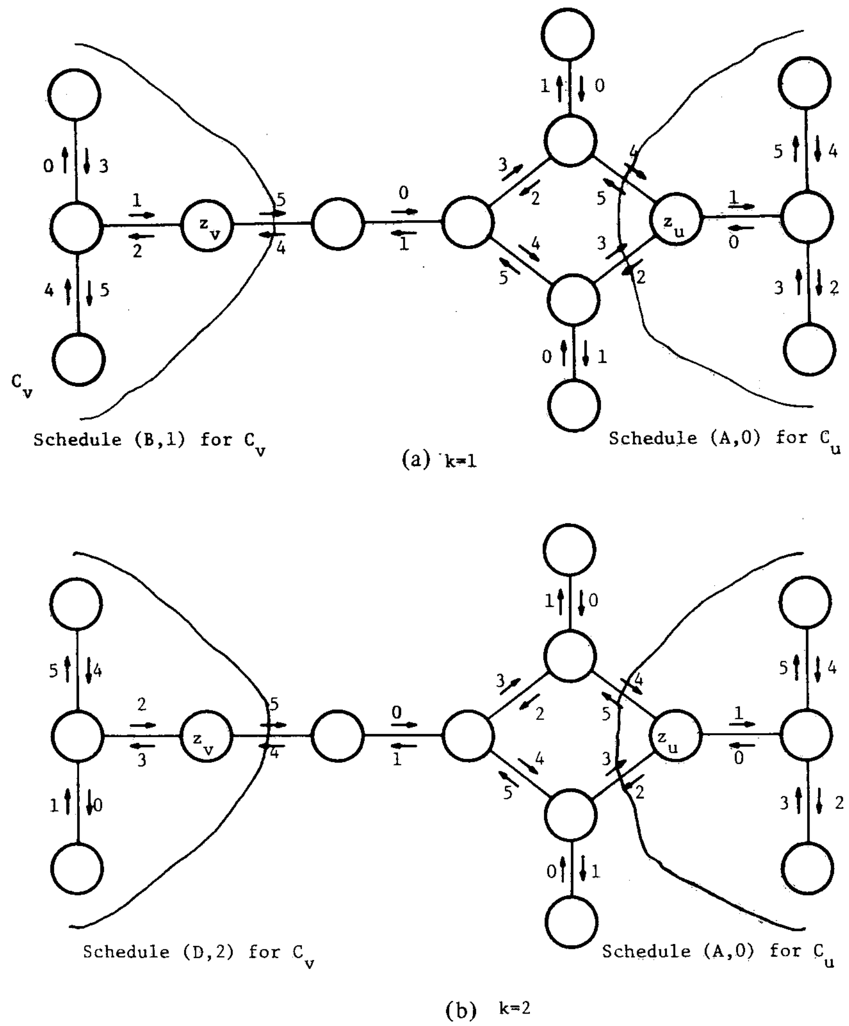

FIG. 16. (a) $k = 1$, (b) 2, (c) 3, (d) 4, (e) 5.

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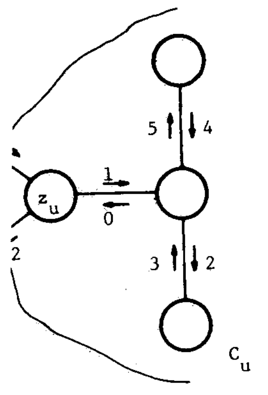

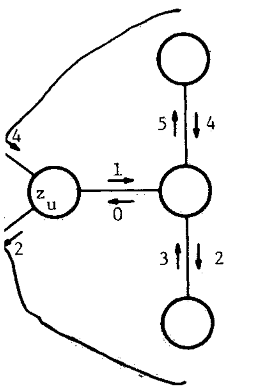

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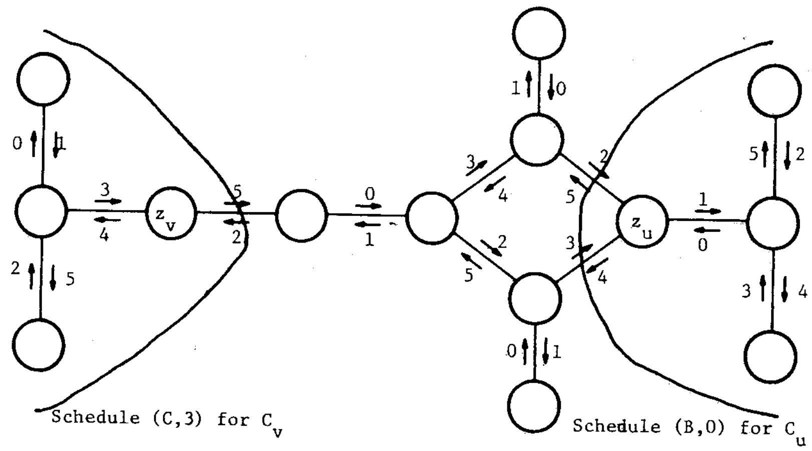
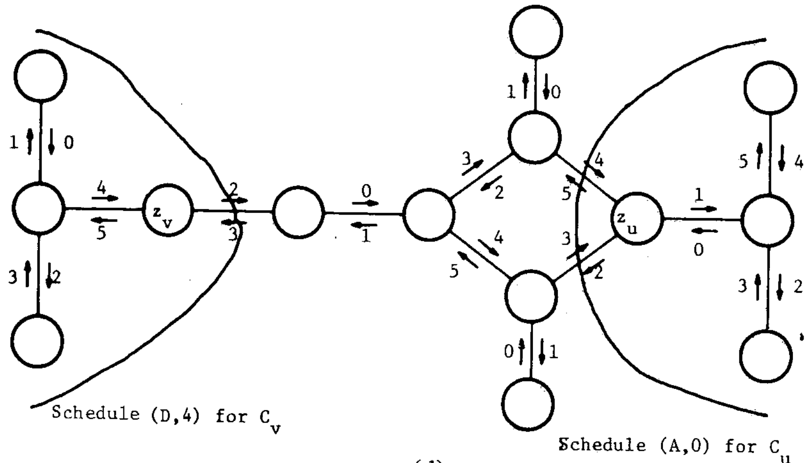
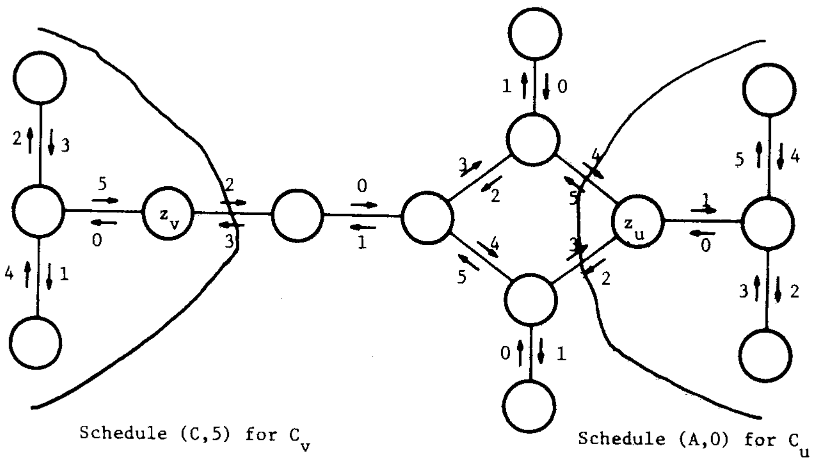


(c) k=3



(d) k=4



(e) k=5

FIG. 16. (*Continued*)

Proof. By the NP-completeness of P_5, which follows by Lemmas 5 and 7 and the fact that the graph G' described in these lemmas is bipartite. ■

Note. In an earlier version of this paper [10] it was shown that the decision problem which corresponds to problem 31 for k phases is NP-complete for all even integers $k \geqslant 8$. In Theorem A1 in the Appendix the problem is shown to be in P for $k \leqslant 5$ phases. The question whether the decision problems for k phases remain NP-complete for odd k greater than 5 is still open.

Finally, consider problem 30 (MIN, BG, ST, INA, PA). We shall show that the corresponding decision problem, P_6, for 4 phases, is NP-complete. In Theorem A2 in the Appendix we shall show that the decision problem for $k \leqslant 3$ phases can be solved in polynomial time.

Define an "outlet" to be the subgraph in Fig. 17(a), and a "multiplier" to be the subgraph in Fig. 17(b). Vertex x in both graphs is the "center node," and the edges adjacent to x are "center edges," while the remaining edges are "peripheral edges." Node z in the outlet is "out-node," and edge $z - y$ is an "out-edge." The vertices of these graphs may be partitioned to "transmitters" and "receivers" in either of the two possible ways.

Lemma 8. A multiplier or an outlet can be tested in 4 phases iff all of the peripheral edges are tested at the same phase.

Proof. By the observation that none of the peripheral edges can be tested at the same phase with any of the center edges. ■

A "coordinator degree d" is a (binary) tree decomposed of some multiplier and d outlets, such that each multiplier shares at least 2 of its peripheral edges with other multipliers or outlets, and the out-nodes of the outlet are leaves in the tree. Figure 18 shows a coordinator of degree 3, decomposed of 2 multipliers and 3 outlets. Two of the out-nodes of this coordinator are transmitters, and one is a receiver.

Let $G(V, E)$ be an input to the 4C problem. We reduce it to P_6 in the following way: Replace each $v \in V$ of degree d by a coordinator of degree d. Let $u - v \in E$, and let z_u and z_v be two out-nodes in G_u (G_v). WLG assume that exactly one out of z_u and z_v is a transmitter. (If this is not the case, we replace the outlet corresponding to z_u by a multiplier, and add a new outlet, adjacent to that multiplier. The out-node of this new outlet will represent z_u.) Connect z_u and z_v by an edge.

Let $G'(V', E')$ be the output of the above reduction. Then we have

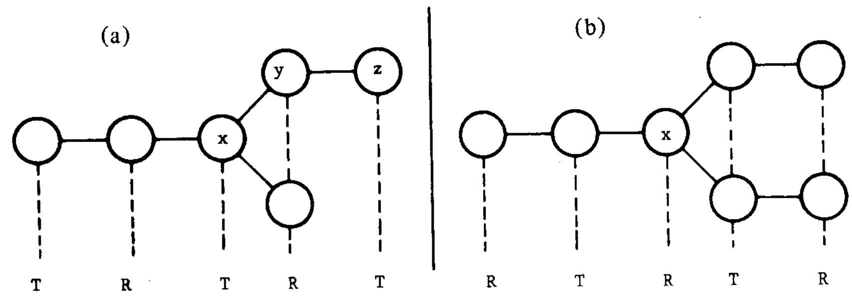


FIG. 17. T, transmitters; R, receivers.

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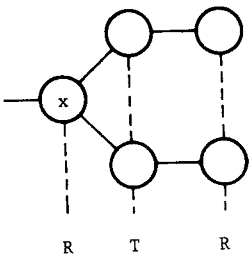

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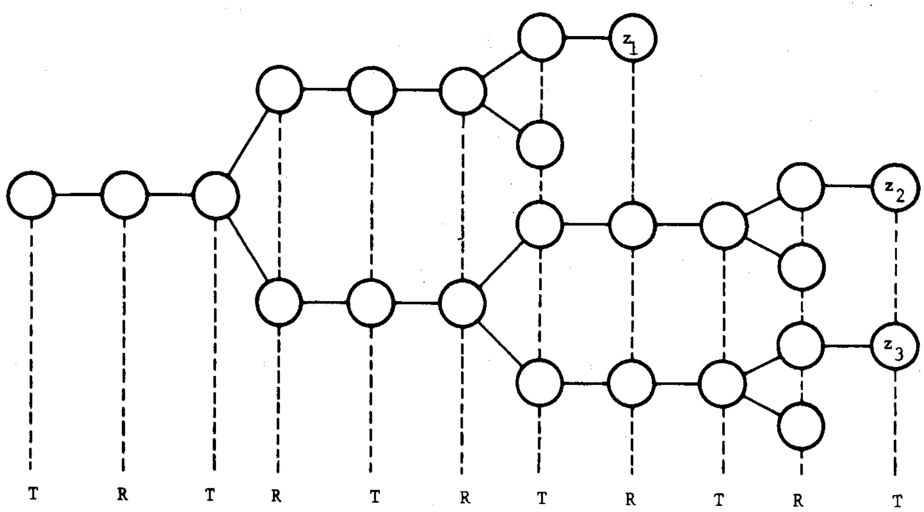


FIG. 18.

Lemma 9. If G' can be tested in 4 phases, G is 4-colorable.

Proof. By Lemma 8, for each subgraph G_u of G', all the out-edges of G_u are tested at the same phase. Let this phase denote the color of u. If $u - v \in E$, then there is an edge $z_u - z_v$ in E', where z_u is an out-node in G_u and z_v is an out-node in G_v. Hence the out-nodes of G_u and G_v cannot be tested at the same phase, which means that u and v do not have the same color, and hence the coloring is proper. ■

Define a standard schedule of a multiplier to be the one given in Fig. 19. A k-standard schedule, for $0 \leqslant k \leqslant 3$, is obtained by replacing phase i in Fig. 19 by $i + k$ (mod 4) $(0 \leqslant i \leqslant 3)$.

Lemma 10. If G is 4-colorable, G' can be tested in 4 phases.

Proof. Let $f\colon V \to \{0, 1, 2, 3\}$ be a 4-coloring of G. A 4-phase testing of G' is obtained as follows:

(a) For each $v \in V$, test the edges of the multipliers in G_v by an $f(v)$-standard schedule.

(b) For each $u - v$ in E, let $f(v) = f(u) + k$ (mod 4). WLG $1 \leqslant k \leqslant 2$.

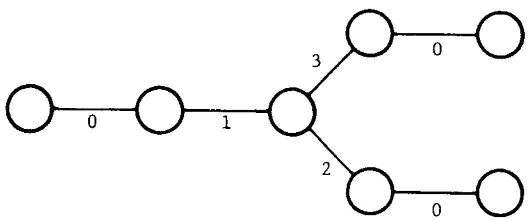


FIG. 19.

The testing of the edges of the corresponding outlets and the edge $z_u - z_v$ connecting the out-nodes is determined by the value of k, as shown in Fig. 20. It is easily checked that this schedule properly tests the edges of G' in 4 phases. ■

By Lemmas 9 and 10 we obtain

Theorem 3. Problem 30 (MIN, BG, ST, INA, PA) is NP-hard. ■

Note. By defining a multiplier to have $k - 1$ center edges (and $k - 1$ peripheral edges), and an outlet to have $k - 1$ center edges and one peripheral edge, one can show, using the techniques of Lemmas 8–10, that the k-phase decision version of problem 30 is NP-complete for all $k \geq 4$.

APPENDIX

We show here that the decision problems that correspond to problem 31 for 5 or less phases and the decision problems that correspond to problem 30 for 3 or less phases can be solved in polynomial time (in fact, in linear time), and hence that Theorems 1 and 2 of Section VI give a sharp bound between NP-completeness and polynomial-time algorithms (provided P ≠ NP). As a by-product, we also provide a method to test cycles in minimum number of phases.

Theorem A1. The decision problem that corresponds to problem 31 for k-phases is solvable in polynomial time for $k \leq 5$.

Proof. The proof is trivial for $k = 2$ and $k = 3$ (a connected graph which can be tested in less than 4 phases cannot have more than one edge).

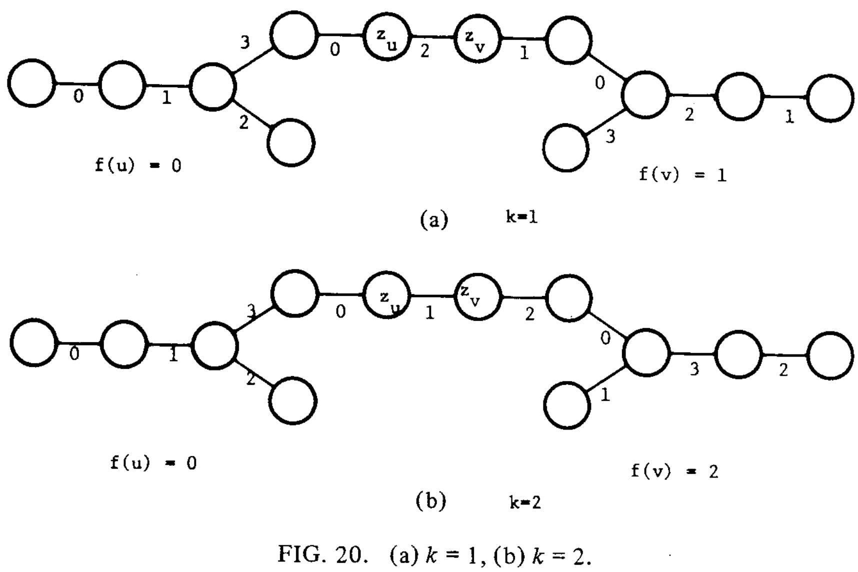
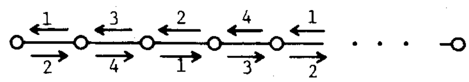


FIG. 20. (a) $k = 1$, (b) $k = 2$.

For $k = 4$ and $k = 5$, we have to consider only connected graphs which do not have a node of degree 3, that is, paths or cycles. Every path (o-o-o- · · · -o) can be tested in 4 phases by repeating the schedule



In fact, it is easily verified that this is the only possible schedule to test the edge of the path in 4 phases (up to permutation of the phases). This also implies that a cycle can be tested in 4 phases only if it has $4n$ edges for some integer n.

The only nontrivial part of this theorem is for $k = 5$. First, we give in Fig. 21 schedules to test in 5 phases cycles of length 5, 7, and 10. These schedules can be extended to tests of cycles with $5 + 4k$, $7 + 4k$, and $10 + 4k$ edges by inserting $4k$ edges into

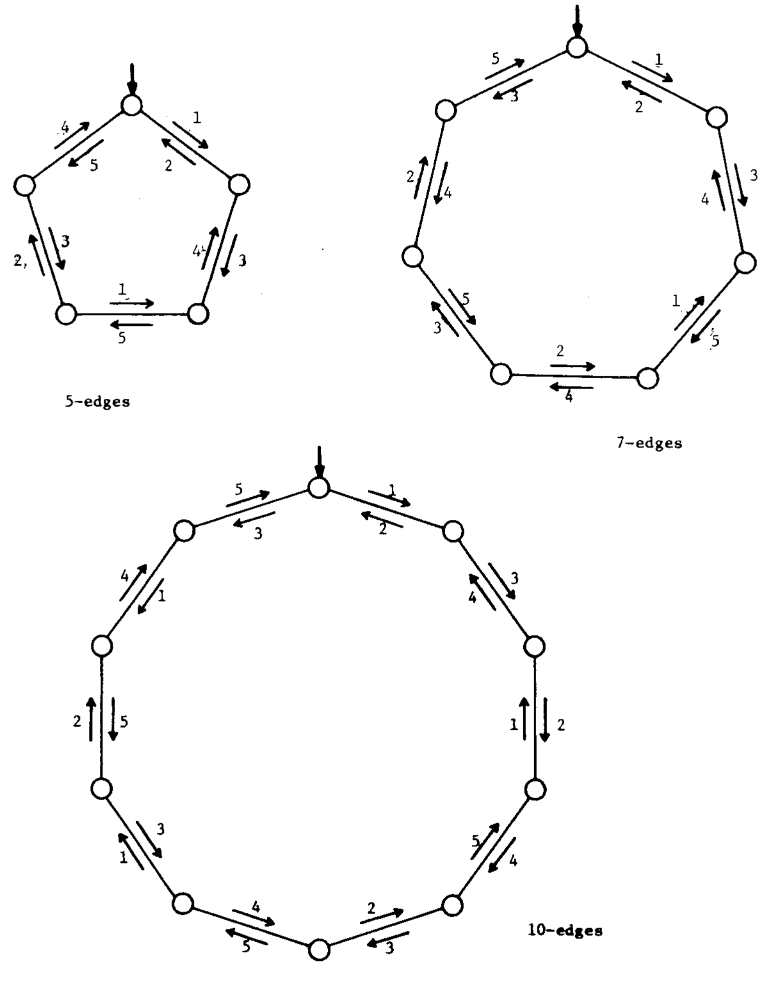


FIG. 21.

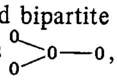
these cycles at the places indicated by bold arrows in Fig. 21, where these edges are tested in 4 phases by the same schedule as indicated above. The only cycles which cannot be tested in 5 phases (but are easily tested in 6) are the triangle and the hexagon. We leave the verification of this fact to the reader. ■

Theorem A2. The decision problem that corresponds to problem 30 for k-phases is solvable in polynomial time for $k \leqslant 3$.

Proof. It is left to the reader to check that

(a) A connected graph can be tested in a single phase only if it has at most one edge.

(b) A connected graph which can be tested in 2 phases (but not in one phase) is a path of two arcs.

(c) A connected bipartite graph can be tested in 3 phases (but not in two) only if it is a star of 3 edges , a path of more than two edges, or a cycle of $6n$ edges.

The theorem follows from (a)-(c) above. ■

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