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Deductive Verification of Distributed Protocols in First-Order Logic

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by

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Abstract

Distributed algorithms and distributed systems are a critical component of modern computing infrastructure. However, ensuring that these algorithms and systems are correct under all operating scenarios is notoriously challenging. Formal verification is a way to mathematically prove that algorithms and systems are correct by using a computer to check all scenarios and corner cases, providing a level of certainty that is beyond manual mathematical proofs. However, applying formal verification to distributed algorithms and distributed systems is nontrivial, since they usually have infinitely-many states and, in general, automatically checking their correctness is undecidable.

This thesis explores formal verification of infinite-state systems, and distributed algorithms in particular, using a decidable fragment of first-order logic. The main idea is to express infinite-state systems, their correctness properties, and their inductive invariants, in a decidable fragment of first-order logic. Thus, the undecidable verification problem is essentially decomposed into decidable sub-problems, which can be solved by existing mature and optimized automated solvers.

Theoretical contributions are developed in several aspects, including the surprising expressiveness of the decidable fragment considered, the decidability question of the problem of inductive invariant inference, and the applicability of first-order logic for liveness and temporal proofs. Based on these ideas, a practical methodology is also developed for verification of both safety and liveness properties of infinite-state systems. The methodology is used to verify several challenging distributed algorithms, including variants of the Paxos algorithm, and several distributed algorithms that are formally verified for the first time.

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Chapter 1

Introduction

Program verification is one of the fundamental problems of computer science. In program verification one seeks to prove, with mathematical certainty, that a program (or more generally computer system), behaves correctly, with respect to a given specification. Turing already addressed this issue in 1949 [175, 225], suggesting essentially what is now known as deductive verification, or Floyd-Hoare style verification, as it was developed in the seminal works of Floyd [80] and Hoare [105]. In Floyd-Hoare style verification, a program is annotated with local assertions that should imply global correctness. To cite Turing's 1949 paper "Checking a large routine" [225]:

> How can one check a routine in the sense of making sure that it is right? In order that the man who checks may not have too difficult a task the programmer should make a number of definite assertions which can be checked individually, and from which the correctness of the whole programme easily follows.

1.1 Logic-Based Deductive Verification

In deductive verification [161], assertions are expressed as formulas in a logical formalism and refer to the program's state. For example, the assertion $x > 0$ means that the value of the program variable `x` is positive. Program statements are also given a logical meaning, either using Dijkstra's weakest precondition calculus [66], or alternatively using a two vocabulary transition formula expressing state transitions. For example, the program statement `x := x + 1` corresponds to the transition formula $x' = x + 1$, where x (resp. x') denotes the value of the program variable `x` before (resp. after) executing the statement.

A *Hoare triple*, $\{ P \}\ C\ \{ Q \}$, states that if command C is executed in a state satisfying assertion P, then its final states satisfy assertion Q. For example, the Hoare triple

$\{ x \geq 0 \}$ `x := x + 1` $\{ x > 0 \}$ is valid. A program annotated with assertions naturally defines a collection of Hoare triples, which are meant to be valid. Validity of Hoare triples can be reduced to validity of logical formulas, using the logical meaning of program statements. For the example, validity of the aforementioned Hoare triple reduces to validity of the formula $(x \geq 0 \wedge x' = x + 1) \rightarrow x' > 0$. Therefore, once a program is annotated with assertions, the problem of checking these assertions can be mechanically translated to the problem of checking validity of logical formulas, called *verification conditions*. Establishing validity of logical formulas is also amenable to mechanization, using (interactive or automated) theorem provers. This approach has become known as deductive program verification, and its end result is a mechanized proof that a given program is correct. Since the proof is mechanized, it can be trusted with a higher degree than a manual "paper proof".

The problem of establishing validity of logical verification conditions (VCs) remains computationally hard. The difficulty of this problem depends on the logical formalism used to express assertions and VCs. For software systems, the assertions needed to prove the program commonly use a rich logical formalism that includes quantifiers, arithmetic, sets, cardinalities, transitive closure, and more. As a result, checking validity of VCs is usually undecidable, and in most cases not even recursively enumerable (i.e., no complete proof system exists). In the verification community, this situation is coped with using two ways (and their combination): 1. Interactive theorem proving, where the proof is done manually and the computer merely checks proof steps; and 2. Automated theorem proving, where heuristics are employed to attempt to automatically solve an undecidable problem.

Interactive theorem proving Interactive theorem provers, also known as proof assistants, usually support a very rich logical formalism based on dependent type theory or high-order logic, which can express most of mathematics. Examples include ACL2 [117], Agda [33], Coq [26], HOL [92] Isabelle [181], Lean [63], PVS [184], and more. While these tools support a very expressive logical formalism, they usually require great manual effort. Proof construction is based on the user interactively applying *tactics*, reducing the current proof goal to simpler goals, and applying previously proven lemmas and theorems. This essentially requires the user to provide a very detailed proof, while the tool ensures soundness.

The result is that practical verification of computer systems is very tedious and demands tremendous manual effort. For example, the seL4 [119] project developed a verified operating system micro-kernel. This project involves a 20 person-year effort. The code base comprises of about 10,000 lines of implementation code, and almost half a million lines of proofs and specifications using Isabelle/HOL. Closer to the application domain of this thesis is the Verdi

project [235, 237], which verified an implementation of Raft [183], a distributed protocol of the Paxos [135] family considered in this thesis. The Verdi project comprises of 530 lines of implementation code, and 50,000 lines of specifications and proofs using Coq.

Automated theorem proving Automated theorem provers, and SMT (satisfiability modulo theories) solvers [23, 62], usually support a logical formalism based on first-order logic modulo theories. Theories can include variants of arithmetic (integer, real, bit-vector, linear, non-linear), and theories that allow reasoning about common data structures, such as arrays, lists, strings, algebraic data types, and more. Examples of SMT solvers include Z3 [61], Yices [71], CVC4 [22], and more. Some solvers also target pure first-order logic, i.e., without theories. Examples include Vampire [207], iProver [123], SPASS [234], and more. What these tools all share in common is that they attempt to solve an undecidable problem,[1] and employ heuristics that are meant to be successful on problem instances that arise in practice.

Deductive verification tools based on SMT solvers have become very successful in the programming languages and verification communities. A prominent example is Dafny [148], which uses Z3 to discharge verification conditions. When using Dafny, the user only needs to annotate the program with assertions, and proving the resulting verification conditions is completely automated, potentially making verification easier and more practical compared to interactive theorem proving. For distributed systems, the IronFleet [100] project has successfully used Dafny and Z3 to verify an implementation of Multi-Paxos [135], a protocol also considered in this thesis.

Automated solver instability in deductive verification The use of SMT solvers and automated theorem provers to automatically discharge VCs significantly reduces the proof burden of the user. However, it also introduces great instability, since the process becomes sensitive to the solver's heuristic. Recent systems verification projects such as IronClad [99], IronFleet [100], and Komodo [75] (using Dafny) identified the underlying solver's instability as a major hurdle to practical deductive verification. If the solver succeeds to validate the user provided assertions, then the program is shown correct and verification succeeds. However, minor perturbations, e.g. changes to the random seeds used in solvers' heuristics, can unpredictably cause the solver to diverge, in which case verification fails. A quotation from [75, Section 9.1] illustrates the problem from the developers' perspective:

[1] While some of the theories supported by the solvers are decidable when restricted to the quantifier-free fragment, software verification in general and verification of distributed protocols in particular requires quantifiers, which usually lead to undecidability.

> The most frustrating recurring problem was proof instability. [...] Even once fixed, the proof may easily timeout again due to minor perturbations. Worse, minor changes can trigger timeouts in seemingly unrelated proofs.

The fact that the underlying problem is undecidable also means that there is no theoretical guarantee that breaking verification into smaller problems (e.g., by splitting the code into several procedures) would improve the solver's performance. Worst of all, when the solver is not able to prove the VCs, the user is left wondering whether the assertions they provided are incorrect, or the solver is just unable to prove them. In such cases the determined user's only choice is to trace and debug the solver's heuristics and figure out what went wrong, a task only few users are capable of, and even for them it is tedious and time consuming.

This situation is further aggravated by the incremental and iterative process of software development. That is to say that verification cannot be a huge one-time effort, and it must be continuous and maintainable with reasonable effort.

Decidable logic for deductive verification This thesis explores the use of a *decidable fragment* of first-order logic for deductive verification. Namely, verification conditions are discharged using an automated theorem prover, and verification is designed to ensure that VCs are expressed in a decidable logic for which the solver is a decision procedure. The use of a decidable fragment of first-order logic brings both theoretical and practical benefits, as well as challenges that must be addressed.

Theoretically, if verification can be structured such that VCs fall in a decidable fragment, then the automated solver is guaranteed to terminate with a definite answer to the question "Are the assertions that annotate the program valid or not?". The fragment considered in this thesis also has a *finite model property*. This means that whenever the annotations are not valid, there is a *finite* first-order structure that is a counterexample to the validity of the annotations. The automated solver can find such a model, which serves as a concrete counterexample that can be communicated to the user.

Practically, using decidable logics can improve the productivity of the verification process. For the fragment and applications considered in this thesis, initial experience suggests that Z3's instability is all but eliminated. This results in a verification process that is significantly more productive (and enjoyable), since the user reliably receives feedback from the automated solver: either confirmation that the assertions are valid, or a concrete counterexample to the contrary. The concrete counterexamples guide the user towards fixing the annotations and finding valid assertions, forming a quick loop of progress.

In light of the theoretical and practical appeal of decidable logics for verification, many such logics have been considered by the verification community, from as early as the 1980s. Examples include Presburger arithmetic and a decidable extension to arrays [218], monadic second-order logic over strings and trees for reasoning about inductive data structures [102], the BAPA logic (Boolean Algebra and Presburger Arithmetic) for reasoning about set cardinalities [131], the array property fragment [35], a decidable fragment of separation logic [25] and the STRAND [158] logic for reasoning about heap data structures, to name a few.

Any decidable logic poses some restrictions on what can be expressed, in order to obtain decidability. Therefore, the main challenge in using a decidable logic for deductive verification is that of *expressiveness*. Namely, is the logic powerful enough to capture programs of interest, and express the assertions required to verify them. Indeed, most decidable logics considered by the verification community are quite powerful, and include some arithmetic (e.g., addition but not multiplication), and also some form of (possibly restricted) quantification.

This thesis also employs a decidable logical fragment for deductive verification. However, unlike most decidable fragments considered by the community before, it is a fragment of pure, uninterpreted, first-order logic. This is unusual, since even full first-order logic (which is undecidable, but recursively enumerable) is widely considered too weak for program verification. Pure first-order logic cannot capture notions that are commonly needed to verify programs and algorithms. For example, it cannot express arithmetic, graph reachability, or inductive data structures.

This thesis posits that a decidable fragment of first-order logic is well-suited for verification of distributed protocols, as uninterpreted relations and functions, as well as quantifiers, can be used to reason about the multiple nodes or threads, messages, values, and other objects of distributed systems. Quite surprisingly, this thesis shows that using suitable encodings, pure first-order logic, and even a decidable fragment thereof, can capture proofs of several complex distributed protocols. The protocols considered are beyond reach of any other decidable fragment commonly considered by the verification community in the past.

Therefore, we identify the first challenge that this thesis addresses:

Challenge 1 (Expressiveness)**.** *How can a decidable fragment of first-order logic capture the reasoning needed to verify interesting distributed protocols?*

1.2 Inductive Invariants

An essential feature of computation is *iteration*, and its most basic manifestation in programming is a loop. The assertion annotation that corresponds to a program loop is of prime importance in deductive verification. This annotation is known as a *loop invariant* or *inductive invariant*, and it is the most direct application of mathematical induction in program verification.

A valid loop invariant is an assertion that is true in all states that can occur when the program enters the loop, and also preserved by the loop body. That is, whenever the loop body executes in a state satisfying the invariant (and also the loop condition, if any), the post-state must also satisfy the invariant. Therefore, by induction on the number of loop iterations, it follows that the inductive invariant is satisfied by all states reachable at the loop head. For example, $x > 0$ is an inductive invariant for the loop in the following code segment:

```
x := 1
while true:
    x := x + 1
```

This follows from the validity of the following two Hoare triples:

$$\{\, true \,\}\ \texttt{x := 1}\ \{\, x > 0 \,\} \qquad\qquad \{\, x > 0 \,\}\ \texttt{x := x + 1}\ \{\, x > 0 \,\}$$

A perspective advocated by Naur [178], Floyd [80], Dijkstra [65, 66], and Hoare [105], and even earlier by Goldstine and von Neumann [91], and Turing [225], is that knowing the inductive invariant is essential to writing a correct loop, and a correct program. From this perspective of programming as a mathematical activity, it follows that the programmer should be able to provide inductive invariants rather easily. For if the programmer does not know the inductive invariant, how could they hope to write the loop code correctly?

However, as we know today, most programmers do not write programs in the same way a mathematician proves theorems. Indeed, the requirement that inductive invariants be preserved by the loop body makes them non-intuitive for most programmers. For example, $x > 0$ is not an inductive invariant for the loop in the following code segment:

```
x := 1
y := 1
while true:
```

```
x := x + y
y := y + 2
```

Even though $x > 0$ is satisfied by all states reachable at the loop head, it is not an inductive invariant, since it is not preserved by the loop body. That is, the Hoare triple

$$\{\, x > 0 \,\}\ \texttt{x := x + y ; y := y + 2}\ \{\, x > 0 \,\}$$

is not valid (e.g., consider executing the code starting at program state $[\texttt{x} \mapsto 1, \texttt{y} \mapsto -1]$). One inductive invariant for the above loop is $x > 0 \wedge y > 0$. This invariant is preserved by the loop body, and the following Hoare triple representing this is valid:

$$\{\, x > 0 \wedge y > 0 \,\}\ \texttt{x := x + y ; y := y + 2}\ \{\, x > 0 \wedge y > 0 \,\}$$

Adding the fact $y > 0$ to the invariant makes it inductive. This is a special case of a common situation in mathematics, where one must strengthen the hypothesis for a proof by induction.

Another inductive invariant for the above loop is $x = 1 + \left(\frac{y-1}{2}\right)^2$. This invariant is more tight, and it also implies $x > 0$. In fact, this invariant, when combined with $y \geq 0$, precisely captures the reachable states of the loop. However, the VCs that result from this invariant require non-linear integer arithmetic and are thus harder to check. It is also harder to come up with this inductive invariant in the first place. As this example illustrates, the most useful inductive invariant is not necessarily the most precise, and useful inductive invariants often abstract many details of the actual set of reachable states. However, they must still imply the property we are trying to prove, and of course be preserved by the loop body.

Another difficulty involved with inductive invariants is that they are hard to maintain as part of the development process. Indeed, a small change to the body of a loop (e.g., for the purpose of optimization or adding a new feature) could lead to a drastic change in the required inductive invariant.

As verification requires inductive invariants, and as they are very tricky for programmers to provide, the verification community has devoted great effort to developing methods for automatic discovery of inductive invariants.

A continuous and fruitful effort in this direction evolved from Cousot and Cousot's Abstract Interpretation [56, 57]. Static analysis by abstract interpretation provides a multitude of techniques used to analyze software in various domains. Simple abstract interpretation techniques are employed by standard compilers, and more advanced abstract interpretation techniques are used in particular applications domains (e.g., [58]), and are

an active area of research. However, most techniques based on abstract interpretation find inductive invariants that prove basic and universal correctness properties (e.g., memory safety, avoiding of division-by-zero), and are usually not sufficient to prove full functional correctness.

The problem of finding inductive invariants that allow full functional correctness proofs is fundamental for deductive verification, and is of both theoretical and practical interest. In the context of this thesis, checking a given inductive invariant is decidable, since it reduces to checking a VC in a decidable fragment of first-order logic. In this setting, a natural question is the decidability of the problem of finding an inductive invariant for a given program and property. Can the class of programs, properties, and potential invariants be restricted to make this problem decidable as well? This thesis considers this question, and explores a connection between the modeling in first-order logic and the well established theory of well-structured transition systems [6, 76] and well-quasi-orders [130].

From a more practical perspective, a natural question is how to exploit the benefits of decidability of VC checking to develop a practical interactive methodology for finding inductive invariants. This thesis explores this direction as well, and shows that using the decidable fragment we can provide more automated help for finding inductive invariants, even in an interactive, rather than a fully automated setting.

Thus, we identify the second challenge considered in this thesis:

Challenge 2 (Inductive Invariants)**.** *How to find inductive invariants for full functional correctness proofs? Under what conditions is this problem theoretically decidable? How can it be approached in practice, exploiting the decidable logical fragment?*

1.3 Liveness and Temporal Verification

Inductive invariants can be used to prove *safety* properties, but they do not suffice to prove *termination*, or more generally *liveness* properties. The distinction between safety and liveness was introduced by Lamport [134]. Informally, a safety property asserts that something bad does not happen during system execution. A liveness property asserts that something good eventually does happen. A particular case of a liveness property is program termination. However, the notion is more general. Some programs and systems are not meant to terminate, but to run indefinitely and interact with their environment (e.g., a server that serves requests, an operating system). Such systems are known as reactive systems [163], and some of their most important correctness requirements are liveness properties. For

example, a typical liveness property for a reactive system assets that "every request is eventually followed by a response". In concurrent and distributed systems, it is typical for liveness properties to depend on *fairness assumptions*, such as fair thread scheduling, eventual message delivery by the network, etc.

A violation of a safety property is a finite execution trace (showing something bad can happen), while a violation of a liveness property is an infinite execution trace (showing an infinite execution where something good never happens). This gives the safety vs. liveness distinction an elegant topological interpretation, such that every property of execution traces can be expressed as the intersection of a safety property and a liveness property [15].

The canonical way to prove termination and other liveness properties is to use a ranking function (variant function) into a well-ordered set[2] (this dates back to [80, 225]). For example, to prove termination of a loop, one can provide a function to the natural numbers that decreases with each loop iteration. For more general liveness properties, the ranking functions must decrease until the "good thing" happens.

Given a program annotated with a ranking function, a suitable verification condition can be generated that is valid if and only if the ranking function is decreasing as it should. However, since the range of the ranking function is a well-ordered set, the logical formalism used to express the VC must contain a domain that represents the well-ordered set. The most common examples for ranking functions are functions whose range is the natural numbers (the ordinal ω), or lexicographic ranking functions with natural components (essentially functions to ω^n). This presents a challenge to verification with first-order logic, since first-order logic cannot capture the natural numbers, or the notion of a well-founded relation. We therefore identify the third challenge addressed in this thesis:

Challenge 3 (Liveness)**.** *How can a proof technique based on first-order logic, and a decidable fragment thereof, be used to verify liveness and temporal properties? How can the fact that first-order logic cannot express well-founded relations be circumvented?*

1.4 Distributed Protocols

The main application domain considered in this thesis is verification of distributed protocols. Distributed protocols (or distributed algorithms) allow a set of discrete computing units that communicate with each other (e.g., by message passing or shared memory) to achieve a common task [16]. Distributed protocols are increasingly important, as they are used in

[2] A set equipped with a well-founded relation is also sufficient.

many computer systems, both for performance and for fault tolerance reasons. Distributed protocols are used in a variety of scales: from multiple computing cores on the same chip, to global scale cloud infrastructures employed by companies such as Google [37], Amazon [180], and more.

In addition to their importance, distributed protocols are notoriously hard to design, implement, test, debug, and formally verify. Indeed, distributed protocols are often hard to understand even when they appear simple. One reason for this is that a distributed protocol must function correctly under any behavior of the communication network, relative speeds of processors, and possible failures. This concurrency and asynchrony introduces a great deal of non-determinism, in the form of various interleavings of events. For example, messages can be dropped, delayed, reordered, and duplicated by the network. Further complication is introduced by the possibility of process failures, either benign (i.e., processes can crash) or Byzantine (i.e, processes can act arbitrarily, including maliciously). This leads to a potentially large number of corner cases, and a correct algorithm must handle all of them, including their combinations.

Another implication is that bugs can occur on rare scenarios, making both testing and debugging extremely difficult. Testing often cannot be made exhaustive, and a bug may occur in production due to a rare combination of failures and interleaving of asynchronous events. In this case, the bug would also be very difficult to reproduce and diagnose. This makes formal verification of distributed protocols very appealing, as a formal proof ensures the correctness under all scenarios.

Two anecdotal stories from recent years illustrate the difficulty of designing distributed protocols and reasoning about their correctness informally:

1. The Chord protocol operates a peer-to-peer distributed hash table [216]. Since its publication in 2001, Chord had great impact on the distributed computing and systems community, and in 2011 it won the ACM SIGCOMM Test of Time Paper Award. The original paper claimed a correctness proof, and a further publication in PODC [154] presented a more detailed correctness proof. In spite of the original claims, a 2012 paper by Pamela Zave, followed by a series of works [239, 241, 242], shows that the originally claimed invariants of Chord are incorrect, and that protocol correctness requires further assumptions and adjustments beyond what is stated in the original publications.

2. The Zyzzyva protocol for speculative Byzantine fault tolerance was published in SOSP 2007 [125], received a Best Paper Award, and also reported in a CACM article in

2008 [126] and a journal paper in 2009 [127]. The FaB protocol for fast Byzantine fault tolerance was published in 2005 [166], won the Best Paper Award, and also reported in a journal paper in 2006 [167]. In 2017, roughly ten years after the original publications, Ittai Abraham et al. [12] found a safety bug in the Zyzzyva protocol (and demonstrated it in concrete scenarios), as well as a liveness bug in FaB. Recently they also proposed a fix to the discovered bugs [13].

These anecdotes demonstrate that informal reasoning about distributed protocols and their invariants, even by expert researchers and peer-review, is not always sufficient to avoid subtle bugs and overlooked corner cases.

This thesis focuses on verification of distributed protocols at the protocol (algorithm) design level, and not at the level of a concrete systems implementation. Protocol designs are an appealing object for formal verification, since even protocols with rather compact description (a few pages at most) can have very subtle behaviors, such that informal reasoning done by humans can take years to uncover subtle bugs. Moreover, uncovering and fixing a bug at the design level is much easier than later on at the system implementation level.

While not at the focus of this thesis, the results developed in this thesis also open a path to verification of protocol implementations in distributed systems. This is already being explored by further graduate students in Tel Aviv University, and promising results of extending the approach developed in this thesis to systems verification are published in [219].

1.5 Contributions

Below we outline the contributions of this thesis. These contributions were published in [185–190]. The techniques developed in this thesis are also integrated into the Ivy deductive verification system [171, 186], which is open-source software and freely available under an MIT license [169].

1.5.1 Modeling in a Decidable Fragment of First-Order Logic

This thesis develops the use of a decidable fragment of first-order logic for deductive verification. To achieve this, we must be able to express protocols, properties, and inductive invariants, in the decidable fragment. We approach this in two steps. First, we consider expressing protocols, properties, and inductive invariants in first-order logic, which raises significant challenges. We then consider the restrictions of the decidable fragment, most importantly restricted use of quantifier alternations, and how to mitigate these restrictions.

1.5.1.1 Modeling in First-Order Logic

Our first step is to expressing protocols, properties, and inductive invariants, in first-order logic. This may seem impossible, since first-order logic cannot even express transitive closure which is essential to express network topologies (e.g., ring topology), or properties of sets and cardinalities which are essential for reasoning about distributed consensus protocols. This was expressed above as Challenge 1, and we outline our approach for addressing it below. This approach appears in more detail in Chapter 3.

Deterministic paths One of the main hurdles to using first-order logic in verification, is the fact that it cannot express transitive closure. Several previous works [111–113, 133, 194, 195, 223, 224] showed that reachability for linked data structures such as linked lists and trees can be encoded in a decidable fragment of first-order logic. This thesis exploits these ideas to model reachability and transitive closure for verification of distributed protocols. Modeling reachability in first-order logic can be done for general graphs with outdegree one (i.e., graphs where the outdegree of every vertex is at most one), with trees and rings as special cases.

If we represent graphs by a binary relation that captures edges, then we cannot express graph reachability in first-order logic. The key idea exploited in this thesis is to represent graphs in a different way, by a *path* relation. For trees, the path relation is a binary relation that captures graph reachability, i.e., the transitive closure of the graph edges. From this relation, graph edges can be expressed using a first-order formula with a single universal quantifier. It is well known that the successor relation of a total order can be defined in first-order logic from the order relation: $s(x, y) \equiv x < y \wedge \forall z.\ x < z \rightarrow y \leq z$, and this is the idea used to express graph edges from the path relation.

While the path relation is binary for trees (and also forests), for rings and general graphs we use a ternary relation. For a ring, the transitive closure is trivial (a clique), and does not contain information about the order of the nodes in the ring. In this setting, the non-trivial facts about paths in the ring involve three elements. For distinct nodes x, y, z, it may be the case that y is *between* x and z in the ring, or that it is not (in which case z is between x and y). Formally, y is between x and z if y is part of the shortest (i.e., acyclic) path from x to z. We define the ternary *btw* relation to capture this fact. For general graphs of outdegree one, we use a ternary relation that generalizes both the graph reachability relation used for lines and forests, and the *btw* used for rings. This ternary relation allows to query, with a quantifier free formula, whether a node is reachable from another node, whether a node is

part of a cycle, and whether a node is between two other nodes on a cycle.

In the setting considered in this thesis, the modeling in first-order logic is complete, due to the finite model property of the decidable fragment used. This means that every counterexample provided by the solver corresponds to an actual graph (with the path relation interpreted according to actual paths in the graph), and is never spurious. Further details on the encoding of paths in first-order logic, and its completeness property, appear in Section 3.3.

Quorums and cardinality thresholds Many distributed protocols employ majority sets or quorums that are defined by thresholds on set cardinalities. For example, a protocol may wait for at least $\frac{N}{2}$ nodes to confirm a proposal before committing a value, where N is the total number of nodes. This is often used to ensure consistency. In Byzantine failure models, where processes can act arbitrarily (including maliciously), a common threshold is $\frac{2N}{3}$, where at most a third of the nodes may be Byzantine. Another common threshold is at least $t+1$ nodes, where at most t node may be faulty.

First-order logic cannot completely capture set cardinalities and thresholds. However, this thesis exploits the fact that protocol correctness relies on rather simple properties that are implied by the cardinality threshold, and that these properties can be encoded in first-order logic. The idea is to use a variant of the standard encoding for second-order logic in first-order logic (see e.g. [203]). We introduce a sort for quorums, i.e., sets of nodes with the appropriate cardinality, and use a binary relation *member* to capture set membership. Then, properties that are needed for protocol correctness can be axiomatized in first-order logic.

For example, the fact that any two sets of at least $\frac{N}{2}$ nodes intersect is crucial for many consensus protocols. This property can be expressed in first order logic:

$$\forall q_1, q_2 : \mathsf{quorum}_{\mathsf{i}}.\ \exists n : \mathsf{node}.\ member(n, q_1) \wedge member(n, q_2)$$

For Byzantine consensus algorithms that use sets of at least $\frac{2N}{3}$ nodes, the key property is that any two quorums intersect at a non-Byzantine node. This can also be expressed in first-order logic:

$$\forall q_1, q_2 : \mathsf{quorum}_{\mathsf{ii}}.\ \exists n : \mathsf{node}.\ \neg byz(n) \wedge member(n, q_1) \wedge member(n, q_2)$$

As a final example, protocols that use sets of $t+1$ nodes (when at most t nodes can be faulty) often rely on the fact that any such set contains at least one non-faulty node. This

can also be expressed in first-order logic:

$$\forall q : \mathsf{quorum}_{\mathrm{iii}}.\ \exists n : \mathsf{node}.\ \neg byz(n) \wedge member(n, q)$$

This thesis exploits this idea to verify distributed protocols that use a variety of thresholds, and in all cases we observed the properties that are needed to verify the protocol are actually expressible in first-order logic. This idea greatly simplifies deductive verification of distributed protocols, since it cleanly separates out the reasoning about set cardinalities (which is usually trivial). This allows for a clean formulation of the protocol and its inductive invariants, as well as the resulting verification conditions, all in first-order logic, greatly simplifying verification.

Network models Distributed protocols that operate via message passing may assume any one of a number of network models. One common example is that of a network that may drop, duplicate, and re-order messages. Another example is a network that may drop and re-order messages, but may not duplicate them. Another common example is a lossy FIFO network, in which messages may be dropped, but are otherwise delivered in the order they were sent. This thesis exploits the fact that all of these network models can be encoded in first-order logic in a natural way. This facilitates expressing distributed protocols and their inductive invariants in first-order logic. For example, to represent a FIFO channel, one uses the axioms for a linear order to capture the order of messages in the channel.

For liveness properties, one usually needs fairness assumptions regarding the network. Common examples are that every message is eventually delivered, i.e., no dropping; or that if a message is sent infinitely often then it is eventually delivered, i.e., message dropping is allowed in a restricted way. The techniques developed in this thesis for liveness and temporal verification also allow to naturally express such fairness assumptions about the network model.

1.5.1.2 Eliminating Quantifier Alternation Cycles

This thesis develops the use of a *decidable fragment* of first-order logic for deductive verification. The decidable fragment considered in this thesis is a many-sorted generalization of the classical Bernays-Schönfinkel-Ramsey class, also called EPR (effectively propositional reasoning). The classical (single-sorted) fragment is restricted to relational first-order formulas (i.e., formulas over a vocabulary that contains constant symbols and relation symbols but no function symbols) with quantifier prefix $\exists^*\forall^*$ in prenex normal form. Satisfiability of this

fragment is decidable [153, 205]. Moreover, formulas in this fragment enjoy the *finite model property*, meaning that a satisfiable formula is guaranteed to have a finite model. Moreover, it must have a model with no more elements than the sum of the number of constants and the number of existential quantifiers in the formula.

While the classical single-sorted EPR fragment does not allow any function symbols nor quantifier alternation except $\exists^*\forall^*$, in a many-sorted setting it can be naturally extended to allow stratified or acyclic function symbols and quantifier alternation (see e.g., [1, 124]). For example, a function from sort A to sort B is allowed, but not together with a function from sort B to sort A. A $\forall\exists$ quantifier alternation essentially introduces a (Skolem) function. This many-sorted EPR extension, henceforward called EPR in this thesis, maintains both the finite model property and the decidability of the satisfiability problem (albeit increasing the bound on the model's size).

The restriction of many-sorted EPR to acyclic functions and quantifier alternations may seem quite limiting. Indeed, for many protocols considered in this thesis, most notably Paxos and its variants, a natural encoding in first-order logic results in VCs that contain quantifier alternation cycles. To mitigate this, this thesis develops a systematic way to soundly eliminate the cycles, using *derived relations* and *rewrites*. These allow the user to transform the original system such that the resulting system can be verified in EPR, while the soundness of the transformation itself is also checked using EPR. Thus, a system that contained quantifier alternation cycles is essentially verified by decomposition into several problems, each resulting in an EPR check.

Derived relations A useful idea explored in this thesis for eliminating quantifier alternation cycles is *derived relations*. The idea is to capture an existential formula by a new relation symbol, called a derived relation, and then to use the derived relation as a substitute for the formula, both in the code and in the inductive invariant, thus eliminating some quantifier alternations. The user is responsible for defining the derived relations and modifying the code and invariants to use it. It is then possible to automatically check the soundness of these changes, as well as to generate update code for the derived relation. Note that a derived relation's meaning is not exposed to the underlying theorem prover (as it would result in a quantifier alternation cycle), and its meaning is only manifested in its update code. This is a form of abstraction, and may lead to spurious counterexamples. However, if spurious counterexample arise, they are presented to the user, who can address them by adapting the derived relations' definitions, or via rewrites as explained below.

Rewrites Another useful technique developed in this thesis is to allow the user to soundly rewrite program conditions. This is useful both to eliminate quantifier alternation cycles, and to eliminate spurious counterexamples that result from introducing derived relations. The idea is that if the original model of the protocol contained a formula φ (e.g., as the condition of an `if` statement), then the user can replace it by another condition ψ, provided that at that point of the program, φ and ψ are equivalent (or in some cases that φ implies ψ). This is useful, since it may be the case that one can prove this holds using EPR, and then prove the resulting program using EPR, while the proof of the original program cannot be carried in EPR directly. Effectively, this breaks down a verification problem that contains quantifier alternation cycles into two problems that are both acyclic.

Transformations using derived relations and rewrites provide a powerful methodology for verification using EPR that allows to verify a surprisingly wide set of challenging protocols. Moreover, in the protocols considered in this thesis, the transformations maintain the simplicity and readability of both the code and the inductive invariants. Another encouraging fact demonstrated in this thesis is that the transformation can be reusable across many protocols that share some common structure. This is the case for verification of several protocols from the Paxos family (outlined further in Section 1.5.5), where the transformations needed to eliminate quantifier alternation cycles were completely reusable across protocols.

1.5.1.3 Benefits of Using EPR

One of the key goals for the methodology developed in this thesis is to reach a high level of productivity in the verification process. This is achieved by a combination of *transparency* and *stability*. Transparency means that whenever verification fails, the user is provided with a simple and accessible explanation for the failure, and they have a way forward to resolve the issue. Stability means that the automated solver used in the process is not sensitive to minor perturbations, e.g., random seeds used by heuristics. As mentioned in Section 1.1, solver instability has been identified as a major hurdle for practical deductive verification [75, 100].

As an initial evaluation of these ideas, the methodology developed in this thesis is implemented as part of the Ivy deductive verification system. Ivy uses the Z3 automated theorem prover for discharging verification conditions in the EPR fragment. As outlined further in Section 1.5.5, this methodology is evaluated on several challenging distributed protocols, with promising results. The initial evaluation suggests that using EPR for verification, as described above, enables to achieve both transparency and stability.

Transparency Transparency means that verification failures are not opaque to the user. If verification fails, the user should be able to diagnose the failure and apply their creativity and ingenuity to resolve the issue. In the methodology developed in this thesis, transparency is maintained at three key points.

First, once an initial model of the protocol is created in first-order logic, Ivy checks if its verification conditions are in the decidable fragment. If they are not, this failure is reported to the user in a clear way, by describing the quantifier alternation cycles. The user can then use derived relations and rewrites, as described above, to eliminate some quantifier alternations and break the cycles. The modeling language is designed to make the quantifier structure of the resulting VCs clearly apparent from the program source, facilitating this process.

Second, when the user applies derived relations and rewrites, their soundness is checked using EPR queries to Z3. Third, verification of the transformed model is also reduced to EPR queries (i.e., its resulting verification conditions). The decidability and finite model property of EPR ensure that if any of the EPR checks fail (either checks for soundness of the transformations, or for verifying the transformed model), a finite concrete counterexample will be found by the automated solver. The finite counterexamples are presented to the user, and guide them towards fixing the problem, thus achieving transparency.

Stability Stability means that the performance of the automated solver used to check verification conditions is not sensitive to minor perturbations, e.g., random seeds used by heuristics. The many-sorted EPR decidable fragment used in this thesis is supported by Z3 via model based quantifier instantiation [84]. The experience reported in this thesis shows that for this fragment, the solver instability of Z3 is all but eliminated.

Across the many challenging verification projects reported in this thesis, Z3 was able to decide all verification conditions, and most often terminated within a few seconds. This is the case for both valid VCs, and invalid VCs that arise during the development process. Even more importantly, the performance was not sensitive to small perturbations or changes to the random seed used. This results in a verification process that is much more productive and enjoyable, since the user reliably receives feedback from the automated solver: either confirmation that the transformations and inductive invariants are valid, or a concrete counterexample to the contrary. The fact that the solver reliably responds in a few seconds provides for a quick loop of progress in the verification process, significantly improving productivity.

1.5.2 Decidability of Invariant Inference

When using a decidable fragment for modeling systems and their invariants, checking if an invariant is inductive is decidable. It is therefore natural to investigate the decidability of the problem of invariant *inference*, i.e., the problem of finding a suitable inductive invariant for proving that a given system satisfies a given safety property. This problem is parameterized by a language (i.e., infinite set of first-order sentences) L, the search space of potential inductive invariants. This thesis explores this problem in a general context, develops restrictions under which this problem is decidable, and also obtains undecidability results that show the restrictions are necessary.

A key observation is that due to the restriction of the possible inductive invariants to a given language, the invariant inference problem is different from the safety verification problem, and hence may be decidable even in cases where safety verification is not. For invariant inference, the expected outcome is not "safe/unsafe", but rather "inductive invariant exists/does not exist in the given language". Investigating the decidability of this problem is important to better understand the foundation of existing methods for invariant inference (e.g., abstract interpretation [56, 57], IC3/PDR [34, 116]), since such methods are only able to infer inductive invariants in a certain language, so the underlying problem they solve is actually not safety verification but invariant inference in that language. Therefore, whenever the invariant inference problem is undecidable, a tool cannot be complete even for the language in which it is searching; in contrast, when the problem is decidable, tool developers can aim to have complete algorithms for a restricted language.

This thesis formulates the general problem of inferring inductive invariants in a restricted language L, applies the technique of well-structured transition systems [5–7, 9, 76] based on well-quasi-orders (wqo's) to get sufficient conditions for decidability of invariant inference. The formalization is parametric in the language L, and associates with each language L a quasi-order $\sqsubseteq_L$ on the state space, such that if $\sqsubseteq_L$ is a wqo, then invariant inference in L is decidable. This leads to a (parametric) connection between languages in first-order logic and the theory of decidability of verification based on wqo's.

This thesis develops the following results for decidability of the invariant inference problem, which are presented in Chapter 5.

Decidability of universally quantified invariants for deterministic paths This thesis proves that for programs that manipulate graphs with outdegree one, modeled as discussed in Section 1.5.1.1, and restricting to universally quantified inductive invariants, the

invariant inference problem is decidable (while safety is still undecidable). This class includes many programs manipulating singly-linked-lists [112]. The technical proof builds on Kruskal's Tree Theorem [129] to show that the suitable $\sqsubseteq_L$ is a wqo, as it corresponds to homeomorphic embedding of graphs. Being formulated in logic, this result naturally extends to capture programs with additional structure beyond graph reachability (e.g., sorting algorithms for linked lists). The complexity of inferring universally quantified invariants even for linked-list programs is shown in this thesis to be non-elementary, by reduction from the reachability problem of lossy counter machines [211, 212].

Undecidability of alternation-free invariants for deterministic paths In the same setting as above, inferring alternation-free invariants is shown to be undecidable. This demonstrates that the invariant inference problem is theoretically harder than invariant checking, since in this setting checking inductiveness of an alternation-free invariant is decidable. This also shows that mixing universal and existential information even without alternation makes invariant inference undecidable, so in this setting the restriction to universal invariants is necessary for decidability of invariant inference.

Undecidability of universally quantified invariants in general If the program is allowed to manipulate a single unrestricted binary relation, then invariant inference is undecidable even when restricting to universally quantified invariants. This is shown by constructing a safe transition system that has a universally quantified inductive invariant if and only if a given counter machine halts. This shows that the restrictions we develop are necessary.

Systematic constructions for decidability To overcome the general undecidability while allowing unrestricted relations, this thesis develops systematic ways to construct classes of systems and languages for invariants for which invariant inference is decidable. These constructions start with some established wqo, (e.g. the deterministic paths class with universal invariants) and gradually extend it to construct new languages with suitable wqo's. The constructions utilize the connection between logic and wqo, via the $\sqsubseteq_L$ definition, and employ Higman's Lemma [104] to show that the new languages induces a wqo. Each construction allows to handle richer system (i.e., in terms of the logical vocabulary), while decidability is maintained by further restricting the language of potential invariants. The constructions can be applied iteratively, and this is demonstrated by obtaining a decidable fragment of the invariant inference problem that captures a nontrivial example of a network

learning switch.

1.5.3 Interactive Inference of Universal Invariants

When invariant inference is undecidable, and also when it is decidable and practical techniques do not (yet) provide full automation, the user must be involved in the process of finding inductive invariants. Even in such cases, this thesis shows that using a decidable logical fragment can have further benefits beyond that of providing counterexamples.

Manually finding inductive invariants is one of the most creative and challenging parts of deductive verification. This was identified above as Challenge 2. For the special case of universally quantified inductive invariants, this thesis develops an interactive process that allows the user to incrementally obtain an inductive invariant, where user interaction is based on a graphical representation of invariants and counterexamples. This methodology is explained in detail in Chapter 6, and implemented in the Ivy deductive verification system.

The key idea is to check a candidate invariant (which initially could be just the safety property to be proven), and in case it is not inductive, to graphically present a counterexample to induction (CTI) to the user, and also allow the user to graphically guide generalization from the counterexample, which results in modifications to the invariant. This exploits both a graphical representation of finite structures as graphs (relying on EPR's finite model property), and on a graphical representation of universally quantified formulas as *excluded substructures*. It also exploits the decidability of EPR to offer further assistance to the user in the form of well-defined decidable checks that assist in generalization. For example, checking a potential generalization using bounded unrolling, or minimizing counterexamples according to user selected criteria. The result of this process is that the user gets much automated help, but the user is kept in the loop, so the verification process is never stuck trying to automatically solve an undecidable problem.

1.5.4 Liveness and Temporal Verification

Safety properties can be proven using inductive invariants. In contrast, liveness properties, and general temporal properties of infinite-state systems are usually proven using ranking functions or well-founded relations. However, pure first-order logic (without theories) cannot express the required rankings or the notion of a well-founded relation or well-ordered set. Therefore, it may seem that liveness verification cannot be done in pure first-order logic, as identified in Challenge 3. This thesis addresses this challenge and shows that on the contrary, the formalism of first-order logic provides a unique opportunity for proving liveness, using a

new technique developed in this thesis, and presented in Chapters 7 and 8.

The technique exploits the flexibility of representing states as first-order structures, and uses first-order temporal logic (FO-LTL) (e.g., [2, 162]) for temporal specification.

This general formalism provides a powerful and natural way to model temporal properties of infinite-state systems. It naturally supports both *unbounded parallelism*, where the system is allowed to dynamically create processes, and *infinite-state per process*. Unbounded-parallelism usually requires infinitely many (or quantified) fairness assumptions (e.g., that every thread is scheduled infinitely often in a program with dynamic thread creation, where an infinite trace can have infinitely many threads). This is fully supported by the formalism and the developed proof technique.

The technique developed in this thesis is based on a novel liveness-to-safety reduction[3] that transforms temporal verification (expressed in FO-LTL) to safety verification of an infinite-state system expressed in first-order logic without temporal operators. This allows us to leverage existing safety verification techniques, and the other techniques developed in this thesis, to verify liveness and temporal properties. While such a reduction cannot be complete for computability reasons,[4] it is sound, and it was successful in proving liveness of several challenging protocols, including protocols for which this research obtained the first mechanized liveness proof.

The liveness-to-safety reduction is based on an abstract notion of acyclicity, using *dynamic abstraction*. For finite-state systems, liveness can be proven through acyclicity (the absence of fair cycles). This is the classical liveness-to-safety reduction of [27]. This also works for parameterized systems, where the state space is finite (albeit unbounded) for every system instance [196]. For infinite-state systems, the acyclicity condition is unsound (an infinite-state system can be acyclic but non-terminating). The liveness-to-safety reduction with dynamic abstraction defines a finite abstraction that is fine-tuned for each execution trace, while abstracting only the cycle detection aspect (rather than the actual transitions of the system). Such fine-tuned abstraction is made possible by exploiting the symbolic representation of the transition relation in first-order logic, as well as the first-order formulation of the fairness constraints.

The basic observation used to is that, once a finite domain of objects is fixed, there exist only finitely many first-order logic structures over the same signature, providing a natural finite abstraction by projection. The developed technique defines a finite set of objects for

[3] The term *reduction* is not used here to mean a complexity theoretic reduction.

[4] The temporal verification problem in this setting is Π^1_1-complete [2], while safety verification is in the arithmetical hierarchy.

every potential counterexample trace, by further exploiting the first-order representation of fairness constraints.

This thesis makes the following contributions in the context of liveness and temporal verification.

Abstract fair cycle parameterized by dynamic abstraction function This thesis defines a notion of an abstract fair cycle, and this definition is parameterized by a dynamic finite abstraction function, and a fairness selection function. This definition is general, and does not depend on the first-order logic formalism. The definition gives rise to a parametric reduction from liveness (more precisely, from fair termination) to safety that is sound for systems with both infinitely many states and infinitely many fairness constraints.

Realization in first-Order logic This thesis instantiates the parameterized definition of an abstract fair cycle in a uniform way for systems expressed in first-order logic and FO-LTL. The first step uses a tableau construction for FO-LTL to convert any temporal verification problem into fair termination. Next, the *footprint* of transitions and abstraction by projection are used to define a dynamic abstraction function in first-order logic. The footprint of a transition is the finite set of elements that participated in it. The result is an *automatic* transformation from verification of temporal properties to verification of safety properties. Furthermore, the transformation itself does not introduce any quantifier alternations, so it maintains the possibility to prove the resulting safety problem using the EPR decidable fragment, which was indeed used in the protocols considered in this thesis.

Increasing power via a nesting structure To extend the applicability of the reduction, this thesis defines a *nesting structure*, which can be specified by the user. This allows to break the termination argument into several nested levels. In the developed technique, this is the analogue of a lexicographic termination argument, and we show that it strictly increases the power of the reduction.

Temporal prophecy and temporal prophecy witnesses An additional mechanism developed in this thesis to increase the proof power of the liveness-to-safety reduction is *temporal prophecy* and *prophecy witnesses*, which is presented in Chapter 8. Here, the idea is to augment the system with additional temporal formulas that are not part of the specification, and also with additional constant symbols that are essentially Skolem witnesses for temporal formulas. This is shown to strictly increase the proof power of the

safety-to-liveness reduction, and to subsume the nesting structure in this regard. (However, it does not preserve the simplicity of a "lexicographic" argument.) Moreover, the class of temporal properties provable via temporal prophecy and the liveness-to-safety reduction is closed under *first-order reasoning*, with cut elimination as a special case.

In addition to increasing the proof power, temporal prophecy witnesses also facilitate verification of the resulting safety problem using the EPR decidable fragment. By introducing prophecy witnesses, one can often eliminate quantifier alternations in the resulting verification conditions. The idea is that a temporal prophecy witness is used to name a particular element (e.g., the thread that is eventually starved), and then inductive invariants can be specified for this particular constant, rather than with a quantifier. In several cases considered in this thesis, this allowed to eliminate quantifier alternation cycles.

Implementation and integration in Ivy The liveness-to-safety reduction with temporal prophecy has been integrated into the Ivy deductive verification system. In Ivy, temporal prophecy formulas are derived from an inductive invariant provided by the user (for proving the safety property induced by the liveness-to-safety reduction), which provides a seamless way to prove temporal properties.

Evaluation on challenging examples The effectiveness of the approach for liveness verification is demonstrated on some challenging examples. Some cannot be handled without a nesting structure or temporal prophecy, and some examples require temporal prophecy witnesses to allow verification in the EPR decidable fragment. To the best of the author's knowledge, this research presented the first mechanized liveness proof for both TLB Shoot-down [29] and Stoppable Paxos [140]. Interestingly, the liveness of Stoppable Paxos is tricky enough that Lamport et al. prove it using an informal proof of about 3 pages of rigorous temporal-logic reasoning [140, Section A.2].

1.5.5 Verification of Protocols from the Paxos Family

The main application domain considered in this thesis is verification of protocols from the Paxos family. Below we provide some background on Paxos, and later outline the main contributions of this thesis from the perspective of verification of Paxos protocols.

1.5.5.1 Background on Paxos

On of the most important and widely studied algorithms in distributed computing and distributed systems is the Paxos [135, 136] algorithm for distributed consensus, and the Paxos

family of protocols for state machine replication (SMR), which is the most popular approach for implementing fault tolerant distributed systems [210]. In distributed consensus, a network of unreliable processors with unreliable communication try to agree on a single value. In SMR, a (virtual) centralized sequential state machine is replicated across many nodes, providing fault tolerance while exposing a simple (centralized and sequential) semantics to its clients. SMR is implemented by repeatedly solving distributed consensus to agree on the next command to be executed. Fault tolerant distributed computing, achieved using a protocol from the Paxos family, underlies many services used daily by billions of people from companies such as Google, Amazon, and more [37, 41, 180].

Paxos, and similar consensus algorithms such as Viewstamped Replication [182] and Raft [183], are also widely studied (e.g., [94, 159, 232]) because they contain ideas that are fundamental for distributed computing in general. These include leader election, majority voting and quorum systems, two-phase commit, ballots (also called views or rounds), reconfiguration, and more. Note that distributed consensus is not solvable in a general asynchronous setting with failures [77]. However, solutions that provide consistency (i.e., processors cannot decide on different values) under all scenarios, and ensure progress under "good conditions" [96] are possible, and protocols in the Paxos family achieve this.

The Paxos family includes many variants that support various optimizations and features, and employ different trade-offs. Examples include optimizations that reduce the number of messages or nodes needed to commit a new command in certain scenarios (e.g., Fast Paxos [138], Flexible Paxos [107]); reconfiguration, a feature that allows to change the set of operating nodes while the state-machine is running (e.g., Vertical Paxos [141], Stoppable Paxos [140]); and more.

Due to its importance, verification of distributed protocols of the Paxos family, and systems that implement them, is an ongoing research challenge. The systems and programming languages communities have had several recent success stories in verifying Paxos-like protocols and their implementations in projects such as IronFleet [100], Verdi [235], and PSync [70].

1.5.5.2 Contributions

This thesis applies deductive verification using first-order logic and EPR to several protocols in the Paxos family. Quite surprisingly, this thesis shows that protocols as complex as Paxos can indeed be verified in a decidable fragment, and it presented the first verification of Paxos in a decidable fragment. For some of the variants, namely Vertical Paxos, Fast Paxos,

and Stoppable Paxos, this thesis presents the first mechanized verification. In this context, perhaps the greatest feat of this thesis is verification of both safety and liveness of Stoppable Paxos [140]. Stoppable Paxos is arguably the most intricate algorithm in the Paxos family: as acknowledged by Lamport et al., "getting the details right was not easy" [140, Section 1]. Their paper includes 10 pages of rigorous informal proof of safety, and 3 additional pages of a liveness proof composed of rigorous temporal reasoning. Before this thesis, this protocol has never been formally verified, and state-of-the-art verification projects for simpler variants of Paxos (e.g, Multi-Paxos, Raft) take between 5,000 and 50,000 lines of proof [40, 100, 235]. In contrast, the models developed in this thesis are a few hundred lines, with a few dozens of simple invariants that are automatically checked in a decidable logic.

The techniques developed in this thesis have also enabled a recent work [219] that considers verification of systems implementations (versus protocol design). This work, which is outside the scope of this thesis, also successfully applies the techniques developed in this thesis to verify the Raft protocol.

For the protocols in the Paxos family considered here, this thesis makes the following contributions.

Formalization in first-order logic of protocols and inductive invariants For each protocol in the Paxos family considered, the techniques of modeling in first-order logic outlined in Section 1.5.1.1 are employed to model both the protocol and its inductive invariant in first-order logic.

Transformation to EPR and decidable safety verification For the Paxos protocols considered, the model in first-order results in verification conditions that contain cyclic quantifier alternations. This thesis applies the techniques outlined in Section 1.5.1.2 to eliminate the cycles and enable verification in the decidable EPR fragment. The result is the first verification of Paxos (in the full asynchronous setting) using a decidable fragment. Another encouraging observation is that the transformations needed to eliminate the cycles were completely reusable across all protocols considered, despite significant different features and optimizations in each protocol. This demonstrates the flexibility and reusability of the transformation mechanism developed in this thesis.

Liveness verification For several protocols, including the most challenging Stoppable Paxos, this thesis also applies the temporal verification technique outlines in Section 1.3 to verify liveness. As mentioned above, liveness of Paxos is not guaranteed under general

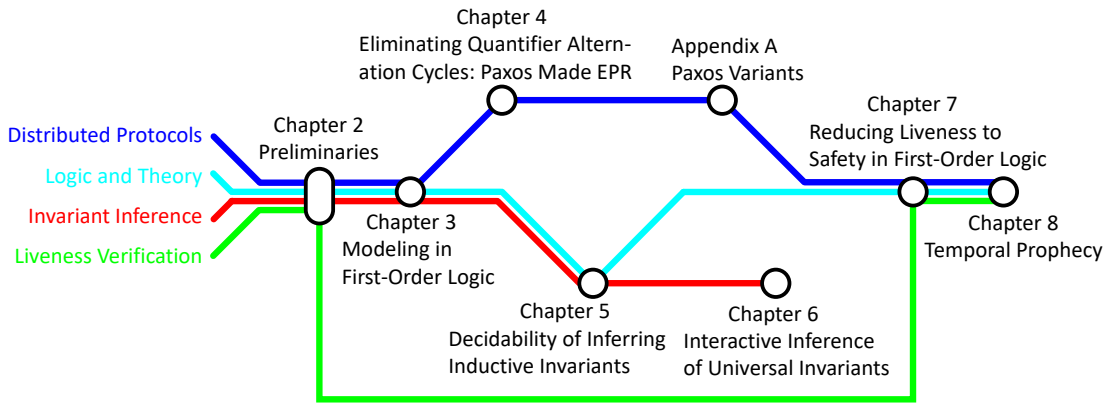


Figure 1.1: Suggested routes for reading this thesis according to the reader's interest. Four routes are depicted: for readers interested in verification of distributed protocols and Paxos protocols, for readers interested in logic and theoretical aspects, for readers interested in automatic or semi-automatic inference of inductive invariants, and for readers interested in liveness and temporal verification. The scenic route going through all chapters in order is not depicted but is nonetheless suggested.

conditions. This thesis proves the liveness property formalized in [140, Section 4]. Namely, that the protocol makes progress if eventually there is a unique leader for a high enough ballot number that is non-faulty and can communicate with a non-faulty quorum. This property is expressible in FO-LTL, and the fact that the temporal verification technique developed in this thesis could handle the liveness proof of a protocol as challenging as Stoppable Paxos demonstrates its applicability.

1.6 How to Read This Thesis

This thesis is organized in three parts, corresponding to the three challenges identified above. Part I addresses the challenge of modeling protocols in a decidable fragment of first-order logic (challenge 1). Part II addresses the challenge of finding inductive invariants (challenge 2). Part III addresses verification of liveness and temporal properties using first-order logic (challenge 3).

As this thesis contains contributions and raises future research directions on several topics, different readers could be interested in different contributions and ideas, and need not read the thesis in its entirety. Below are a few suggested routes for readers with different interests, along with suggested future research directions. Figure 1.1 depicts the various suggested routes for reading this thesis (the route going through all chapters in order is omitted, but is nonetheless suggested).

Reader interested in verification of distributed protocols and Paxos protocols Skim Chapters 2 and 3 for the formalism of using first-order logic to model protocols. Read Chapter 4 for safety verification of Paxos and Multi-Paxos, and Appendix A for verification of Vertical Paxos, Fast Paxos, Flexible Paxos and Stoppable Paxos. Read Chapters 7 and 8 for liveness verification. Check out the open-source Ivy deductive verification system [169], and the work of [219] that builds on the ideas of this thesis and extends them to verify executable implementations for Multi-Paxos and Raft. Try to use Ivy to verify your distributed protocol or distributed system.

Reader interested in logic and theoretical aspects Read Chapters 2 and 3 for the formalism of using first-order logic to model infinite-state transition systems. Read Chapter 5 for results on decidability of invariant inference and the connection to well-quasi-orders. Read Chapters 7 and 8 for an interesting technique to verify liveness properties in first-order logic, without using ranking functions or well-founded relations. Perhaps ask yourself how to extend the completeness-for-finite-models results of Section 3.3 to paths in other classes of graphs, or other properties that are seemingly beyond first-order logic. Ask yourself how to extend the decidability results of Chapter 5 using other well-quasi-orders, or extend them to other classes of invariants other than universal, or come up with further restrictions that also reduce the complexity of invariant inference. Think about the technique developed in Chapters 7 and 8 as a proof system for first-order linear temporal logic (FO-LTL), and its connection to well-known systems of arithmetic such as Primitive Recursive Arithmetic or Peano Arithmetic. The proof system of Chapter 7 cannot prove termination of the Ackermann function (Section 7.5), and when extended with temporal prophecy it admits a robust closure property and a cut-elimination theorem (Section 8.4).

Reader interested in automatic or semi-automatic invariant inference Skim Chapters 2 and 3 for the formalism of using first-order logic to model infinite-state transition systems. Read Chapter 5 for results on decidability of automatic invariant inference, and Chapter 6 for a semi-automatic methodology for finding inductive invariants via *interactive generalization*. Perhaps think about exploiting the decidable classes of Chapter 5 to create practical algorithms for invariant inference. Think about ways to improve the technique of Chapter 6 and make it more automated, or perhaps generalize the idea of interactive generalization to classes of invariants other than universally quantified. If you are developing an invariant inference technique, mine this thesis for challenging benchmarks.

Reader interested in liveness and temporal verification Read Chapter 2 and optionally skim Chapter 3 for the formalism of using first-order logic to model infinite-state transition systems. Read Chapter 7 for a technique to verify liveness properties using an abstract acyclicity test and *dynamic abstraction*, without using ranking functions or well-founded relations. Think about the parametric definition of dynamic abstraction and the abstract acyclicity test (Section 7.3.1), and perhaps apply it in other contexts that are not based on first-order logic. Read Chapter 8 and think about applying the idea of *temporal prophecy* in other contexts not based on first-order logic. Or try to apply your favorite temporal verification technique to the challenging examples that motivated the techniques developed in this thesis (specifically Sections 7.4.1 and 8.1).

Part I

Modeling in a Decidable Fragment of First-Order Logic

Chapter 2

Preliminaries

This chapter presents the formalism used in this thesis for deductive verification of infinite-state systems by specifying both the system and its invariants in first-order logic. Here we focus on safety verification, and definitions that are only needed for liveness and temporal verification are deferred to Chapter 7 (Section 7.2).

2.1 Many-Sorted First-Order Logic

We use standard many-sorted first-order logic with equality [203]. A sorted first-order *vocabulary* or *signature*, denoted Σ, consists of sorted constant symbols, sorted function symbols, and sorted relation symbols. Functions and relation symbols can have any finite arity, and constant symbols are considered function symbols of arity zero, also called nullary function symbols. We sometimes write r^k to indicate that r is a relation of arity k. Terms and formulas are constructed according to the syntax listed in Figure 2.1. We always assume terms and formulas are well-formed and well-sorted. The set of free variables in a term or a formula is defined as usual. A term without free variables is called a *ground term*. A formula without free variables is called a *sentence* or a *closed formula*. We sometimes write $\varphi(x_1, \ldots, x_k)$ (respectively, $t(x_1, \ldots, x_k)$) to indicate that $x_1, \ldots, x_k$ are free in φ (respectively, t). We sometimes omit the sorts of variables in case they are either clear or unimportant.

Given a vocabulary Σ, a *structure* of Σ is a pair $s = (\mathcal{D}, \mathcal{I})$, where $\mathcal{D}$ is a sorted *domain*, and $\mathcal{I}$ is an *interpretation*. The sorted domain $\mathcal{D}$ maps each sort s of Σ to a set $\mathcal{D}(\mathsf{s})$, called the domain or universe of s in s. If all these sets are finite, we say that the domain $\mathcal{D}$ and the structure s are finite. By abuse of notation, we sometimes use $\mathcal{D}$ to denote $\bigcup_{\mathsf{s}\in\Sigma} \mathcal{D}(\mathsf{s})$, that is, the union of the universes of all the sorts. The interpretation $\mathcal{I}$ maps

$\langle t \rangle$	::=		terms
	|	$x : \mathsf{s}$	logical variable x
	|	c	constant symbol c
	|	$f\,(\langle t \rangle, \ldots, \langle t \rangle)$	application of function f
	|	$\mathrm{ite}\,(\langle \varphi \rangle, \langle t \rangle, \langle t \rangle)$	if-then-else term
$\langle \varphi \rangle$	::=		formulas
	|	$r\,(\langle t \rangle, \ldots, \langle t \rangle)$	membership in relation r
	|	$\langle t \rangle = \langle t \rangle$	equality between terms
	|	$\langle \varphi \rangle \wedge \langle \varphi \rangle$	conjunction
	|	$\langle \varphi \rangle \vee \langle \varphi \rangle$	disjunction
	|	$\langle \varphi \rangle \rightarrow \langle \varphi \rangle$	implication
	|	$\langle \varphi \rangle \leftrightarrow \langle \varphi \rangle$	bi-implication
	|	$\neg \langle \varphi \rangle$	negation
	|	$\forall x : \mathsf{s}.\ \langle \varphi \rangle$	universal quantification
	|	$\exists x : \mathsf{s}.\ \langle \varphi \rangle$	existential quantification

Figure 2.1: Syntax of terms and formulas in many-sorted first-order logic.

each symbol in Σ to its meaning in s. (Recall that constant symbols are function symbols with arity zero.) $\mathcal{I}$ associates each k-ary relation symbol $r : \mathsf{s}_1, \ldots, \mathsf{s}_k$ with a relation $\mathcal{I}(r) \subseteq \mathcal{D}(\mathsf{s}_1) \times \ldots \times \mathcal{D}(\mathsf{s}_k)$, and associates each k-ary function symbol $f : \mathsf{s}_1, \ldots, \mathsf{s}_k \to \mathsf{s}$ with a function $\mathcal{I}(f) : \mathcal{D}(\mathsf{s}_1) \times \ldots \times \mathcal{D}(\mathsf{s}_k) \to \mathcal{D}(\mathsf{s})$.

We use the standard semantics of many-sorted first-order logic. Given a vocabulary Σ and a structure $s = (\mathcal{D}, \mathcal{I})$, an assignment σ maps every logical variable $x : \mathsf{s}$ to an element in the domain of s, i.e., $\sigma(x : \mathsf{s}) \in \mathcal{D}(\mathsf{s})$. We write $s, \sigma \models \varphi$ to denote that the structure s and assignment σ *satisfy* the formula φ. We write $s \models \varphi$ to mean that $s, \sigma \models \varphi$ for any σ, and we will reserve this for whenever φ is a closed formula, i.e., without free variables. For a formula $\varphi(x_1, \ldots, x_k)$ whose free variables are $x_1, \ldots, x_k$, and elements $e_1, \ldots, e_k \in \mathcal{D}$ (with appropriate sorts), we write $s \models \varphi(e_1, \ldots, e_k)$ to mean that $s, \sigma \models \varphi$ where σ is an assignment that maps x_i to e_i for each $i \in \{1, \ldots, k\}$. For a set of formulas Γ, we say that $s, \sigma \models \Gamma$ (respectively, $s \models \Gamma$) if $s, \sigma \models \varphi$ (respectively, $s \models \varphi$) for every $\varphi \in \Gamma$. For a set of formulas Γ and a formula φ, we say that Γ *entails* φ, denoted $\Gamma \models \varphi$, to mean that for every s, σ, if $s, \sigma \models \Gamma$ then $s, \sigma \models \varphi$. We use $\psi \models \varphi$ to mean $\{\psi\} \models \varphi$, and $\Gamma, \psi \models \varphi$ to mean $\Gamma \cup \{\psi\} \models \varphi$. We say that two formulas φ and ψ are *equivalent* if $\psi \models \varphi$ and $\varphi \models \psi$.

We say that a formula is in *negation normal form* if negation is only applied to its atomic subformulas, namely $r(t_1, \ldots, t_n)$ or $t_1 = t_2$, and the formula does not contain implications or bi-implications. Every formula can be transformed to an equivalent formula in negation normal form, and the transformation is essentially linear in the size of the formula.[1] For example, the negation normal form of $\neg\exists x, y.\ r(x, y) \wedge x \neq y$ is $\forall x, y.\ \neg r(x, y) \vee x = y$.

We say that a formula is *quantifier-free* if it contains no quantifiers. We say that a formula is in *prenex normal form* if it is of the form $Q_1 . \cdots Q_n . \psi$ where ψ is quantifier-free, and each Q_i is either $\forall x : \mathsf{s}$ or $\exists x : \mathsf{s}$. Every formula can be transformed to an equivalent formula in prenex normal form, and the transformation is linear in the size of the formula. For example, the prenex normal form of $\forall x.\ r(x) \to \exists y.\ p(x, y)$ is $\forall x.\ \exists y.\ r(x) \to p(x, y)$.

We say a formula is *universally quantified* or *universal*, if it is in prenex normal form and has only universal quantifiers. An *existentially quantified* or *existential* formula is similarly defined. We sometimes use a regular expression over quantifiers to refer to the quantifier prefix of a formula in prenex normal form. For example, formulas with quantifier prefix $\exists^* \forall^*$ are those whose quantifier prefix is composed of any number of existential quantifiers followed by any number of universal quantifiers. Whenever an existential quantifier is in

[1] The only caveat, observed by an anonymous referee of this thesis, is that the transformation is exponential in the nesting depth of bi-implications. If this depth is assumed to be a fixed constant (an assumption which is practically justified) then the transformation is linear in the size of the formula.

the scope of a universal quantifier or vice versa, we call this a *quantifier alternation*. A formula is *alternation-free* if it contains no quantifier alternations; namely, if it is a Boolean combination of universal and existential formulas.

We say that a formula φ is *satisfiable* if there are some s and σ such that $s, \sigma \models \varphi$. Otherwise, we say that φ is *unsatisfiable*. We say that a formula φ is valid if $s, \sigma \models \varphi$ for any s and σ. Note that φ is valid if and only if $\neg\varphi$ is unsatisfiable. A formula $\varphi(x_1, \ldots, x_n)$, whose free variables are $x_1, \ldots, x_n$, is valid if and only if the closed formula $\forall x_1, \ldots, x_n.\ \varphi(x_1, \ldots, x_n)$ is valid; it is satisfiable if and only of the closed formula $\exists x_1, \ldots, x_n.\ \varphi(x_1, \ldots, x_n)$ is satisfiable. We say that two formulas φ and ψ are *equisatisfiable* to mean that φ is satisfiable if and only if ψ is satisfiable. Note that if two formulas are equivalent then they are also equisatisfiable, but the converse does not necessarily hold.

Every formula φ can be transformed to an equisatisfiable universal formula, by a process called *Skolemization*. The resulting formula is denoted by $Sk(\varphi)$, and the transformation is essentially linear in the size of φ.[2] To Skolemize φ, we first transform it into negation normal form, and then eliminate existential quantifiers by using fresh function symbols added to the vocabulary in the following way. For every existential quantifier $\exists y$ that resides in the scope of universal quantifiers $\forall x_1, \ldots, \forall x_k$, remove the existential quantifier $\exists y$ and replace every occurrence of y (bound by $\exists y$) by $f(x_1, \ldots, x_k)$, where f is a fresh function symbol added to the vocabulary. The original formula φ and the resulting formula $Sk(\varphi)$ are equisatisfiable. For example, the Skolemized version of $\forall x.\ r(x) \vee (\exists y.\ p(x, y) \wedge \forall z.\ \exists w.\ q(x, y, z, w))$ is $\forall x.\ r(x) \vee (p(x, f_y(x)) \wedge \forall z.\ q(x, f_y(x), z, f_w(x, z)))$, where f_y and f_z are fresh function symbols of appropriate arities and sorts. Note that the vocabulary of $Sk(\varphi)$ is a superset of the vocabulary of φ. The added function symbols are referred to as *Skolem functions*. A nullary Skolem function, which arises whenever an existential quantifier is not in scope of any universal quantifier, is called a *Skolem constant*.

A structure $s_1 = (\mathcal{D}_1, \mathcal{I}_1)$ (over Σ) is a *substructure* of a structure $s_2 = (\mathcal{D}_2, \mathcal{I}_2)$ (over Σ) if for every sort $s \in \Sigma$, $\mathcal{D}_1(\mathsf{s}) \subseteq \mathcal{D}_2(\mathsf{s})$, and for every $a \in \Sigma$, $\mathcal{I}_1(a)$ is the restriction of $\mathcal{I}_2(a)$ to $\mathcal{D}_1$.

2.2 Many-Sorted EPR

This section presents the decidable fragment of first-order logic used in this thesis. We first review the classical unsorted decidable fragment, and then its straightforward many-sorted

[2] As Nikolaj Bjørner pointed out, the worst-case is actually quadratic when considering formula sizes as strings. However, this is not a practical concern, and if it ever may be it can be circumvented by a more efficient representation.

extension.

2.2.1 Classical EPR

The effectively propositional (EPR) fragment of first-order logic, also known as the Bernays-Schönfinkel-Ramsey class, is a fragment of first-order logic for which satisfiability is decidable [205]. The fragment contains first-order formulas over a relational vocabulary (that is, a vocabulary that contains constant symbols and relation symbols but no non-nullary function symbols), where the formula is in prenex normal form and has quantifier prefix $\exists^* \forall^*$. The satisfiability problem for this class is NEXPTIME-complete[3] [153]. Beyond decidability, the EPR fragment has a *finite model property*, meaning that an EPR formula is satisfiable if and only if it is satisfiable by a *finite* structure. The size of this structure is bounded by the total number of existential quantifiers and constants in the formula.

To see this, consider a formula $\varphi = \exists x_1, \ldots, x_n.\ \forall y_1, \ldots, y_m.\ \psi$, where ψ is quantifier free, and a structure $s = (\mathcal{D}, \mathcal{I})$ such that $s \models \varphi$. Then, there is an assignment σ such that $s, \sigma \models \forall y_1, \ldots, y_m.\ \psi$. We can project the structure s to a substructure whose domain contains only $\sigma(x_1), \ldots, \sigma(x_n)$ and $\mathcal{I}(c_1), \ldots, \mathcal{I}(c_k)$ where $c_1, \ldots, c_k$ are the constants symbols present in φ. This substructure also satisfies φ, and its domain is of the required size (and clearly finite). This projection, and the bound on the size of the model, can also be understood by considering Skolemized form $Sk(\varphi)$. The existential quantifiers in φ result in Skolem constants, and there are no non-nullary Skolem functions. Thus, the total number of constants in $Sk(\varphi)$ is precisely the total number of existential quantifiers and constant symbols in φ. The projection described above essentially maintains the interpretation of the Skolem constants.

The above argument can be seen as a special instance of Herbrand's theorem, for a case in which the Herbrand universe is finite. For any universal formula ψ, Herbrand's theorem says that ψ is satisfiable if and only if the set of ground instantiations of ψ is satisfiable. For an EPR formula φ, apply Herbrand's theorem to $Sk(\varphi)$, and observe that the set of ground instantiations of $Sk(\varphi)$ is finite. This also provides a way to reduce satisfiability of an EPR formula to propositional SAT, as the satisfiability of a finite set of ground instantiations amounts to propositional satisfiability.[4]

[3] If we fix the maximal arity of relations then satisfiability of EPR formulas is actually in Σ_2^{P}, i.e., second level of the polynomial-time hierarchy. If we were to further restrict the maximal number of quantifiers used then satisfiability is in NP, i.e., the same as satisfiability of propositional formulas.

[4] We have sidestepped the presence of equality. This is unessential, as we can add the axioms governing the equality symbol to φ, noting that they are universally quantified so φ remains in EPR.

2.2.2 Many-Sorted Extension of EPR

While EPR does not allow any (non-nullary) function symbols or quantifier alternation except $\exists^* \forall^*$, in a many-sorted context it can be easily extended to allow *stratified* function symbols and quantifier alternations (as formalized below). The extension maintains both the finite model property and the decidability of the satisfiability problem (see e.g, [1]). The crux of the idea is to allow functions and quantifier alternations that still ensure that the vocabulary of the Skolemized formula generates only a finite set of ground terms.

Quantifier alternation graph Let φ be a formula in negation normal form over a many-sorted signature Σ. We define the *quantifier alternation graph* of φ as a directed graph where the set of vertices is the set of sorts in Σ, and the set of directed edges, called *quantifier alternation edges* or $\forall\exists$ *edges*, is defined as follows.

- **Function edges:** let f be a function that appears in φ from sorts $\mathsf{s}_1, \ldots, \mathsf{s}_k$ to sort s (i.e., $f\colon \mathsf{s}_1, \ldots, \mathsf{s}_k \to \mathsf{s}$). Then there is a $\forall\exists$ edge from s_i to s for every $i \in \{1, \ldots, k\}$.

- **Quantifier edges:** let $\exists x : \mathsf{s}$ be an existential quantifier that resides in the scope of the universal quantifiers $\forall x_1 : \mathsf{s}_1, \ldots, \forall x_k : \mathsf{s}_k$ in φ. Then there is a $\forall\exists$ edge from s_i to s for every $i \in \{1, \ldots, k\}$.

Intuitively, quantifier edges are the edges that would arise as function edges from Skolem functions in $Sk(\varphi)$.

Many-sorted EPR A formula φ is *stratified* if the quantifier alternation graph of its negation normal form is acyclic. The *many-sorted EPR fragment* consists of all stratified formulas. This fragment maintains both the decidability of the satisfiability problem and the finite model property of unsorted EPR (though the model size bound must be adjusted). The reason for this is that for a stratified formula φ, the vocabulary of $Sk(\varphi)$ generates a finite set of ground terms (whose size is the adjusted model size bound). Therefore, any structure that satisfies φ can be projected to a finite subset of its domain that still satisfies φ (as in unsorted EPR). Similarly, the set of ground instantiations of $Sk(\varphi)$ is finite, so the satisfiability problem of many-sorted EPR can be reduced to propositional SAT (as in unsorted EPR).

In this thesis, whenever we say a formula is (in) EPR, we refer to the many-sorted EPR fragment.

2.3 Transition Systems

A *transition system* is a triple $T = (S, S_0, R)$, where S is a (possibly infinite) set of states called the *state space*, $S_0 \subseteq S$ is the set of *initial states*, and $R \subseteq S \times S$ is the *transition relation*. A *trace* of T is a (finite or infinite) sequence of states $\pi = s_0, s_1, \ldots$, such that $s_0 \in S_0$ and $(s_i, s_{i+1}) \in R$ for every $0 \leq i < |\pi|$. We sometimes denote an infinite trace by $(s_i)_{i=0}^{\infty}$, and a finite trace of n transitions by $(s_i)_{i=0}^{n}$. A state $s \in S$ is *reachable* in T if there exists a finite trace of T, $(s_i)_{i=0}^{n}$, such that $s_n = s$. The *image* of a set $A \subseteq S$, is $R(A) = \{s' \in S \mid \exists s \in A.\, (s, s') \in R\}$.

Safety property A *safety property* is defined by a set $P \subseteq S$ of "good" states.[5] A transition system T *satisfies* the safety property P, denoted $T \models P$, if all the reachable states of T are in P. In this case we say T is *safe with respect to* P.

Inductive invariant Let $T = (S, S_0, R)$ be a transition system and P be a safety property. A set $I \subseteq S$ is an *inductive invariant* for (T, P) if

(i) $S_0 \subseteq I$, that is, all initial states are in I (**initiation**);

(ii) $R(I) \subseteq I$, that is, I is closed under the transition relation (**consecution**); and

(iii) $I \subseteq P$, that is, all states in I are good (**safety**).

A well-known observation is that $T \models P$ if and only if there exists an inductive invariant for (T, P). The if direction follows since I contains all the reachable states of T, by induction on the length of traces. For the only if direction, whenever $T \models P$, the set of all reachable states of T is an inductive invariant for (T, P).

Counterexample to induction (CTI) Let $T = (S, S_0, R)$ be a transition system and P be a safety property, and let $I \subseteq S$ be a set of states. A state $s \in S$ is a *counterexample to induction* (CTI) for I if (i) $s \in S_0$ but $s \notin I$, or (ii) $s \in I$, but there exists $s' \notin I$ such that $(s, s') \in R$, or (iii) $s \in I$ but $s \notin P$. Clearly, a set $I \subseteq S$ is inductive if and only if there is no counterexample to induction for it.

[5] This definition of safety as a set of states specializes the more general definition from [15] of safety as a prefix-closed and limit-closed set of traces. However, the difference is not essential for the purposes of this thesis. In particular, every safety property over traces can be expressed as a safety property over states by augmenting the states with more information.

2.4 Transition Systems in First-Order Logic

We now provide a formalism for specifying transition systems in first-order logic. We call such specifications *first-order transition systems*. We note that this formalism is Turing-complete (as seen later in Section 2.5.4). Furthermore, Section 2.5 presents a toy imperative modeling language whose semantics is given by a transition system specified in first-order logic, and an extended version of this language is supported by the Ivy deductive verification system [171, 186]. Therefore, users can specify transition systems in first-order logic using the familiar concept of an imperative programming language.

2.4.1 Transition Systems

Syntax A first-order logic specification of a transition system (S, S_0, R) is a tuple $T = (\Sigma, \Gamma, \iota, \tau)$, where Σ is a first-order *vocabulary*, Γ is a *background theory* given as a *finite* set[6] of closed formulas over Σ, ι is a closed formula over Σ specifying the initial states, and τ is *two-vocabulary* closed formula specifying the transition relation. That is, τ is a closed formula over $\Sigma \uplus \Sigma'$, where $\Sigma' = \{a' \mid a \in \Sigma\}$.

Semantics A first-order specification $(\Sigma, \Gamma, \iota, \tau)$ defines a class of transition systems, one for each domain $\mathcal{D}$.[7] Let $\mathcal{D}$ be any domain (possibly infinite) for Σ, then the transition system (S, S_0, R) defined by $(\Sigma, \Gamma, \iota, \tau)$ is given by:

$$
\begin{aligned}
S &= \{s = (\mathcal{D}, \mathcal{I}) \mid \mathcal{I} \text{ is an interpretation of } \Sigma \text{ for domain } \mathcal{D} \text{ and } s \models \Gamma\} \\
S_0 &= \{s \in S \mid s \models \iota\} \\
R &= \left\{(s, s') \in S \times S \mid (s, s') \models \tau\right\}
\end{aligned}
$$

Thus, in a transition system specified by $(\Sigma, \Gamma, \iota, \tau)$, the states are first-order structures over vocabulary Σ that satisfy the theory Γ, the initial states are those that satisfy ι, and the transition relation contains the pairs of states that satisfy the two vocabulary formula τ. In the above definition, given $s = (\mathcal{D}, \mathcal{I}) \in S$ and $s' = (\mathcal{D}, \mathcal{I}') \in S$, we use (s, s') as to denote the structure $(\mathcal{D}, \mathcal{I} \uplus \mathcal{I}'')$, where $\mathcal{I}'' = \lambda a' \in \Sigma'.\ \mathcal{I}'(a)$. Namely, the structure defined by (s, s') is a structure over the vocabulary $\Sigma \uplus \Sigma'$ with the same domain as s and s', and where

[6] In this thesis we restrict Γ to be finite. This essentially means we use pure first-order logic, rather than first-order logic modulo a theory, since the conjunction of all formulas in Γ can be expressed as a first-order formula. The definitions naturally extend to the more general case of Γ as a possibly infinite set of formulas, but this thesis considers only finite first-order theories.

[7] While this subtlety is unessential, we note that fixing $\mathcal{D}$ outside of the transition system (S, S_0, R) is required to make S a set of set theory, i.e., without inadvertently defining a set of all sets, as would occur if we allow $\mathcal{D}$ to be any domain over Σ in the definition of S.

the symbols in Σ are interpreted as in s, and the symbols in Σ' are interpreted as in s'.

A *trace* of $(\Sigma, \Gamma, \iota, \tau)$ is a trace of the transition system (S, S_0, R) for some $\mathcal{D}$. As such, a trace is a sequence of first-order structures over Σ. Every state along the trace has its own interpretation of the symbols of Σ, but they all share the same domain $\mathcal{D}$. The *reachable states* of $(\Sigma, \Gamma, \iota, \tau)$ consist of all the states reachable in (S, S_0, R) for some $\mathcal{D}$.

2.4.2 Safety Properties and Inductive Invariants

A closed first-order formula φ_A over Σ can be used to specify a set of states $A \subseteq S$, by the mapping $A = \{s \in S \mid s \models \varphi_A\}$. We use this mapping to specify safety properties and inductive invariants. Thus, safety properties and inductive invariants are represented by closed first-order formulas, over Σ. With overloading of notation, we will denote these formulas by P (for a safety property) and I (for an inductive invariant).

Given a first-order transition system specification $T = (\Sigma, \Gamma, \iota, \tau)$, and a safety property specified by a first-order formula P, we say that T satisfies P, denoted $T \models P$, if all the reachable states of T satisfy P (i.e., for any $\mathcal{D}$). We say that a first-order formula I is an *inductive invariant* for T and P if the following (first-order) entailments hold:

(i) $\Gamma, \iota \models I$, that is, all initial states satisfy I (**initiation**);

(ii) $\Gamma, \Gamma', I, \tau \models I'$, that is, I is closed under the transition relation τ (**consecution**); and

(iii) $\Gamma, I \models P$, that is, I entails P (**safety**).

Note that this coincides with the definition of an inductive invariant from Section 2.3 (requiring it for any $\mathcal{D}$). In the above, we use φ' to denote the formula obtained from φ by substituting every symbol $a \in \Sigma$ by a', its primed version, and we extend this definition to sets of formulas as well.

Verification conditions (VCs) Given a first-order transition system specification $T = (\Sigma, \Gamma, \iota, \tau)$, a safety property specified by a first-order formula P, and a first-order formula I, checking if I is an inductive invariant for T and P can be done by checking that the following formulas are *unsatisfiable*:

(i) $\bigwedge \Gamma \wedge \iota \wedge \neg I$,

(ii) $\bigwedge \Gamma \wedge \bigwedge \Gamma' \wedge I \wedge \tau \wedge \neg I'$, and

(iii) $\bigwedge \Gamma \wedge I \wedge \neg P$.

We refer to these formulas as the *verification conditions* (VCs) associated with T, P, and I. In the above, we relied on the fact that Γ is a finite set of first-order formulas, and used $\bigwedge \Gamma$ to denote the conjunction of all formulas in Γ.

Whenever the verification conditions are in EPR, the problem of checking their satisfiability is decidable, and if they are satisfiable a finite counterexample to induction can be obtained. Note that I appears both positively and negatively in the verification conditions, imposing further restrictions of quantifier alternations used in I for the VCs to be in EPR. An $\exists\forall$ alternation in I results in a $\forall\exists$ alternation in $\neg I'$, which appears in the VCs. Thus, any alternation (both $\forall\exists$ and $\exists\forall$) in I contributes an edge to the quantifier alternation graph of the VCs. Also observe that Γ, ι, and τ appear only positively in the VCs, and P appears only negatively.

2.4.3 Finite vs. Infinite Structures

When defining the semantics of a first-order transition system specification, we allowed $\mathcal{D}$ to be any domain, finite or not. We call this the *first-order semantics*. An alternative definition restricts $\mathcal{D}$ to be any finite domain (but not infinite); let us call this the *finite structure semantics*. The rest of this thesis generally considers the first-order semantics. However, in most parts of the thesis, the distinction between the two semantics is insignificant, and in some cases we use this to change perspective and use the finite structure semantics. In a nutshell, whenever we consider safety properties of transition systems such that the conjunction of Γ, ι, τ, and $\neg P$, is in EPR, then the system is safe under the first-order semantics if and only if it is safe under the finite structure semantics. This is due to the finite model property of EPR, and the fact that safety violations are always finite traces, as we explain below.

Let $T = (\Sigma, \Gamma, \iota, \tau)$ and P be a first-order specification of a transition system and a safety property. For any $k \in \mathbb{N}$, there is an error trace of length k, i.e., a trace of T leading to a state violating P if and only if the following formula is satisfiable:

$$\left(\bigwedge_{i=0}^{k} \wedge\Gamma(\Sigma_i)\right) \wedge \iota(\Sigma_0) \wedge \left(\bigwedge_{i=0}^{k-1} \tau(\Sigma_i, \Sigma_{i+1})\right) \wedge \neg P(\Sigma_k)$$

where $\Sigma_i = \{a_i \mid a \in \Sigma\}$, and $\tau(\Sigma_i, \Sigma_{i+1})$ denotes the formula over vocabulary $\Sigma_i \uplus \Sigma_{i+1}$ obtained from τ when every symbol $a \in \Sigma$ is replaced by a_i and every symbol $a' \in \Sigma'$ is replaced by a_{i+1}. The formulas $\bigwedge \Gamma(\Sigma_i)$, $\iota(\Sigma_0)$, and $P(\Sigma_k)$, are defined similarly. Thus, $(\Sigma, \Gamma, \iota, \tau) \models P$ if and only if the above formula is unsatisfiable for every k.

In this thesis, we mostly consider transition systems where the combination of Γ, ι, τ, and $\neg P$, is in EPR. This means that the above formula is also in EPR for every k. In this case, due to the finite model property of EPR, satisfiability over finite structures and over general structures coincide. Therefore, when considering safety properties of such transition systems, the system is safe under the first-order semantics if and only if it is safe under the finite structure semantics. In this sense, the distinction between the two semantics is insignificant in this case.

Intuitively, in such a case, every transition requires the existence of a finite number of elements, and since the trace is comprised of a finite number of transitions, we only require finitely many elements in the domain. However, this is not the case for liveness properties (treated in Part III, Chapters 7 and 8), where the fact the we use the first-order semantics *is* important. The reason for this is that a counterexample to a liveness property is an infinite trace. Such a trace may require an infinite domain, even when every transition only requires the existence of a finite number of elements.

2.5 RML: Relational Modeling Language

In this section we define a simple imperative modeling language, called *relational modeling language* (RML). RML is used in this thesis to model infinite-state systems, and distributed protocols in particular. The semantics of an RML program (sometimes called an RML model) is given by a first-order transition system as defined in Section 2.4. Thus, RML can be seen as a convenient syntax for specifying transition systems in first-order logic. RML is further designed to make the quantifier structure of the resulting verification conditions apparent from the program source, facilitating verification using EPR. As we shall see, RML is Turing-complete, and remains so even without allowing any quantifier alternations. The Ivy deductive verification system [171, 186] implements an extended version of RML with many additional features (e.g., modules), but maintains its key design principles.

2.5.1 Syntax

Figure 2.2 presents the abstract syntax of RML. An *RML program*, or *RML model*, consists of *declarations* and *actions*, and defines a first-order transition system $T = (\Sigma, \Gamma, \iota, \tau)$. The declarations determine the state space by determining a first-order vocabulary Σ and a theory Γ. The declarations also determine the initial states by determining the formula ι. The transition relation τ is determined by the actions of the program. Each action consists of loop-free code, and a transition of τ corresponds to (non-deterministically) selecting an

$\langle \text{rml} \rangle$	::=	$\langle \text{decls} \rangle$; $\langle \text{actions} \rangle$	
$\langle \text{decls} \rangle$	::=	ϵ | $\langle \text{decls} \rangle ; \langle \text{decls} \rangle$	
	|	**sort** s	
	|	**relation** r : $\bar{\mathsf{s}}$	
	|	**function** f : $\bar{\mathsf{s}} \to \mathsf{s}$	
	|	**individual** v : s	
	|	**axiom** φ	
	|	**init** φ	
$\langle \text{actions} \rangle$	::=	ϵ | $\langle \text{actions} \rangle$; $\langle \text{actions} \rangle$	
	|	**action** A { $\langle \text{cmd} \rangle$ }	
$\langle \text{cmd} \rangle$	::=	**skip**	do nothing
	|	**abort**	terminate abnormally
	|	$r(\overline{x}) := \varphi(\overline{x})$	first-order update of relation r to formula $\varphi(\overline{x})$
	|	$f(\overline{x}) := t(\overline{x})$	first-order update of function f to term $t(\overline{x})$
	|	$v := *$	havoc of individual v
	|	**assume** φ	assume formula φ holds
	|	$\langle \text{cmd} \rangle$; $\langle \text{cmd} \rangle$	sequential composition
	|	$\langle \text{cmd} \rangle$ | $\langle \text{cmd} \rangle$	non-deterministic choice

Figure 2.2: Core syntax of RML. s denotes a sort identifier and $\bar{\mathsf{s}}$ denotes a tuple of sort identifiers separated by commas. r denotes a relation identifier. f denotes a function identifier. v denotes an individual identifier. φ denotes a closed first-order formula. A denotes an action identifier. $\overline{x}$ denotes a tuple of logical variables. $t(\overline{x})$ denotes a term with free logical variables $\overline{x}$ and $\varphi(\overline{x})$ denotes a first-order formula with free logical variables $\overline{x}$.

action and executing its code atomically. Thus, an RML program can be understood as a single loop, where the loop body is a non-deterministic choice between all the actions. The restriction of each action to loop-free code simplifies the presentation, and it does not reduce RML's expressive power, as nested loops can always be converted to a flat loop.

Declarations The declarations of an RML program define:

(i) a set of sorts given by **sort** declarations;

(ii) a set of sorted relations given by **relation** declarations;

(iii) a set of sorted functions given by **function** declarations, and by **individual** declarations that define nullary functions;

(iv) a set of first-order axioms given by **axiom** declarations; and

(v) a set of first-order initial-state formulas that specify the initial states, given by **init** declarations.

An RML program defines a first-order transition system specification $(\Sigma, \Gamma, \iota, \tau)$. The many-sorted first-order vocabulary Σ consists of the declared sorts, relations, and function symbols. The theory Γ consists of the formulas declared as axioms. The initial states formula ι is the conjunction of all initial-state formulas.

Actions The transition relation formula τ is determined by the actions of the RML program. Each action consists of a name and a loop-free body, given by an RML command. The transition relation formula τ of the whole program is given by the disjunction of the transition relation formulas associated with each action of the program. Effectively, this means that each τ-transition is a non-deterministic choice between all the actions of the program, and that each action is executed atomically. Below we given an intuitive description of RML commands, and Section 2.5.2 presents their axiomatic semantics and explains how to translate a command C to its associated transition relation formula $\tau[C]$. Formally, if the bodies of actions are $C_1, , \ldots, , C_k$ then the transition relation of the whole program τ is given by:

$$\tau = \bigvee_i \tau[C_i]$$

Commands Each command investigates and potentially updates the state, i.e., the value of the relation and function symbols. The semantics of **skip** is standard. The semantics of **abort** is also standard, and it can be used to define a safety property as part of the RML program, i.e., the safety property says that the program does not execute an **abort** command. The command $r(x_1, \ldots, x_n) := \varphi(x_1, \ldots, x_n)$ is used to update the n-ary relation r to the set of all n-tuples of elements that satisfy the first-order formula φ. For example, $r(x_1, x_2) := (x_1 = x_2)$ updates the binary relation r to the identity relation; $r(x_1, x_2) := r(x_2, x_1)$ updates r to its inverse relation (or transpose); $r_1(x) := r_2(x, v)$ updates r_1 to the set of all elements that are related by r_2 to the current value (interpretation) of the individual v.

The command $f(x_1, \ldots, x_n) := t(x_1, \ldots, x_n)$ is used to update the n-ary function f to map every n-tuple of elements to the element given by the term t. Note that while relations are updated to first-order formulas, functions are updated to first-order terms. For example, $f(x) := x$ updates the function f to the identity function; $f(x_1, x_2) := f(x_2, x_1)$ updates f to its transpose; $f(x) := \text{ite}(r(x), x, f(x))$ updates f to be the identity function for all elements in r, and leaves it unchanged for all elements not in r.

The *havoc* command $v := *$ performs a non-deterministic assignment to v, where it can be assigned to any value (element) of the appropriate sort. The **assume** command is used to restrict the executions of the program to those that satisfy the given (closed) first-order formula. Sequential composition and non-deterministic choice are defined in the usual way.

Syntactic sugars The commands given in Figure 2.2 are the core of RML. Figure 2.3 provides several useful syntactic sugars for RML, including an **assert** command, an **if-then-else** command, and convenient update commands for relations and functions. Syntactic sugar also allows to declare individuals inside actions, either as action parameters or via a **local** declaration. In both cases, individuals are added to Σ, and use of these individuals is restricted to the scope of a specific action.

In addition to the syntactic sugars of Figure 2.3, when presenting RML code (e.g., Figure 3.1) we allow ourselves to use some shorthands that improve clarity, and are straightforward to translate to the official syntax. (In fact, most of these shorthands are supported by Ivy.) Examples include omission of semicolons, use of infix notation (e.g., $u \leq v$ rather than $\leq (u, v)$), and using $u < v$ as a shorthand for $u \leq v \wedge u \neq v$. We also allow ourselves to omit sorts if they can be easily inferred (e.g., declare v : `s` and later use v).

When presenting RML code, we use capital letters for logical variables that are used in assignment statements. We also allow ourselves to mix variables and ground terms in assignments, by a natural extension of the assignment syntactic sugar pro-

Syntactic Sugar	**Desugared RML**
action A $(v_1 : \mathsf{s}_1, \ldots, v_n : \mathsf{s}_n)\,\{C\}$	**action** A $\{v_1 := *; \ldots; v_n := *; C\}$ Also add declarations for new individuals $v_1 : \mathsf{s}_1, \ldots, v_n : \mathsf{s}_n$ that are local to the scope of A
local $v : \mathsf{s} \;:=\; e$	$v := e$ Also add a declaration for a new individual $v : \mathsf{s}$ that is local to the action scope
assert φ	{**assume** $\neg\varphi$; **abort**} | **skip**
if φ C_1 **if** φ C_1 **else** C_2	{**assume** φ ; C_1} | {**assume** $\neg\varphi$} {**assume** φ ; C_1} | {**assume** $\neg\varphi$; C_2}
$r\,(\overline{g}) := \varphi$ $f\,(\overline{g}) := g$	$r\,(\overline{x}) := (\overline{x} = \overline{g} \wedge \varphi) \vee (\overline{x} \neq \overline{g} \wedge r\,(\overline{x}))$ $f\,(\overline{x}) := \mathrm{ite}\,(\overline{x} = \overline{g}, g, f\,(\overline{x}))$

Figure 2.3: Syntactic sugars for RML. In addition to using the notations of Figure 2.2, g denotes a ground term, $\overline{g}$ denotes a tuple of ground terms, and equality and between tuples denotes the conjunction of the component-wise equalities.

vided by Figure 2.3. For example, we write $r(\mathrm{V}, c) \;:=\; p(\mathrm{V})$ as syntactic sugar for $r(x, y) \;:=\; (y = c \wedge p(x)) \vee (y \neq c \wedge r(x, y))$. This convenience is also supported by Ivy.

2.5.2 Axiomatic Semantics

We now provide a formal semantics for RML commands by defining a weakest precondition operator, which also allows us to define the transition relation formula of an RML command.

Weakest precondition of RML commands We define the *weakest precondition* operator for RML commands with respect to assertions expressed as closed first-order formulas over Σ. In this section, we use assertions and formulas interchangeably, and they are always assumed to be closed. A state satisfies an assertion if it satisfies it in the usual semantics of many-sorted first-order logic.

Figure 2.4 presents the definition of the weakest precondition operator for RML commands. The weakest precondition [66] of a command C with respect to an assertion Q, denoted $wp(C, Q)$, is an assertion Q' such that every execution of C starting from a state that satisfies Q' leads to a state that satisfies Q. Further, $wp(C, Q)$ is the weakest such assertion. Namely, $Q' \models wp(C, Q)$ for every Q' as above. (Note that RML commands are loop-free, so termination of an RML command is trivial.)

$$
\begin{aligned}
wp\left(\mathbf{skip}, Q\right) &= Q \\
wp\left(\mathbf{abort}, Q\right) &= \mathit{false} \\
wp\left(r\left(\overline{x}\right) := \varphi\left(\overline{x}\right), Q\right) &= \left(\bigwedge \Gamma \rightarrow Q\right)\left[\varphi\left(\overline{s}\right) \ / \ r\left(\overline{s}\right)\right] \\
wp\left(f\left(\overline{x}\right) := t\left(\overline{x}\right), Q\right) &= \left(\bigwedge \Gamma \rightarrow Q\right)\left[t\left(\overline{s}\right) \ / \ f\left(\overline{s}\right)\right] \\
wp\left(v := *, Q\right) &= \forall x.\, \left(\bigwedge \Gamma \rightarrow Q\right)\left[x \ / \ v\right] \\
wp\left(\mathbf{assume}\ \varphi, Q\right) &= \varphi \rightarrow Q \\
wp\left(C_1 \ ; \ C_2, Q\right) &= wp\left(C_1, wp\left(C_2, Q\right)\right) \\
wp\left(C_1 \mid C_2, Q\right) &= wp\left(C_1, Q\right) \wedge wp\left(C_2, Q\right)
\end{aligned}
$$

Figure 2.4: Rules for weakest precondition of RML commands. $\varphi\left[\beta \ / \ \alpha\right]$ denotes φ with occurrences of α substituted by β. $\overline{s}$ denotes a tuple of terms. $\bigwedge \Gamma$ denotes the conjunction of all formulas in the theory Γ.

The rules for wp of **skip** and **abort** are standard, as are the rules for **assume**, sequential composition and non-deterministic choice. The rules for updates of relations and functions and for havoc are instances of Hoare's assignment rule [105], applied to the setting of RML and adjusted for the fact that state mutations are restricted by the axioms in Γ.

Transition relation formulas of RML commands The weakest precondition operator is closely related to transition relation formulas. Recall that a transition relation formula has vocabulary $\Sigma \uplus \Sigma'$, where the primed symbols represent the state after executing the command. For further details on efficient ways to encode verification conditions (using weakest precondition operators and transition formulas), see [79, 147]. Here, we use the following connection to define the transition relation of a command C, denoted by $\tau[C]$, using the weakest precondition operator (as defined in Figure 2.4):

$$\tau[C] = \neg wp\left(C, \neg \psi_{\Sigma=\Sigma'}\right)$$

where:

$$\psi_{\Sigma=\Sigma'} = \bigwedge_{r \in \Sigma} \forall \overline{x}.\ r(\overline{x}) \leftrightarrow r'(\overline{x}) \ \wedge \ \bigwedge_{f \in \Sigma} \forall \overline{x}.\ f(\overline{x}) = f'(\overline{x})$$

This makes a slight abuse of the definition of wp, since it applies wp to a formula over $\Sigma \uplus \Sigma'$. However, in this context, the symbols in Σ' can be treated as additional auxiliary symbols, without special meaning.

Intuitively, there is a transition from s to s' if and only if s does not satisfy the weakest precondition of "not being s'". This is captured by the above connection, and "not being s'" is captured by $\neg \psi_{\Sigma=\Sigma'}$. Note further that non-deterministic choice between commands results

in a conjunction in the weakest precondition, and a disjunction in the transition relation. Similarly, a havoc command results in a universal quantifier in the weakest precondition, and an existential quantifier in the transition relation.

Recall that an RML program defines a first-order transition system specification: $(\Sigma, \Gamma, \iota, \tau)$. The transition relation τ given by the disjunction of the transition relations of each action in the RML program, where the transition relation of each action is computed from its body via the above definition. Note that when using the syntactic sugar of Figure 2.3 to define an actions with parameters, these parameters are existentially quantified in the transition relation.

Safety properties The **abort** commands in the RML program also define a safety property P. Intuitively, the safety property is that the program does not execute an **abort** command. Formally, if the bodies of actions are $C_1, , \ldots, , C_k$ then the safety property P is given by:

$$P = \bigwedge_i wp\left(C_i, true\right)$$

2.5.3 Quantifier Alternation Structure

One of the design goals of RML is to facilitate verification using EPR. For this, the user must have visibility and control over the quantifier alternation graph of verification conditions that result from RML programs and inductive invariants. In the VCs (defined in Section 2.4.2), the formulas from Γ, ι, and τ appear positively. In RML, the user has direct control over Γ and ι via declarations. In contrast, τ is computed from the program source. Thus, it is important that the quantifier structure of τ is clearly apparent from the program source.

In RML, the only quantifiers in τ that are not explicit in the program source are those that arise from havoc commands (action parameters included), which lead to existential quantifiers in τ. The close connection between the program source and the quantifier alternation structure of VCs can be observed in Figure 2.4, and is also manifested by the following lemma, showing that under certain conditions $\forall^* \exists^*$-formulas are closed under *wp*. (Recall that τ is given by a negation of *wp*, so $\forall^* \exists^*$ in *wp* corresponds to $\exists^* \forall^*$ in τ.)

Lemma 2.5. *Let C be an RML command, and suppose that all assignments to relations use quantifier-free formulas, and all **assume** commands, as well as the theory Γ, contain formulas with $\exists^* \forall^*$ prenex normal form. Under these conditions, if Q is a $\forall^* \exists^*$-formula, then so is the prenex normal form of $wp(C, Q)$.*

Proof. Straightforward from the rules of Figure 2.4. □

2.5.4 Turing-Completeness

To see that RML is Turing-complete, we can encode a (Minsky) counter machine in RML. The finite state (program counter) can be encoded using nullary relations. Each counter c_i can be encoded with a unary relation r_i. The value of counter c_i is the number of elements in r_i. Testing for zero, incrementing, and decrementing counters can all be easily expressed by RML commands that only use alternation-free formulas in **assume** statements (see below). Therefore, RML, as well as the formalism of first-order transition systems, are Turing-complete, even when restricted not to use any quantifier alternations.

Testing for zero can be done by checking $\forall x.\ \neg r_i(x)$. Incrementing can be done by **local** $v := *$; **assume** $\neg r_i(v)$; $r_i(v) := \mathit{true}$. Decrementing can be done by **local** $v := *$; **assume** $r_i(v)$; $r_i(v) := \mathit{false}$.

Chapter 3

Modeling in First-Order Logic

This chapter is partially based on work published in [186, 187].

This chapter is concerned with the following question: how can we model distributed algorithms as first-order transition systems? Moreover, we seek to do this in a way that facilitates verification using the EPR decidable fragment. Recall that we use uninterpreted, many-sorted, first-order logic. Therefore, modeling a distributed protocol as a first-order transition system involves some abstraction, since protocols usually employ concepts that are not definable in uninterpreted first-order logic. We thus aim for a sound abstraction that is precise enough to allow verification. (As we shall see, in some cases we can exploit the finite model property of EPR to obtain a form of completeness.)

In order to express concepts in first-order logic, we use suitable sorts, relation symbols, and function symbols. The sorts and symbols have an intended meaning, but we use them in uninterpreted first-order logic. To capture *part* of the intended interpretation, we add a finite set of first-order axioms. These are first-order formulas that are valid in the intended interpretation. By adding them to the first-order transition system, we allow the proof of verification conditions to rely on these axioms. By using only axioms that are valid in the intended interpretation, we guarantee that the first-order transition system is a sound abstraction of the actual system or protocol, so any invariant proved for the first-order transition system is also an invariant of the protocol.

In this chapter, we present encodings for transitive closure of deterministic paths (Section 3.3), quorums and cardinality constraints (Section 3.4), and network semantics (Section 3.5). Similar encodings for deterministic paths have been used in the past in the context of heap verification, and we present them in detail for uniformity and to make this thesis self contained. The encodings for cardinalities and network semantics are more straightforward,

and the key novelty is not the encodings themselves, but rather their use as part of an overall scheme for verification of distributed algorithms in decidable logic, as presented in the following chapters of this thesis.

3.1 Motivating Examples

To illustrate both the challenges and our solutions, we present two simple examples of distributed protocols, and their modeling and verification in first-order logic and EPR.

3.1.1 Leader Election in a Ring Protocol

Consider a simple protocol for leader election in a ring [43]. The protocol assumes a unidirectional ring of unbounded size. That is, nodes are organized in a ring, and messages can only be sent from each node to its immediate successor in the ring. The protocol also assumes every node has a unique ID — a natural number known to the node (but initially not to other nodes), such that no two nodes have the same ID. The protocol performs leader election by decentralized extrema-finding. Every node starts by sending its own ID to its successor in the ring. A node forwards messages that contain an ID higher than its own ID. When a node receives a message with its own ID, it concludes that it has the maximal ID, and declares itself as a leader. The key safety property of this protocol is that it elects *at most* one leader.

This example poses several challenges to modeling in first-order logic. First, note that the IDs are natural numbers. It is well-known that no finite (or even recursively enumerable) axiomatization of the natural numbers (including arithmetic and/or uninterpreted relations) in first-order logic can be complete. Second, the protocol assumes a finite ring topology. This notion is also not first-order definable. Also, as we shall see, the inductive invariant of the protocol requires concepts such as paths in the ring — transitive closure of ring edges — and transitive closure is also not first-order definable. However, using the techniques we develop in this chapter, we can express this protocol, and its safety proof, in first-order logic.

Figure 3.1 presents an RML model of the leader election protocol. We use a sort node to represent nodes in the ring network, and a sort id to represent node IDs. The function *id* maps every node to its ID. We define a binary relation $\leq$ on IDs, with axioms for $\leq$ being a total order, as explained in Section 3.2. That is, we abstract the real protocol, which uses natural numbers, to a protocol that uses an arbitrary totally ordered set. The ring topology is represented by a ternary *btw* relation, as explained in Section 3.3.3. The idea is that $btw(x, y, z)$ holds if y is between x and z in the ring, that is, if y is part of the shortest

```
1  sort node
2  sort id
3
4  function id : node → id
5  relation ≤ : id, id
6  relation btw : node, node, node
7  relation leader : node
8  relation msg : id, node
9
10 axiom Γ_total order[≤]
11 axiom Γ_ring[btw]
12 axiom ∀n1 : node, n2 : node. n1 ≠ n2 → id(n1) ≠ id(n2)
13
14 init ∀x : node. ¬leader(x)
15 init ∀x : node, y : id. ¬msg(x, y)
16
17 action SEND(n : node, m : node) {
18   assume φ_s(n, m)
19   msg(id(n), m) := true
20 }
21 action RECEIVE(n : node, i : id) {
22   assume msg(i, n)
23   msg(i, n) := *
24   if id(n) = i {
25     leader(n) := true
26   } else {
27     if id(n) ≤ i then {
28       local m := *
29       assume φ_s(n, m)
30       msg(i, m) := true
31     }
32   }
33 }
```

Figure 3.1: RML model of the leader election in a ring protocol. The theory $\Gamma_{\text{total order}}$ axiomatizes a total order relation, and is explained in Section 3.2. The theory Γ_{ring} axiomatizes a ring graph, and is explained in Section 3.3.3. The formula $\varphi_s(x, y)$ is used to check if y is the successor of x in the ring according to the *btw* relation, and it is also explained in Section 3.3.3.

(i.e., acyclic) path from x to z.

The state of the protocol is represented by the *leader* relation, indicating which nodes consider themselves leaders; and the *msg* relation, modeling the network by storing which messages are in transit. As explained in Section 3.5, this actually models a network with message dropping, reordering, and duplication; and this protocol is indeed correct under such network semantics. The model contains two actions: SEND that models a node sending its ID to its successor in the ring, and RECEIVE that models a node receiving a message.

The safety property of the protocol is expressed by the following first-order formula:

$$\forall x : \mathsf{node},\ y : \mathsf{node}.\ leader(x) \wedge leader(y) \rightarrow x = y \tag{3.2}$$

Most importantly, the safety of the protocol can be proven with the following inductive invariant expressed in first-order logic:

$$\forall x : \mathsf{node},\ y : \mathsf{node}.\ leader(x) \rightarrow id(y) \leq id(x) \tag{3.3}$$

$$\forall x : \mathsf{node},\ y : \mathsf{node}.\ msg(id(x), x) \rightarrow id(y) \leq id(x) \tag{3.4}$$

$$\forall x : \mathsf{node},\ y : \mathsf{node},\ z : \mathsf{node}.\ \ btw(x, y, z) \wedge msg(id(x), z) \rightarrow id(y) \leq id(x) \tag{3.5}$$

Equation (3.3) states that a leader has a maximal ID, and Equation (3.4) similarly states that a node whose ID is self-pending has a maximal ID. Equation (3.5) involves three nodes, and it states that a message cannot bypass a node with a higher ID. Note that the invariant is universally quantified, and the VCs for checking it are in EPR (the only quantifier alternation edge results from the *id* function, from node to id). Indeed, this example takes under two seconds to verify with Ivy and Z3 (on a standard laptop).

The fact that the protocol's inductive invariant is expressible in first-order logic depends on the fact that we use *btw* to represent the ring topology. Note that Equation (3.5) states that if the ID of node x is pending at node z, then the ID of x is greater than that of every node between x and z. It is crucial to express this fact, regardless of the length of the path between x and z. Using an alternative representation of the ring by a binary relation that represents graph edges, this fact is not first-order expressible. It can be expressed, e.g., using transitive closure. Thus, the use of the *btw* relation, with its definitions axioms Γ_{ring}, is key to verifying this protocol in first-order logic.

The details of Γ_{ring} and the *btw* relation, as well as similar representations for other classes of graphs, appear in Section 3.3.

```
1  sort node
2  sort value
3  sort quorum
4
5  relation member : node, quorum
6  relation vote_msg : node, value
7  relation decision : value
8
9  axiom ∀q1, q2 : quorum. ∃n : node. member(n, q1) ∧ member(n, q2)
10
11 init ∀x : node, y : value. ¬vote_msg(x, y)
12 init ∀x : value. ¬decision(x)
13
14 action VOTE(n : node, v : value) {
15   assume ∀x : value. ¬vote_msg(n, x)
16   vote_msg(n, v) := true
17 }
18
19 action DECIDE(v : value) {
20   assume ∃q : quorum. ∀n : node. member(n, q) → vote_msg(n, v)
21   decision(v) := true
22 }
```

Figure 3.6: RML model of the majority vote toy protocol. For this protocol, we wish to prove *decision* never contains two different values. We use a sort for quorums with a suitable axiom stating that any two quorums intersect, as explained in Section 3.4.

3.1.2 Majority Vote Protocol

Consider a set of N nodes that can send messages to each other (all to all communication). They can decide on a unique value, by having each node vote for a single value, and requiring a majority for a decision. That is, a value is considered decided if more than $N/2$ nodes voted for it. This protocol does not guarantee that some value will be decided (the votes may split in such a way that no value obtains a majority), but it is safe, in the sense that *at most* one value is decided.

This toy protocol illustrates an essential part of the safety argument of many consensus protocols, such as Paxos [135, 136]. The argument is that since each node only votes for a single value, and since any two sets of more than $N/2$ nodes intersect, then at most one value can obtain a majority of the votes. While straightforward, this argument uses concepts that are not directly expressible in first-order logic. It uses sets defined by predicates (the nodes that voted for a value), and relies on the cardinalities of these sets. However, using the techniques we develop in this chapter, this protocol, and other protocol relying on similar arguments (including Paxos), can be modeled and verified in first-order logic.

Figure 3.6 presents an RML model of the majority vote protocol. It uses sorts for nodes and values (node and value), and also a sort quorum for *quorums*, i.e., majority sets. This technique is explained in detail in Section 3.4. In a nutshell, we use a relation

member : node, quorum to represent set membership, and to use an axiom to state that any two quorums intersect (line 9). Then, the condition of having at least $N/2$ votes for a single value can be expressed in first-order logic (line 20). This involves an abstraction, since other than the axiom of line 9, the quorum sort and the *member* relation are uninterpreted. This is nevertheless sound, and also precise enough to prove the safety of the majority vote protocol. The safety property of the protocol, i.e., that at most one value is decided, can be expressed by the following first-order formula:

$$\forall v_1 : \mathsf{value},\ v_2 : \mathsf{value}.\ \mathit{decision}(v_1) \wedge \mathit{decision}(v_2) \rightarrow v_1 = v_2 \tag{3.7}$$

The safety of the majority vote protocol can be proven by the following inductive invariant expressed in first-order logic:

$$\forall n : \mathsf{node},\ v_1 : \mathsf{value},\ v_2 : \mathsf{value}.\ \mathit{vote\_msg}(n, v_1) \wedge \mathit{vote\_msg}(n, v_2) \rightarrow v_1 = v_2 \tag{3.8}$$

$$\forall v : \mathsf{value}.\ \mathit{decision}(v) \rightarrow \exists q : \mathsf{quorum}.\ \forall n : \mathsf{node}.\ \mathit{member}(n, q) \rightarrow \mathit{vote\_msg}(n, v) \tag{3.9}$$

Equation (3.8) states that every node votes for at most one value, and Equation (3.9) states that every decided value must be due to a majority vote, i.e., there is a quorum such that all member of the quorum voted for the decided value. This invariant contains some quantifier alternations, but the resulting VCs are in EPR. The axiom of line 9 introduces a quantifier alternation edge from quorum to node, and Equation (3.9) introduces an edge from value to quorum, and another edge from quorum to node, arising from the negation of the invariant in the VCs (see Section 2.4). When combined, these edges do not create a cycle, so the VCs are in EPR. Indeed, this example also takes under two seconds to verify with Ivy and Z3 (on a standard laptop).

The rest of this chapter explores the issues illustrated by these two toy protocol examples in more detail, and provides techniques for modeling systems and protocols in first-order logic. Section 3.2 begins with a short discussion of expressing total orders in first-order logic, which provides a basis for the technique presented in Section 3.3 for representing paths in several classes of graphs with outdegree one, in a way that is both sound and complete for finite models. Section 3.4 discussed the issue of expressing higher-order quantification and cardinality thresholds, and Section 3.5 discusses the modeling of various network semantics in first-order logic. We note that while this chapter illustrates the issues using toy examples, the techniques we present also apply to complex protocol such as Paxos and its variant, and have led to the first mechanized verification of several protocols, as we shall see in Chapter 4

Transitivity	$\forall x, y, z.\ x \leq y \wedge y \leq z \rightarrow x \leq z$
Antisymmetry	$\forall x, y.\ x \leq y \wedge y \leq x \rightarrow x = y$
Totality	$\forall x, y.\ x \leq y \vee y \leq x$
Successor	$\varphi_s(x, y) \equiv x \leq y \wedge x \neq y \wedge \forall z.\ x \leq z \wedge x \neq z \rightarrow y \leq z$

Figure 3.10: Encoding of a total order in first-order logic. Axioms stating the relation $\leq$ is a total order; and a formula for defining the successor relation from $\leq$.

(for safety proofs), and later in Chapter 7 (for liveness proofs).

3.2 Total Orders

One important example of axioms expressible in first-order logic is the axiomatization of total orders. In many cases, protocols use natural numbers or integers. A common use is to enforce a total order on a set of elements. The leader election protocol of Section 3.1.1, which uses IDs that are natural numbers, is an example of this. Such use of integers or natural numbers can be captured in first-order logic by adding a binary relation $\leq$, along with the axioms listed in Figure 3.10, which precisely capture the properties of a total (linear) order.

These axioms are sound for a finite totally ordered set (e.g., the set of node IDs in the leader election in a ring protocol of Figure 3.1), and also for infinite totally ordered sets, such as the natural numbers or the integers. Moreover, the successor relation is expressible as a first-order formula φ_s over the order relation $\leq$. This formula, also listed in Figure 3.10, contains one universal quantifier.

The axioms of Figure 3.10 are universally quantified and do not contain any quantifier alternations. However, the fact that the successor formula contains a universal quantifier may lead to quantifier alternations that create a cycle in the quantifier alternation graph. For example, one may wish to state the following property in an inductive invariant: $\forall x, y.\ \varphi_s(x, y) \rightarrow p(x, y)$. However, this formula's prenex normal form has quantifier prefix $\forall^* \exists^*$, and it is outside of the EPR fragment. In contrast, the formula $\forall x, y.\ p(x, y) \rightarrow \varphi_s(x, y)$, or the formula $\varphi_s(a, b) \rightarrow p(a, b)$ where a and b are constant symbols, do not create any quantifier alternations. Thus, the restrictions of EPR allow some use of the successor formula, but not general use.

When the total order axioms and successor formula are used for verification such that the resulting VCs are in EPR, the obtained counterexamples will always be finite (due to EPR's finite model property). This means that even if the total order is used as an abstraction

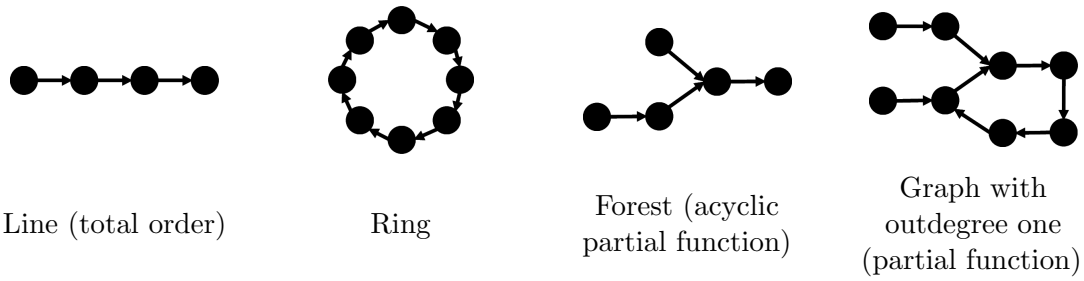


Figure 3.11: Illustration of classes of graphs with outdegree one. Paths in these graphs can be captured in the EPR decidable fragment of first-order logic.

for the integers or natural numbers, in the obtained counterexamples its domain will be a finite totally ordered set. Note that for such models, the $\leq$ relation is always precisely the transitive closure of the successor relation defined by the φ_s formula. However, this is not the case for infinite models. For example, consider the set $\{0,1\} \times \mathbb{N}$ ordered lexicographically. It satisfies the total order axioms, but $\leq$ is not equivalent to the transitive closure of the successor relation. This observation inspires the next section, where we present several adaptations of the total order axioms and successor formula to represent paths in graphs with outdegree one.

Another observation worth noting is that it is not possible to keep both the successor relation and the order relation in the vocabulary and have a complete axiomatization of the connection between them. That is, if one takes the vocabulary $\Sigma = \{s^2, \leq^2\}$, then there is no first-order theory Γ that is satisfied by a structure $(\mathcal{D}, \mathcal{I})$ if and only if $\mathcal{I}(\leq)$ is a total order and is also the reflexive transitive closure of $\mathcal{I}(s)$. This can easily be shown by compactness of first-order logic (e.g., [203]). Interestingly, the following formula (when taken together with the total order axioms of Figure 3.10) does provide a complete characterization of transitive closure for *finite* structures (i.e., as above but restrict $\mathcal{D}$ to be finite) [152]: $\forall x, y.\, x \leq y \leftrightarrow (x = y \vee \exists z.\, s(x,z) \wedge z \leq y)$. However, since this formula contains $\forall\exists$ quantifier alternation, it is outside of EPR, and does not have a finite model property. In the next section, we will maintain both completeness for finite structures and remain in EPR, by using a vocabulary that does not contain the successor (or edge) relation s, and using φ_s in lieu of it as needed and under the restrictions of EPR.

3.3 Deterministic Paths

One of the main hurdles to using first-order logic in verification is the fact that it cannot express transitive closure, which is required to express properties and invariant that involve

paths in graphs. In this section we present encodings for four classes of graphs, all with outdegree one, that allow to express graph paths in first-order logic. We say that a graph has outdegree one if the outdegree of every vertex is at most one (i.e., each vertex has at most one outgoing edge). The classes we consider are depicted in Figure 3.11. The encodings in first-order logic are inspired by the total order axioms and the successor formula discussed in the previous section. These encodings exploit ideas that have been used in the past in the context of verification of heap manipulating programs [111–113, 133, 194, 195, 223, 224]. However, for the purpose of uniformity, and to make this thesis self contained, we present the encodings in detail.

For each class of graphs, we present axioms of a *path relation*, a *successor formula* that allows to express graph edges from the path relation, update formulas for manipulating the graph, and soundness and completeness theorems. The completeness theorems state that any *finite* model of the axioms is isomorphic to a graph of the corresponding class. Together with the finite model property of EPR, this leads to a complete reasoning procedure for deductive verification of programs manipulating such graphs. That is, every obtained counterexample corresponds to an actual graph, and the encoding of paths in first-order logic does not lead to spurious counterexamples.

3.3.1 Line

The first class of graphs we consider is *line* graphs. That is, a directed graph where every node has both indegree one and outdegree one, except for two extreme nodes: the minimal node has indegree zero, and the maximal node has outdegree zero. Since we are interested in transition systems manipulating this graph, we do not assume that all elements of the domain are part of the graph structure. This allows us to also express addition and removal of nodes from the graph. Example uses of this encoding are to represent a queue or a stack data-structure, in first-order logic. Thus, pushing and popping elements corresponds to adding and removing elements from the line graph.

The idea for the encoding is to represent the line graph with a transitive relation $\leq$ that is total and reflexive only on the elements that are part of the line. For elements that are part of the line, $\leq$ represents the reflexive transitive closure of graph edges. Other elements in the domain that are not part of the line are left outside the $\leq$ relation. To add or remove elements from the line, the $\leq$ relation is updated. Figure 3.12 lists the axioms for the $\leq$ relation (which is the path relation in this case), a successor formula φ_s that derives graph edges from the $\leq$ relation, several useful first-order queries, and update code for adding and

Transitivity	$\forall x, y, z.\ x \leq y \wedge y \leq z \rightarrow x \leq z$
Antisymmetry	$\forall x, y.\ x \leq y \wedge y \leq x \rightarrow x = y$
Partial totality	$\forall x, y.\ x \leq x \wedge y \leq y \rightarrow x \leq y \vee y \leq x$
Partial reflexivity	$\forall x, y.\ x \leq y \rightarrow x \leq x \wedge y \leq y$
Successor	$\varphi_s(x, y) \equiv x \leq y \wedge x \neq y \wedge \forall z.\ x \leq z \wedge x \neq z \rightarrow y \leq z$
x is in the line	$x \leq x$
x is minimal	$x \leq x \wedge \forall y.\ y \leq y \rightarrow x \leq y$
x is maximal	$x \leq x \wedge \forall y.\ y \leq y \rightarrow y \leq x$
Remove v	$x \leq y := x \leq y \wedge x \neq v \wedge y \neq v$
Add v as minimum	**assert** $v \not\leq v$; $x \leq y := x \leq y \vee (x = v \wedge y = v) \vee (x = v \wedge y \leq y)$
Add v as maximum	**assert** $v \not\leq v$; $x \leq y := x \leq y \vee (x = v \wedge y = v) \vee (x \leq x \wedge y = v)$
Add v after u	**assert** $v \not\leq v \wedge u \leq u$; $x \leq y := x \leq y \vee (x = v \wedge y = v) \vee$ $(x \leq u \wedge y = v) \vee (x = v \wedge u \leq y \wedge u \neq y)$

Figure 3.12: Encoding of a line graph in first-order logic. Axioms stating that $\leq$ is a total order on a subset of its domain; a formula for defining the successor relation from $\leq$; first-order queries to test if an element is in the line, and if an element is maximal or minimal; and RML update code for removing and adding an element at various positions. Note that adding an element requires that it is not already in the line.

removing nodes from the line.

The soundness and completeness of this encoding are given by the following theorems.

Theorem 3.13 (Line Soundness)**.** *For any line graph $G = (V, E)$, and any additional set of nodes U disjoint from V, let $s = (\mathcal{D}, \mathcal{I})$ be the first-order structure given by $\mathcal{D} = V \uplus U$, $\mathcal{I}(\leq) = E^* \subseteq V \times V$, where E^* is the reflexive transitive closure of E (when considered over the carrier set V). Then, s satisfies the axioms of Figure 3.12, and in addition we have the following (items i and ii are by construction, and stated only for uniformity with the next theorem):*

(i) $\mathcal{D} = V \uplus U$;

(ii) for any two elements $u, v \in \mathcal{D}$, $s \models u \leq v$ *if and only if* $(u, v) \in E^*$;

(iii) for any two elements $u, v \in \mathcal{D}$, $s \models \varphi_s(u, v)$ *(where* φ_s *is taken from Figure 3.12) if and only if* $(u, v) \in E$; *and*

(iv) for any element $v \in \mathcal{D}$, $s \models v \leq v$ *if and only if* $v \in V$.

Moreover, let $G' = (V', E')$ and U' be obtained from G, U by one of the updates of Figure 3.12, and let s' be the structure defined from G', U' as s is defined from G, U, then $s, s' \models \tau$, where τ is the transition relation of the update of Figure 3.12.

Theorem 3.14 (Line Completeness)**.** *For any finite structure $s = (\mathcal{D}, \mathcal{I})$ that satisfies the axioms of Figure 3.12, there is a line graph $G = (V, E)$ and additional set of nodes U, such that the following holds:*

(i) $\mathcal{D} = V \uplus U$;

(ii) for any two elements $u, v \in \mathcal{D}$, $s \models u \leq v$ *if and only if* $(u, v) \in E^*$; *and*

(iii) for any two elements $u, v \in \mathcal{D}$, $s \models \varphi_s(u, v)$ *(where* φ_s *is taken from Figure 3.12) if and only if* $(u, v) \in E$; *and*

(iv) for any element $v \in \mathcal{D}$, $s \models v \leq v$ *if and only if* $v \in V$.

Moreover, let s and s' be two structures with the same domain that satisfy the axioms of Figure 3.12 such that $s, s' \models \tau$, where τ is the transition relation of one of the updates of Figure 3.12. Then, the line graph corresponding to s' is obtained from the line graph corresponding to s by the update from Figure 3.12 corresponding to τ.

Reflexivity	$\forall x.\ x \preceq x$
Transitivity	$\forall x, y, z.\ x \preceq y \wedge y \preceq z \rightarrow x \preceq z$
Antisymmetry	$\forall x, y.\ x \preceq y \wedge y \preceq x \rightarrow x = y$
Partial totality	$\forall x, y, z.\ x \preceq y \wedge x \preceq z \rightarrow y \preceq z \vee z \preceq y$
Successor	$\varphi_s(x, y) \equiv x \preceq y \wedge x \neq y \wedge \forall z.\ x \preceq z \wedge x \neq z \rightarrow y \preceq z$
x has no outgoing edge	$\forall y.\ x \preceq y \rightarrow x = y$
Remove edge outgoing from u	$x \preceq y := x \preceq y \wedge (x \preceq u \rightarrow y \preceq u)$
Add edge from u to v	**assert** $v \not\preceq u \wedge \forall x.\ u \preceq x \rightarrow u = x$; $x \preceq y := x \preceq y \vee (x \preceq u \wedge v \preceq y)$

Figure 3.15: Encoding of a forest (i.e., acyclic graph with outdegree one) in first-order logic. Axioms for $\preceq$, representing graph reachability (the reflexive transitive closure of graph edges); a formula for defining the successor relation (graph edges) from $\preceq$; a first-order query to test if an element has no outgoing edge (i.e., it is a root); and RML update code for removing and adding edges. Note that adding an edge from u to v requires that u has no outgoing edge, and that $v \not\preceq u$, i.e., that the new edge does not create a cycle.

Proof. Given $s = (\mathcal{D}, \mathcal{I})$, define $V = \{v \in \mathcal{D} \mid (v, v) \in \mathcal{I}(\leq)\}$, $U = \mathcal{D} \setminus V$, and $E = \{(u, v) \in \mathcal{D} \times \mathcal{D} \mid s \models \varphi_s(u, v)\}$, where φ_s is taken from Figure 3.12. Observe that the axioms force $\leq$ to be a total order on V, and since it is a finite total order, the order relation is the transitive closure of the successor it defines. From this, the rest of the theorem clearly follows. □

3.3.2 Forest: Acyclic Partial Function

The next class of graphs we encode in first-order logic is forests, i.e., directed graphs with outdegree one that are also *acyclic*. Another way to view these graphs is that they represent an acyclic partial function from a finite domain to itself. The encoding of this class in first-order logic, and specifically in EPR, has been developed in [111, 112] in the context of verification of linked data structures. We present it here in a general context, and in a way that is uniform with the other classes of graphs we handle.

The idea for the encoding is to represent the reflexive transitive closure of the acyclic partial function (i.e., graph edges) with a binary relation $\preceq$, which serves as the path relation. (In [111, 112] this binary relation is denoted n^*.) To add or remove edges, the $\preceq$ relation is updated. Figure 3.15 lists the axioms for the $\preceq$ relation, a successor formula φ_s that derives graph edges from the $\preceq$ relation, and update code for adding and removing edges. The code for adding an edge assumes that no edge is currently outgoing from the source vertex. Thus, to update the function, we first remove the existing edge, and then add a new edge.

The code further requires that the new edge does not create a cycle, i.e., that the source of the new edge is not reachable from its target, and this condition is expressible using the $\preceq$ relation.

The soundness and completeness of this encoding are given by the following theorems.

Theorem 3.16 (Forest Soundness)**.** *For any directed forest $G = (V, E)$, i.e., an acyclic directed graph where the outdegree of any vertex is at most one, let $s = (\mathcal{D}, \mathcal{I})$ be the first-order structure given by $\mathcal{D} = V$, $\mathcal{I}(\preceq) = \{(u, v) \in V \times V \mid (u, v) \in E^*\}$, where E^* is the reflexive transitive closure of E. Then, s satisfies the axioms of Figure 3.15, and in addition we have the following:*

(i) for any two elements $u, v \in \mathcal{D}$, $s \models u \preceq v$ if and only if $(u, v) \in E^*$; and

(ii) for any two elements $u, v \in \mathcal{D}$, $s \models \varphi_s(u, v)$ (where φ_s is taken from Figure 3.15) if and only if $(u, v) \in E$.

Moreover, let $G' = (V, E')$ be a directed forest obtained from G by addition or removal of an edge, and let s' be the structure defined from G' as s is defined from G, then $s, s' \models \tau$, where τ is the transition relation of the suitable update code in Figure 3.15.

Theorem 3.17 (Forest Completeness)**.** *For any finite structure $s = (\mathcal{D}, \mathcal{I})$ that satisfies the axioms of Figure 3.15, there is a directed forest $G = (\mathcal{D}, E)$ such that the following holds:*

(i) for any two elements $u, v \in \mathcal{D}$, $s \models u \preceq v$ if and only if $(u, v) \in E^*$; and

(ii) for any two elements $u, v \in \mathcal{D}$, $s \models \varphi_s(u, v)$ (where φ_s is taken from Figure 3.15) if and only if $(u, v) \in E$.

Moreover, let s and s' be two structures with the same domain that satisfy the axioms of Figure 3.15 such that $s, s' \models \tau$, where τ is the transition relation of the edge addition or removal code listed in Figure 3.15. Then, the graph corresponding to s' is obtained from the graph corresponding to s by a suitable edge addition or removal corresponding to τ.

Proof. Given $s = (\mathcal{D}, \mathcal{I})$, define $E = \{(u, v) \in \mathcal{D} \times \mathcal{D} \mid s \models \varphi_s(u, v)\}$, where φ_s is taken from Figure 3.15. With this definition, and since $\mathcal{D}$ is finite, the theorem easily follows. For further details see [111]. □

3.3.3 Ring

The next class of graphs we consider is *rings*. That is, a directed connected graph where every vertex has both indegree and outdegree exactly one. Any such ring is isomorphic to $\mathbb{Z}_n$,

Cyclicity	$\forall x, y, z.\ btw(x, y, z) \rightarrow btw(y, z, x)$
Transitivity	$\forall w, x, y, z.\ btw(w, x, y) \wedge btw(w, y, z) \rightarrow btw(w, x, z)$
Antisymmetry & Irreflexivity	$\forall x, y, z.\ btw(x, y, z) \rightarrow \neg btw(x, z, y)$
Totality	$\forall x, y, z.\ x \neq y \wedge x \neq z \wedge y \neq z \rightarrow btw(x, y, z) \vee btw(x, z, y)$
Successor	$\varphi_s(x, y) = \forall z. z \neq x \wedge z \neq y \rightarrow btw(x, y, z)$

Figure 3.18: Encoding of a ring in first-order logic. Axioms stating that *btw* represents a ring comprising the entire domain; and a formula for defining the successor relation (ring edges) from *btw*. The query $btw(x, y, z)$ represents the fact that x, y, and z are distinct elements, and that y is *between* x and z in the ring. That is, y is part of the shortest (i.e., acyclic) path from x to z.

i.e., $\{0, \ldots, n-1\}$ with arithmetic modulo n. For $\mathbb{Z}_n$, the ring edges (i.e., successor relation) are defined by $E = \left\{(x, y) \in {\mathbb{Z}_n}^2 \mid y = x + 1 \mod n\right\}$. In this topology, any element is reachable from any element via the successor relation. That is, the transitive closure of the successor relation is trivial (a clique). In this setting, the non-trivial facts about paths in the ring involve three elements. For distinct elements x, y, z, it may be the case that y is *between* x and z in the ring, or that it is not (in which case z is between x and y). Formally, y is between x and z if y is part of the shortest (i.e., acyclic) path from x to z. We define the ternary *btw* relation to capture this fact. For $\mathbb{Z}_n$, *btw* is defined by: $btw(x, y, z) \equiv x < y < z \vee z < x < y \vee y < z < x$.

We shall now see that a ring can be modeled in way that is similar to the modeling of a line (Section 3.3.1) and a forest (Section 3.3.2), by using the ternary *btw* relation as the path relation. That is, the *btw* relation admits a first-order axiomatization via universal formulas, the successor relation is definable from *btw* using a single universal quantifier, and the axiomatization is complete for finite structures. Figure 3.18 lists the axiomatization of *btw* and the definition of the successor relation (graph edges) from *btw*. The axioms comprise Γ_{ring} used in the leader election protocol of Figure 3.1. Recall that for that example, this encoding allowed us to verify the protocol using an inductive invariant expressible in first-order logic, i.e., without using a logic that supports transitive closure.

The soundness and completeness of the encoding are given by the following theorems.

Theorem 3.19 (Ring Soundness)**.** *For any $n \in \mathbb{N}$, let $s = (\mathcal{D}, \mathcal{I})$ be the first-order structure given by $\mathcal{D} = \mathbb{Z}_n$ and $\mathcal{I}(btw) = \{(x, y, z) \in {\mathbb{Z}_n}^3 \mid x < y < z \vee z < x < y \vee y < z < x\}$. Then, s satisfies the axioms of Figure 3.18, and in addition we have the following:*

(i) for any three elements $u, v, w \in \mathcal{D}$, $s \models btw(u, v, w)$ if and only if

$u < v < w \vee w < u < v \vee v < w < u$; *and*

(ii) for any two elements $u, v \in \mathcal{D}$, $s \models \varphi_s(u, v)$ *(where* φ_s *is taken from Figure 3.18) if and only if* $v = u + 1 \mod n$.

Theorem 3.20 (Ring Completeness)**.** *Any finite structure* $s = (\mathcal{D}, \mathcal{I})$ *that satisfies the axioms of Figure 3.18 is isomorphic to* $\mathbb{Z}_{|\mathcal{D}|}$. *That is, there exists a bijection* $f : \mathcal{D} \to \mathbb{Z}_{|\mathcal{D}|}$ *such that:*

(i) for any three elements $u, v, w \in \mathcal{D}$, $s \models btw(u, v, w)$ *if and only if* $f(u) < f(v) < f(w) \vee f(w) < f(u) < f(v) \vee f(v) < f(w) < f(u)$; *and*

(ii) for any two elements $u, v \in \mathcal{D}$, $s \models \varphi_s(u, v)$ *(where* φ_s *is taken from Figure 3.18) if and only if* $f(v) = f(u) + 1 \mod |\mathcal{D}|$.

Proof. Let a denote an arbitrary element in $\mathcal{D}$. Define an order relation $\sqsubseteq$ on $\mathcal{D}$ by $\sqsubseteq = \{(x, y) \in \mathcal{D}^2 \mid x = y \vee x = a \vee btw(a, x, y)\}$. The axioms of Figure 3.18 imply that this relation is a total order. Therefore, $(\mathcal{D}, \sqsubseteq)$ is isomorphic to $(\mathbb{Z}_{|\mathcal{D}|}, \leq)$. Let f be the isomorphism, then (i) and (ii) hold for this f. □

Note that for a ring, we do not present update formulas. In this thesis, we use the ring encoding to model a static ring network topology. For an evolving ring, it may be the case that the graph will not always maintain a ring form (e.g., edge removal must break the ring). In such cases, the general encoding presented in the next section can be used.

3.3.4 General Partial Function

We now generalize the ideas of the previous sections to encode paths of a general graph where the outdegree of every vertex is at most one (i.e., the graph may contain cycles). Another way to view such a graph is as a partial function from V to V, which may be cyclic. As before, the idea is to use a path relation, here a ternary relation p, with universally quantified axioms, from which the graph edges can be recovered using a first-order query with a single universal quantifier. Moreover, updates to the graph such as addition and removal of edges can be expressed as updates to p.

Unlike in the acyclic case of Section 3.3.2, when the graph may be cyclic, reachability (i.e., transitive closure of edges) is not enough to recover the graph. Instead, we combine the ideas of Sections 3.3.1 to 3.3.3 and define a ternary relation p, such that for every w, the binary relation $p(w, \cdot, \cdot)$ satisfies the axioms of a line graph from Section 3.3.1. Intuitively, the elements of the line are the elements reachable from w, and they are ordered by the

Transitivity	$\forall w, x, y, z.\ p(w, x, y) \wedge p(w, y, z) \rightarrow p(w, x, z)$
Antisymmetry	$\forall w, x, y.\ p(w, x, y) \wedge p(w, y, x) \rightarrow x = y$
Partial totality	$\forall w, x, y.\ p(w, x, x) \wedge p(w, y, y) \rightarrow p(w, x, y) \vee p(w, y, x)$
Partial reflexivity	$\forall w, x, y.\ p(w, x, y) \rightarrow p(w, x, x) \wedge p(w, y, y)$
Cycle maximality	$\forall x, y.\ p(x, x, y) \rightarrow x = y$
Transitivity of reachability	$\forall x, y, z.\ p(x, y, y) \wedge p(y, z, z) \rightarrow p(x, z, z)$
Path consistency	$\forall w, x, y, z.\ p(x, y, z) \wedge p(x, z, w) \wedge y \neq z \rightarrow p(y, z, w)$
Successor	$\varphi_s(x, y) = p(x, y, y) \wedge \forall z.\ p(x, z, z) \rightarrow p(x, y, z)$
y reachable from x	$p(x, y, y)$
x is on a cycle	$p(x, x, x)$
y is between x and z	$x \neq y \wedge x \neq z \wedge y \neq z \wedge p(x, y, z)$
x has no outgoing edge	$\forall y. \neg p(x, y, y)$
Remove edge outgoing from u	$p(x, y, z) := p(x, y, z) \wedge x \neq u \wedge (p(x, u, u) \rightarrow p(x, z, u))$
Add edge from u to v	**assert** $\forall x. \neg p(u, x, x)$; $p(x, y, z) := p(x, y, z) \vee ((p(x, u, u) \vee x = u) \wedge \neg p(x, z, z) \wedge$ $(((p(x, y, u) \vee y = v) \wedge (p(v, z, z) \vee z = v)) \vee$ $(p(v, y, z) \wedge z \neq v))$

Figure 3.21: Encoding of a general (i.e., possibly cyclic) graph with outdegree one in first-order logic. Axioms for a ternary path relation p that captures graph reachability as well as the structure of cycles; a formula for defining the successor relation (graph edges) from p; first-order queries to test graph reachability, cycles, order of elements on cycles (i.e., betweenness), and if an element has no outgoing edge (i.e., it is a root); and RML update code for removing and adding edges. Note that adding an edge from u to v requires that u has no outgoing edge. For every element w, the binary relation $p(w, \cdot, \cdot)$ is a reflexive total order on the elements reachable from w, ordered by their shortest (i.e., acyclic) distance from w. The first (i.e, minimal) element in the line is w's successor, and if w is on a cycle, w itself will be the last (i.e., maximal) element on the line.

(shortest) distance from w. The first (i.e, minimal) element in the line is w's successor. If w is on a cycle, then w itself will be the maximal element in the line defined by $p(w,\cdot,\cdot)$. Note that it may be the case that $p(w,\cdot,\cdot)$ is empty, in which case w has no successor. It may also be the case that w is both the first and last element (i.e., $p(w,\cdot,\cdot) = \{(w,w)\}$), which means w is its own successor (i.e., it has a self edge).

Formally, let (V,E) be a directed graph where the outdegree of each vertex is at most one. Let us define the following ternary relation over V:

$$P = \left\{ (x,y,z) \in V^3 \;\middle|\; \exists n \in \mathbb{N}.\, \exists x_1,\ldots,x_n \in V. \right. \\ \left. \left(\bigwedge_{i=1}^{n-1} (x_i, x_{i+1}) \in E \right) \wedge \left(\bigwedge_{1 \le i < j \le n} x_i \neq x_j \right) \wedge (x,x_1) \in E \wedge \left(\bigvee_{i=1}^{n} x_i = y \right) \wedge x_n = z \right\} \tag{3.22}$$

That is, P contains the (x,y,z)'s such that both y and z are reachable from x by one or more applications of E, and the distance from x to y is less than or equal to the distance from x to z. In other words, y is on the acyclic path from x to z. This is similar to *btw* of Section 3.3.3, but it may also hold when x, y and z are not distinct. Note that $(x,y,y) \in P$ iff y is reachable from x, and that $(x,x,x) \in P$ iff x is on a cycle. In this case, P will also contain (x,y,x) for every y reachable from x, and will not contain (x,x,y) for any $y \neq x$.

A ternary path relation p whose intended interpretation is according to P defined above can be axiomatized by universally quantified formulas. Figure 3.21 lists the axiomatization for p, along with a successor formula φ_s that derives graph edges from p with a single universal quantifier, several useful first-order queries, and update code for adding and removing edges. The code for adding an edge assumes that no edge is currently outgoing from the source vertex. Thus, to update the function, we first remove the existing edge, and then add a new edge. (Unlike Figure 3.15, the code does not require that the new edge does not create a cycle.)

The soundness and completeness of this encoding are given by the following theorems.

Theorem 3.23 (General Partial Function Soundness)**.** *For any directed graph $G = (V,E)$ where the outdegree of any vertex is at most one, let $s = (\mathcal{D},\mathcal{I})$ be the first-order structure given by $\mathcal{D} = V$, $\mathcal{I}(p) = P$, where P is given by Equation (3.22). Then, s satisfies the axioms of Figure 3.21, and in addition we have the following:*

(i) for any three elements $u,v,w \in \mathcal{D}$, $s \models p(u,v,w)$ if and only if $(u,v,w) \in P$ where P is defined from G by Equation (3.22); and

(ii) for any two elements $u, v \in \mathcal{D}$, $s \models \varphi_s(u, v)$ (where φ_s is taken from Figure 3.21) if and only if $(u, v) \in E$.

Moreover, let $G' = (V, E')$ be a directed graph where the outdegree of any vertex is at most one, obtained from G by addition or removal of an edge, and let s' be the structure defined from G' as s is defined from G, then $s, s' \models \tau$, where τ is the transition relation of the suitable update code in Figure 3.21.

Proof. The theorem follows directly from Figure 3.21 and Equation (3.22). The update formulas require careful consideration of graph cycles, but are straightforward. □

Theorem 3.24 (General Partial Function Completeness)**.** *For any finite structure $s = (\mathcal{D}, \mathcal{I})$ that satisfies the axioms of Figure 3.21, there is a directed graph $G = (\mathcal{D}, E)$ where the outdegree of any vertex is at most one, such that the following holds:*

(i) for any three elements $u, v, w \in \mathcal{D}$, $s \models p(u, v, w)$ if and only if $(u, v, w) \in P$ where P is defined from G by Equation (3.22); and

(ii) for any two elements $u, v \in \mathcal{D}$, $s \models \varphi_s(u, v)$ (where φ_s is taken from Figure 3.21) if and only if $(u, v) \in E$.

Moreover, let s and s' be two structures with the same domain that satisfy the axioms of Figure 3.21 such that $s, s' \models \tau$, where τ is the transition relation of the edge addition or removal code listed in Figure 3.21. Then, the graph corresponding to s' is obtained from the graph corresponding to s by a suitable edge addition or removal corresponding to τ.

Proof. The first four axioms of Figure 3.21 (transitivity, antisymmetry, partial totality, and partial reflexivity) are precisely the axioms of Figure 3.12 applied to $p(w, \cdot, \cdot)$, with w also universally quantified. So, for each w, $p(w, \cdot, \cdot)$ defines a line. The last three axioms of Figure 3.21 (cycle maximality, transitivity of reachability, path consistency) ensure that these lines are consistent and define a single graph. The successor formula of Figure 3.21 defines the successor of x to be the minimal element in the line represented by $p(x, \cdot, \cdot)$ (compare it to the minimal element query of Figure 3.12). To construct the required graph, let $E = \{(u, v) \in \mathcal{D} \times \mathcal{D} \mid s \models \varphi_s(u, v)\}$, where φ_s is taken from Figure 3.21. □

Note that while the encoding of a general partial function subsumes all the other encodings presented in Sections 3.3.1 to 3.3.3, it is more complicated, both for the user in charge of expressing queries (e.g., to explore the graph in transitions) and invariants, and for the automated solver used to discharge verification conditions. Therefore, the four encodings

presented are all useful, and as a rule one should use the simplest encoding possible. For example, if a routing protocol maintains a forwarding graph that never contains cycles, it is preferable to use the encoding of Section 3.3.2 (which uses a binary relation), rather than the encoding presented here (which uses a ternary relation).

3.4 Higher-Order Logic and Cardinality Thresholds

An additional hurdle to using first-order logic in verification is the fact that algorithms and their invariants often use higher-order quantification, as well as set cardinalities. A common pattern in distributed algorithms is to use cardinality thresholds. For example, an algorithm may wait until an operation is acknowledged by at least half of the nodes in the system. Expressing such an algorithm requires defining sets by predicates (e.g., the set of nodes from which some message was received), and also expressing constraints over cardinalities of such sets (e.g., that this set contains at least $\frac{N}{2}$ nodes, where N is the total number of nodes).

Another reason an algorithm may require higher-order quantification is if it uses sets or functions as values. For example, consider an algorithm in which messages contain a set of nodes as one of the message fields. Then, the set of messages sent so far (which may be part of the state of the system) is a set of tuples, where one of the elements in the tuples is itself a set of nodes. Similarly, messages may contain maps, which are naturally modeled by functions (e.g., a message may contain a map from nodes to values). In such cases, the invariants needed to prove the algorithms will usually include higher-order quantification, i.e., quantification over sets and functions.

3.4.1 Expressing Higher-Order Logic

While higher-order logic cannot be fully reduced to first-order logic, it is well-known [203] that we can partly express higher-order concepts in first-order logic in the following way.

Sets Suppose we want to express quantification over sets of nodes. We add a new sort called $\mathsf{nodeset}$, and a binary relation $member : \mathsf{node}, \mathsf{nodeset}$. We then use $member(n, s)$ instead of $n \in s$, and express quantification over sets of nodes as quantification over $\mathsf{nodeset}$. Typically, we will need to add first-order assumptions or axioms to correctly express the algorithm and to prove its inductive invariant. For example, the algorithm may set s to the empty set as part of a transition. We can express this by the following (RML) code: $s := *$; **assume** $\forall x : \mathsf{node}. \neg member(x, s)$.

Functions Functions can be encoded as first-order elements in a similar way. Suppose messages in the algorithm contain a map from nodes to values. In this case, we can add a new first-order sort called map, and a function symbol $\mathit{apply} : \mathsf{map}, \mathsf{node} \to \mathsf{value}$. Then, we can use $\mathit{apply}(m, n)$ instead of $m(n)$, and replace quantification over functions with quantification of the first-order sort map. As before, we may need to add axioms that capture some of the intended second-order meaning of the sort map.

Incompleteness While this encoding is sound (as long as we only use axioms that are valid in the higher-order interpretation), it is not complete. The reason is that the encoding does not force the interpretation of sorts like $\mathsf{nodeset}$ or map to include *all* the sets or functions according to the higher-order semantics. In the context of deductive verification, this means that an inductive invariant that is correct in the higher-order semantics can actually have a counterexample to induction in the first-order semantics. However, in the verification projects presented in this thesis, we did not experience this incompleteness to be a practical obstacle for verification in first-order logic. Namely, we did not encounter a spurious counterexample arising from the incompleteness of encoding higher-order concepts (sets, functions) in first-order logic.

3.4.2 Quorums and Cardinality Thresholds

Many distributed protocols use *cardinality thresholds* to define *quorums* required for various operations. This was illustrated in Section 3.1.2 for a toy protocol. Realistic protocols also use similar principles. For example, a protocol may wait for at least $\frac{N}{2}$ nodes to confirm a proposal before committing a value, where N is the total number of nodes. This is often used to ensure consistency in consensus algorithms. In Byzantine failure models, a common threshold is $\frac{2N}{3}$, where at most a third of the nodes may be Byzantine.

First-order logic cannot completely capture set cardinality thresholds. However, this thesis exploits the fact that protocol correctness commonly relies on rather simple properties that are implied by the cardinality threshold, and that these properties can be encoded in first-order logic. The idea is to use a variant of the encoding presented in Section 3.4.1, and to introduce a sort for quorums defined by a certain threshold, and a membership relation *member* between nodes and quorums. Then, properties that are needed for protocol correctness can be axiomatized in first-order logic. Below we present several examples of this general principle.

Note that once a sort $\mathsf{quorum}_\mathsf{t}$ has been created for sets of nodes with more than t nodes, the condition $|\{n : \mathsf{node} \mid \varphi(n)\}| > t$, i.e., there are more than t nodes for which φ holds,

can be expressed in first-order logic as: $\exists q : \mathsf{quorum_t}.\ \forall n : \mathsf{node}.\ member(n, q) \rightarrow \varphi(n)$. This technique has been used in the example of Figure 3.6, and is one of the key ingredients to verification of Paxos protocols presented in Chapter 4 and appendix A.

Below we present several example of first-order properties of quorums that can be expressed as first-order axioms and used for verifying distributed protocols in first-order logic. We use N to denote the total number of nodes in the system, and we overload the relation symbol *member* as a binary relation between nodes and quorums of various sorts.

Majority sets Many protocols use sets of at least $\frac{N}{2}$ nodes. This threshold ensures that any two such sets intersect, a property that is crucial for many consensus protocols in order to maintain consistency. This property can be expressed in first order logic (as was done in Figure 3.6):

$$\forall q_1, q_2 : \mathsf{quorum}_{\frac{\mathsf{N}}{2}}.\ \exists n : \mathsf{node}.\ member(n, q_1) \wedge member(n, q_2)$$

Byzantine majorities Many Byzantine consensus algorithms use sets of at least $\frac{2N}{3}$ nodes as quorums, assuming that less than $\frac{N}{3}$ nodes may be Byzantine. The key property of this threshold is that any two quorums intersect at a non-Byzantine node. This can also be expressed in first-order logic, using the relation $byz : \mathsf{node}$, which represents Byzantine nodes:

$$\forall q_1, q_2 : \mathsf{quorum}_{\frac{2\mathsf{N}}{3}}.\ \exists n : \mathsf{node}.\ \neg byz(n) \wedge member(n, q_1) \wedge member(n, q_2)$$

Fast quorums Another interesting example is Fast Paxos [138], a variant of Multi-Paxos that improves its latency. This protocol uses two kinds of quorums, *fast quorums* and *classical quorums*. These quorums have the property that any two classic quorums intersect, and any classic quorum and two fast quorums intersect. This can be expressed using two sorts, $\mathsf{quorum_f}$ and $\mathsf{quorum_c}$, as follows:

$$\forall q_1, q_2 : \mathsf{quorum_c}.\ \exists n : \mathsf{node}.\ member(n, q_1) \wedge member(n, q_2)$$

$$\forall q_c : \mathsf{quorum_c},\ q_1, q_2 : \mathsf{quorum_f}.\ \exists n : \mathsf{node}.\ member(n, q_c) \wedge member(n, q_1) \wedge member(n, q_2)$$

Reliable broadcast As a final example, we consider a classical protocol for reliable broadcast that tolerates up to $\frac{N}{3}$ Byzantine faults (see e.g., [144, Figure 6]). The protocol uses two types of quorums, with thresholds of $\frac{N}{3}$ and $\frac{2N}{3}$. For our purposes, the important point is that the correctness of the protocol relies on the following three properties:

(i) every set of at least $\frac{N}{3}$ nodes contains a non-Byzantine node;

(ii) there is a set of more than $\frac{2N}{3}$ non-Byzantine nodes; and

(iii) every set of at least $\frac{2N}{3}$ nodes contains a subset of at least $\frac{N}{3}$ non-Byzantine nodes.

The correctness of the protocol relies solely on these properties, and fortunately, they can be expressed in first-order logic as follows:

(i) $\forall q : \mathsf{quorum}_{\frac{\mathsf{N}}{3}}.\ \exists n : \mathsf{node}.\ member(n, q) \wedge \neg byz(n)$

(ii) $\exists q : \mathsf{quorum}_{\frac{2\mathsf{N}}{3}}.\ \forall n : \mathsf{node}.\ member(n, q) \rightarrow \neg byz(n)$

(iii) $\forall q : \mathsf{quorum}_{\frac{2\mathsf{N}}{3}}.\ \exists q' : \mathsf{quorum}_{\frac{\mathsf{N}}{3}}.\ \forall n : \mathsf{node}.\ member(n, q') \rightarrow \neg byz(n) \wedge member(n, q)$

Soundness and trusted base The presented technique abstracts cardinality thresholds in first-order logic by adding axioms to the first-order transition system (i.e., to Γ). This is sound, so long as the properties are indeed implied by the cardinality thresholds themselves. For example, the fact that any two quorums intersect is implied if the quorums are defined with the threshold $\frac{N}{2}$, but not if they are defined with $\frac{N}{3}$. From the viewpoint of the entire verification process, adding these properties as axioms creates a new proof obligation, to show that they follow from the thresholds. In this thesis, we focus on the first-order part of the verification, and treat the axioms as part of the "trusted base" of the verification process. We also note that these axioms are reusable across many protocols, and having a trusted library of axioms is a reasonable approach in this case.

However, it is entirely possible to eliminate the first-order properties from the trusted base by mechanizing the proof that shows they are implied by the thresholds. This can be done either by interactive theorem proving, which is already partially supported by Ivy (and used in [219] to prove that $\frac{N}{2}$ majorities intersect); or by using other decidable logics that support direct reasoning about set cardinalities, such as BAPA (Boolean Algebra and Presburger Arithmetic) [131].

3.5 Network Semantics

We now discuss two issues related specifically to modeling of distributed algorithms: modeling the semantics of the underlying communication network, and modeling concurrency and asynchrony.

3.5.1 Communication Channel Semantics

In distributed computing, any message passing algorithm must assume some semantics for the communication channels it uses. When verifying the algorithm, it should be verified under this semantics. For example, one algorithm may only be correct if messages are received in the same order as they were sent, and another algorithm may be correct even if messages can be *reordered* by the network. Another common distinction is whether messages may be *duplicated* by the network, i.e., is it possible for a message to be received more times than it was sent. In this section we show that network models can be easily encoded in first-order logic, and present the encodings of three common asynchronous communication models: reordering with duplication, reordering without duplication, and FIFO (first-in-first-out, i.e., no reordering and no duplication).

Figure 3.25 presents the encoding of the three network models in first-order logic using RML syntax. Note that each model may also be modified to allow message dropping, by adding receive transitions that do nothing with the received message. The encoding presented here is focused on safety proofs. For liveness, we must also record which messages have been received, and add suitable fairness assumptions for the network. For example, that every message sent is eventually received, or that every message that is sent infinitely often is received infinitely often. These can also naturally be modeled in using a first-order formalism, as presented in Chapter 7.

Below we provide more details on each of the encodings listed in Figure 3.25.

Reordering with duplication A channel with reordering and duplication is represented by a relation. Since messages may be duplicated, there is no need to "count" how many times a message was sent. Accordingly, the receive operation may non-deterministically remove the received message from the relation, or leave it there.

Reordering without duplication A channel with reordering but no duplication is also represented by a relation, but here we also use a new sort msg that serves as a unique message identifier. The send code obtains a fresh identifier and inserts a tuple to the relation, and the receive code removes a single tuple from the relation. Essentially, we encode a multiset in a set using unique tags.

FIFO A FIFO (first-in-first-out) channel is one that neither duplicates nor reorders messages. We represent it using a line graph of messages, exploiting the technique presented in Section 3.3.1. Messages are represented by a sort msg, with immutable functions $field_1$:

Network Model	Declarations and Send Code	Receive Code
Reordering w/ Duplication	**relation** $msg : \mathsf{s}_1, \ldots, \mathsf{s}_k$ **init** $\forall x_1, \ldots, x_k.\ \neg msg(x_1, \ldots, x_k)$ $msg(f_1, \ldots, f_k) := \text{true}$	$f_1, \ldots, f_k := *$ **assume** $msg(f_1, \ldots, f_k)$ $msg(f_1, \ldots, f_k) := *$
Reordering w/o Duplication	**sort** msg **relation** $msg : \mathsf{msg}, \mathsf{s}_1, \ldots, \mathsf{s}_k$ **init** $\forall m, x_1, \ldots, x_k.\ \neg msg(m, x_1, \ldots, x_k)$ **local** $m : \mathsf{msg} := *$ **assume** $\forall x_1, \ldots, x_k.\ \neg msg(m, x_1, \ldots, x_k)$ $msg(m, f_1, \ldots, f_k) := \text{true}$	$f_1, \ldots, f_k := *$ **local** $m : \mathsf{msg} := *$ **assume** $msg(m, f_1, \ldots, f_k)$ $msg(m, x_1, \ldots, x_k) := \text{false}$
FIFO	**sort** msg **function** $field_1 : \mathsf{msg} \to \mathsf{s}_1, \ldots, field_k : \mathsf{msg} \to \mathsf{s}_k$ **relation** $\leq : \mathsf{msg}, \mathsf{msg}$ **axiom** $\Gamma_{\text{line}}[\leq]$ **init** $\forall x.\ x \not\leq x$ **local** $m : \mathsf{msg} := *$ **assume** $m \not\leq m \wedge \bigwedge_{i=1}^{k} field_i(m) = f_i$ INSERT-AS-MIN$[\leq](m)$	$f_1, \ldots, f_k := *$ **local** $m : \mathsf{msg} := *$ **assume** IS-MAX$[\leq](m)$ **assume** $\bigwedge_{i=1}^{k} field_i(m) = f_i$ REMOVE$[\leq](m)$

Figure 3.25: Encoding of a network models in first-order logic. Declarations, send code, and receive code, are presented for encoding message channels for messages with multiple fields $f_1 : \mathsf{s}_1, \ldots, f_k : \mathsf{s}_k$. The message source and destination can be encoded as fields if needed (e.g., in broadcast communication, only the message source is needed). The send code assumes the individuals $f_1, \ldots, f_k$ contain the message to be sent, and the receive code sets them to the message received. Encodings are presented for three kinds of channels: reordering with duplication; reordering without duplication; and FIFO (first-in-first-out), i.e., no reordering and no duplication. We use Γ_{line}, INSERT-AS-MIN, IS-MAX, and REMOVE to denote the axioms and update code for the encoding of a line graph in first-order logic, presented in Section 3.3.1 (Figure 3.12).

$\mathsf{msg} \to \mathsf{s}_1, \ldots, \mathit{field}_k : \mathsf{msg} \to \mathsf{s}_k$ that map a message to its fields. The send code inserts a message as a minimal element, and the receive code removes the maximal message element. Inductive invariants can use the $\leq$ relation over messages to describe properties of messages in the channel, and also to refer to the ordering between messages. For example, $\forall m.\ m \leq m \to \varphi(m)$ asserts that all messages in the channel must satisfy φ. As another example, $\forall m_1, m_2 : \mathsf{msg}.\ m_1 \leq m_2 \to \psi(m_1, m_2)$ asserts that if both m_1 and m_2 are currently in the channel, and m_1 was sent after m_2, then $\psi(m_1, m_2)$ must hold.

3.5.2 Asynchrony and Concurrency

Another aspect of modeling distributed algorithms is correct handling of *asynchrony* and *concurrency*. Recall that in RML, actions are atomic and sequential; a trace is a sequence of steps, where each step represents a single action. Therefore, in order to faithfully capture a distributed system, we need to make sure that every concurrent execution is equivalent to some sequential execution of actions. In this thesis, we do so in a straightforward way that is well aligned with how distributed protocols are usually described.

To ensure that the semantics of the RML program faithfully captures concurrent executions, we require actions of a distributed protocol to obey the following syntactic restrictions:

(i) every action is *local*, that is, it only investigates and modifies the state of a single node and the communication channels that connect to it; and

(ii) every action first (optionally) receives messages, then performs modifications to the node's state, and then (optionally) sends messages (e.g., an action never does: receive, send, receive).

Due to these restrictions, there is no need to consider concurrent interleavings of actions, as every interleaving is equivalent to some sequential execution. This can be seen as a very simple application of Lipton's theory of left-movers and right-movers [157], where sends are left-movers, receives are right-movers, and local state modifications are both-movers.

Note that while the above restrictions are needed to model the real concurrent semantics of distributed (messages passing) protocols, it is sometimes useful for the verification development process to start with a more abstract, non-local formulation, where nodes have direct access to the state of other nodes. Such a model is useful, e.g., to find an inductive invariant or a suitable first-order abstraction, and it can later be refined to a model that faithfully represents asynchronous executions. To accommodate this, RML is kept flexible to allow modeling and verifying non-local formulations, which can be gradually developed until

they become local and meet the above restrictions.

3.6 Related Work for Chapter 3

Transitive closure Modeling transitive closure in first-order logic has a rich history in several communities, including the database and logic communities [68, 103, 109, 155]. In the verification community, this has been addressed, e.g., in [133, 152, 179]. Our work here follows the principles of [111–113] of using EPR to obtain a sound and complete modeling of deterministic paths. Similar principles have also been used in [194, 195, 223, 224], in the context of local theory extension [108, 215], which like EPR, obtains decidability by a finite complete instantiation of the axioms.

Set cardinalities Modeling set cardinalities for verification has been extensively addressed by the research community, mostly due to its importance for verification of heap data structures as well as distributed protocols. Examples include [14, 21, 69, 70, 131, 143, 231]. These works treat set cardinalities as a primitive of the logic, and thus rely on more powerful logics compared to EPR and first-order logic. In contrast, our approach calls on the user to express reusable first-order properties that capture the use of set cardinalities in protocols. As a result, we are able to use EPR without adding set cardinalities to the logic itself, and we are able to use existing solvers that support EPR. It is also worth noting that we gain flexibility, as none of the other approaches that model cardinalities directly support a logic that is flexible enough to verify the complex protocols that are addressed in this thesis. This is discussed further in Section 4.8.

Chapter 4

Eliminating Quantifier Alternations Cycles: Paxos Made EPR

This chapter is based on the results published in [187, 188].

The techniques presented in the previous chapter allow to express many challenging protocols and their inductive invariants in first-order logic. However, first-order logic is undecidable, and automated solvers are therefore not guaranteed to terminate on the resulting verification conditions. Moreover, in full first-order logic, counterexamples are not guaranteed to be finite. For these reasons, and as explained in Chapter 1, we would like to reduce the verification problem to verification conditions in the many-sorted EPR fragment, which is both decidable and ensures finite counterexamples (that can be presented to the user).

As explained in Section 2.2, a formula is in the many-sorted EPR fragment if it does not contain quantifier alternation cycles (i.e., its quantifier alternation graph is acyclic). Most axioms developed in Chapter 3 are purely universal, and do not contain any quantifier alternations. The axioms for quorums do contain a quantifier alternation, resulting in a single edge from quorums to nodes. However, for complex protocols, most notably Paxos and its variants considered here, the inductive invariant of the protocol contains several more quantifier alternations that result in quantifier alternation cycles.

This chapter presents a methodology for systematic elimination of quantifier alternation cycles in a sound manner. The most creative part in our methodology is adding derived relations and rewriting the code to break the cycles in the quantifier alternation graph. The main idea is to capture an existential formula by a derived relation, and then to use the derived relation as a substitute for the formula, both in the code and in the invariant, thus eliminating some quantifier alternations. The user is responsible for defining the derived

relations and performing the rewrites. The verification system automatically generates update code for the derived relations, and automatically checks the soundness of the rewrites. For the generation of update code, we exploit the locality of updates, as relations (used for defining the derived relations) are updated by inserting a single entry at a time. We identify a class of formulas for which this automatic procedure is always possible.

We illustrates the methodology by applying it to verify safety of the Paxos consensus protocol, and the Multi-Paxos protocol for state machine replication. We have also successfully applied the methodology to verify more protocols in the Paxos family: Vertical Paxos, Fast Paxos, Flexible Paxos and Stoppable Paxos. To the best of the author's knowledge, this work was the first to verify these protocols using a decidable logic, and, in the case of Vertical Paxos, Fast Paxos, and Stoppable Paxos, it was also the first mechanized safety proof. The transformations and models developed here are also the basis for liveness proofs of Paxos variants presented in Chapters 7 and 8. We are encouraged by the fact that the transformations needed to reach verification in EPR are reusable across all Paxos variants we consider, and also serve for liveness verification. Furthermore, the transformations maintain the simplicity and readability of both the code and the inductive invariants.

The general methodology for decidable verification by eliminating quantifier alternation cycles is developed in Section 4.1. Section 4.2 reviews the Paxos consensus algorithm, which is the basis for all Paxos-like protocols. We present our model of the Paxos consensus algorithm as a transition system in first-order logic in Section 4.3, and continue to verify it using EPR by applying our methodology in Section 4.4. In Section 4.5, we describe our verification of Multi-Paxos using EPR. We briefly discuss the verification of Vertical Paxos, Fast Paxos, Flexible Paxos, and Stoppable Paxos in Section 4.6. In Section 4.7 we report on our implementation and experimental evaluation. Related work for this chapter is discussed in Section 4.8. More details about the verification of Vertical Paxos, Fast Paxos, Flexible Paxos, and Stoppable Paxos appear in Appendix A.

4.1 Methodology for Decidable Verification

In this section we explain the general methodology that we follow in our efforts to distributed protocols, and specifically Paxos, using decidable reasoning. While this chapter focuses on Paxos and its variants, the methodology is more general and can be useful for verifying other systems as well.

4.1.1 Modeling in Uninterpreted First-Order Logic

The first step in our verification methodology is to express the protocol as a transition system in many-sorted uninterpreted first-order logic. This step involves some abstraction, since protocols usually employ concepts that are not directly expressible in uninterpreted first-order logic. We use the technique developed in Chapter 3 to replace interpreted domains (e.g., natural numbers) by abstractions in uninterpreted first-order logic (e.g., a totally ordered set), and to encode higher-order concepts such as sets and functions in first-order logic. Most importantly for Paxos and its variants, we use the axiomatization of majority quorums presented in Section 3.4.2.

Semi-bounded verification After the first step, we can already apply bounded — as well as *semi-bounded* — verification. Given a transition system in first-order logic with a candidate inductive invariant, it may still be undecidable to check the resulting verification conditions. However, bounded verification is decidable, and extremely useful for debugging the model before continuing with the efforts of unbounded verification. Contrary to the usual practice of bounding the number of elements in each sort for bounded verification, we use the quantifier alternation graph (defined in Section 2.2) to determine only a *subset* of the sorts to bound in order to make verification decidable. We call this procedure *semi-bounded verification*, and it follows from the observation that whenever we make a sort bounded, we can remove its node from the quantifier alternation graph. When the resulting graph becomes acyclic, satisfiability is decidable without bounding the sizes of the remaining sorts.

4.1.2 Transformation to EPR Using Derived Relations

The second step in our methodology for decidable *unbounded* verification is to transform the model expressed in first-order logic to a model that has an inductive invariant whose verification condition is in EPR, and is therefore decidable to check. The methodology is manual, but following it ensures soundness of the verification process. The key idea is to use *derived relations* to simplify the transition relation and the inductive invariant. Derived relations extend the state of the system and are updated in its transitions. Derived relations are somewhat similar to ghost variables. However there are two key differences. First, derived relations are typically not used to record the history of an execution. Instead, they capture properties of the current state in a way that facilitates verification using EPR. Second, derived relations are not only updated in the transitions, but can also affect them.

The transformation of the model using derived relations is conducted in steps, as detailed

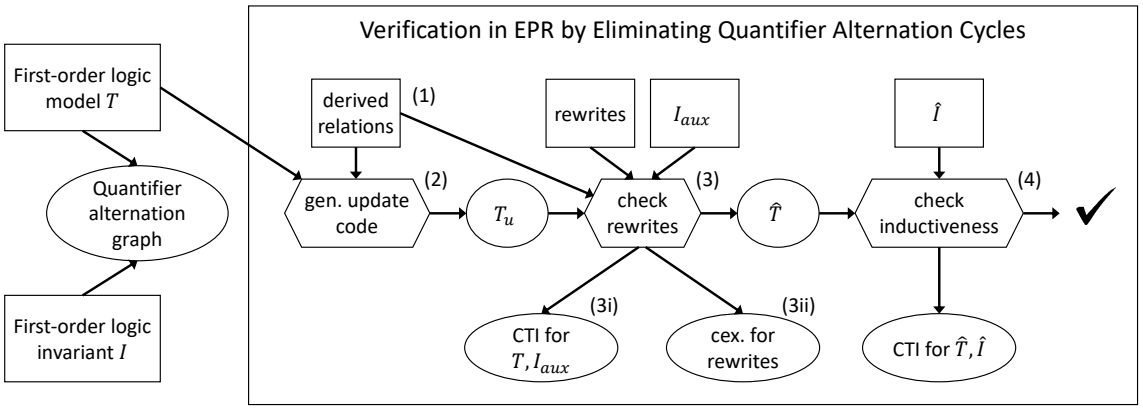


Figure 4.1: Flowchart of the methodology for verification using EPR by eliminating quantifier alternation cycles. User provided inputs are depicted as rectangles, automated procedures are depicted as hexagons, and automatically generated outputs are depicted as ellipses. The original first-order logic model T, and the original (first-order logic) inductive invariant I result in a quantifier alternation graph, which guides the process. The transformation to EPR is carried in steps (1)–(4), detailed in Section 4.1.2. In step (1), the user provides definitions for derived relations. In step (2), update code is automatically generated, resulting in T_u. In step (3), the user provides rewrites that use the derived relations, as well as an auxiliary inductive invariant I_{aux} to prove the soundness of the rewrites. An automated procedure first checks the soundness of the rewrites (and provides a counterexample in case they are unsound, or a counterexample to induction if the I_{aux} is not inductive), and then outputs the transformed model $\widehat{T}$. In step (4), the user provides an inductive invariant $\widehat{I}$ that proves safety of the transformed model $\widehat{T}$, and an automated check either verifies the inductiveness or provides a counterexample to induction. Soundness is ensured by Theorem 4.4, stating that safety of $\widehat{T}$ entails safety of T.

below. The various steps are depicted in Figure 4.1. The inputs provided by the user are depicted by rectangles, while the automated procedures are depicted as hexagons, and their outputs are depicted as ellipses. As illustrated by the figure, the user is guided by the quantifier alternation graph of the verification conditions.

In the sequel, we fix a first-order transition system $T = (\Sigma, \Gamma, \iota, \tau)$ and a safety property P. It is understood that T has an inductive invariant I (for proving P) but its resulting verification conditions are not in EPR, i.e., the associated quantifier alternation graph contains cycles. We will show how to transform T and P into $\widehat{T} = (\widehat{\Sigma}, \widehat{\Gamma}, \widehat{\iota}, \widehat{\tau})$ and $\widehat{P}$, such that $\widehat{T}$ has an inductive invariant $\widehat{I}$ (for proving $\widehat{P}$) whose verification conditions are in EPR, and the transformation ensures that $\widehat{T} \models \widehat{P}$ implies $T \models P$.

(1) Defining a derived relation In the first, and most creative part of the process, the user identifies an existentially quantified formula $\psi(\bar{x})$ that will be captured by a derived relation $r(\bar{x})$. The selection of ψ is guided by the quantifier alternation graph of the verification conditions, with the purpose of eliminating cycles it contains. Quantifier alternations in the verification conditions originate both from the model and the inductive invariant. As we shall see, using r will allow us to eliminate some quantifier alternations. As an example for demonstrating the next steps, consider a program defined with a binary relation p, and let $r(x)$ be a derived relation capturing the formula $\psi(x) = \exists y.\, p(x, y)$.

(2) Tracking ψ by r This step algorithmically extends T into a $T_u = (\Sigma_u, \Gamma_u, \iota_u, \tau_u)$, where $\Sigma_u = \Sigma \cup \{r\}$, and $\Gamma_u = \Gamma$. Intuitively, T_u makes the same transitions as T, but also updates r to capture ψ. Formally, T_u is obtained by adding: (i) an initial condition that initializes r, and (ii) *update code* that modifies r whenever the relations mentioned in ψ are modified. The initial condition and update code are automatically generated in a way that guarantees that the following formula is an invariant of T_u:

$$\forall \bar{x}.\, r(\bar{x}) \leftrightarrow \psi(\bar{x}). \tag{4.2}$$

We call this invariant the *representation invariant* of r. Our scheme for automatically obtaining ι_u and τ_u) and the class of formulas ψ that it supports, are discussed in Section 4.1.3. In our example, suppose that initially p is empty. Then, the ι_u would initialize r to be empty as well. For an action that inserts a pair (a, b) to p, τ_u would insert a to r.

(3) Rewriting the transitions using r In this step, the user exploits r to eliminate quantifier alternations in the verification conditions by rewriting the system's transitions,

obtaining a model $\widehat{T} = (\widehat{\Sigma}, \widehat{\Gamma}, \widehat{\iota}, \widehat{\tau})$ and a safety property $\widehat{P}$, where $\widehat{\Sigma} = \Sigma_u = \Sigma \cup \{r\}$, $\widehat{\Gamma} = \Gamma_u = \Gamma$, and $\widehat{\iota} = \iota_u$. The transition relation $\widehat{\tau}$ differs from τ_u, and the safety property $\widehat{P}$ may also differ from P. The idea is to rewrite the transitions in a way that ensures that the reachable states are unchanged, while eliminating quantifier alternations. This is done by rewriting some program conditions used in **assume** commands in the code. The user performs the rewrites at the RML source level, and the modified program defines $\widehat{\tau}$ and $\widehat{P}$. The simplest example is to use r instead of ψ, but other rewrites are also possible.

While the rewrites are performed by the user, we automatically check that the effect of the modified commands on the reachable states remains the same (under the assumption of the representation invariant). Suppose the user rewrites **assume** φ to **assume** $\widehat{\varphi}$. The simplest way to ensure this has the same effect on the reachable states is to check that the following *rewrite condition* is valid: $\varphi \leftrightarrow \widehat{\varphi}\,[\psi(\bar{x}) \,/\, r(\bar{x})]$. This condition guarantees that the two formulas φ and $\widehat{\varphi}$ are equivalent in any reachable state, due to the representation invariant. In some cases, the rewrite is such that $\widehat{\varphi}\,[\psi(\bar{x}) \,/\, r(\bar{x})]$ is syntactically identical to φ, which makes the rewrite condition trivial.

However, to allow greater flexibility in rewriting the code, we allow using an EPR check to verify the rewrite condition, and also relax the condition given above in two ways. First, we observe that it suffices to verify the equivalence of subformulas of φ that were modified by the rewrite. Formally, if φ is syntactically identical to $\widehat{\varphi}\left[\theta_1(\bar{y}_1) \,/\, \widehat{\theta}_1(\bar{y}_1), \ldots, \theta_k(\bar{y}_k) \,/\, \widehat{\theta}_k(\bar{y}_k)\right]$, then to establish the rewrite condition, it suffices to prove that for every $1 \leq i \leq k$ the following equivalence is valid: $\theta_i \leftrightarrow \widehat{\theta}_i\,[\psi(\bar{x}) \,/\, r(\bar{x})]$. (The case where φ was completely modified is captured by the case where $k = 1$, $\varphi = \theta_1$ and $\widehat{\varphi} = \widehat{\theta}_1$.) Second, and more importantly, recall that we are only interested in preserving the transitions from reachable states of the system. Thus, we allow the user to provide an auxiliary invariant I_{aux} (by default $I_{\text{aux}} = \mathit{true}$) that is used to prove that the reachable transitions remain unchanged after the transformation. Technically, this is done by automatically checking that

(i) I_{aux} is an inductive invariant of T, and

(ii) the following rewrite condition holds for every $1 \leq i \leq k$:

$$\Gamma, I_{\text{aux}}, g \models \theta_i \leftrightarrow \widehat{\theta}_i\,[\psi(\bar{x}) \,/\, r(\bar{x})], \tag{4.3}$$

where g captures additional conditions that guard the modified **assume** command (g is automatically computed from the RML program).

These conditions guarantee that the two formulas φ and $\widehat{\varphi}$ are equivalent whenever the mod-

ified **assume** command is executed. To ensure that these checks can be done automatically, we require that the corresponding formulas are in EPR. We note that verifying I_{aux} for T can be possible in EPR even in cases where verifying safety of T is not in EPR, since I_{aux} can be weaker (and contain less quantifier alternations) than an invariant that proves safety.

In our example, suppose the program contains the command **assume** $\exists y.\, p(a, y)$. Then we could rewrite it to **assume** $r(a)$. For a more sophisticated example, suppose that the program contains the command **assume** $\exists y.\, p(a, y) \wedge q(a, y)$, and suppose this command is guarded by the condition $u(a)$ (i.e., the **assume** only happens if $u(a)$ holds). Suppose further that we can verify that $I_{\text{aux}} = \forall x, y.\, u(x) \wedge p(x, y) \rightarrow q(x, y)$ is an invariant of T. Then we could rewrite the assume command as **assume** $r(a)$ since $(\forall x, y.\, u(x) \wedge p(x, y) \rightarrow q(x, y)) \wedge u(a) \models (\exists y.\, p(a, y) \wedge q(a, y)) \leftrightarrow \exists y.\, p(a, y)$.

(4) Providing an inductive invariant Finally, the user proves that $\widehat{T} \models \widehat{P}$ by providing an inductive invariant $\widehat{I}$, whose verification conditions are in EPR. Usually this is achieved by:

1. Using r in the inductive invariant as a substitute to using ψ. The point here is that using ψ would introduce quantifier alternations, and using r instead avoids them. In our example, the safety proof might require the property that $\forall x.\, u(x) \rightarrow \exists y.\, p(x, y)$, and using r we can express this as $\forall x.\, u(x) \rightarrow r(x)$.

2. Letting the inductive invariant express some properties that are implied by the representation invariant. Note that expressing the full representation invariant would typically introduce quantifier alternations that break stratification. However, some properties implied by it may still be expressible while keeping the verification conditions in EPR. In our example, we may add $\forall x, y.\, p(x, y) \rightarrow r(x)$ to the inductive invariant. Note that adding $\forall x.\, r(x) \rightarrow \exists y.\, p(x, y)$ to the inductive invariant would make the verification conditions outside of EPR.

Given $\widehat{T}$, $\widehat{P}$, and $\widehat{I}$, we can now automatically derive the verification conditions in EPR (as defined in Section 2.4.2) and check that they hold. The following theorem summarizes the soundness of the approach:

Theorem 4.4 (Soundness)**.** *Let $T = (\Sigma, \Gamma, \iota, \tau)$ be a first-order transition system, and P be a safety property (over Σ). If $\widehat{T} = (\widehat{\Sigma}, \widehat{\Gamma}, \widehat{\iota}, \widehat{\tau})$ and $\widehat{P}$ are obtained by the above procedure, and $\widehat{I}$ is an inductive invariant for it that proves $\widehat{T} \models \widehat{P}$, then $T \models P$.*

Proof. Let $B = \{(s, \hat{s}) \mid s \in \mathcal{R} \wedge \hat{s} \in \widehat{\mathcal{R}} \wedge \hat{s}|_\Sigma = s\}$, where $\mathcal{R}$ and $\widehat{\mathcal{R}}$ denote the reachable states of T and $\widehat{T}$ respectively, and $\hat{s}|_\Sigma$ denotes the projection of a state $\hat{s}$ (defined over $\widehat{\Sigma}$) to Σ. Steps 2 and 3 of the transformation above ensure that B is a bisimulation relation between T and $\widehat{T}$, i.e., every transition possible in the reachable states of one of these systems has a corresponding transition in the other. This ensures that $\widehat{T}$ has the same reachable states as T, modulo projection to Σ. This argument extends to include P and $\widehat{P}$, as we may think of a safety violation as a transition to a special error state. Therefore, if $\widehat{T} \models \widehat{P}$, then $T \models P$. □

As shown in the proof of Theorem 4.4, the transformed model $\widehat{T}$ is bisimilar to the original model T. While this ensures that both are equivalent w.r.t. to safety, note that we check safety by checking inductiveness of a candidate invariant. Unlike safety, inductiveness is not necessarily preserved by the transformation. Namely, given a candidate inductive invariant $\widehat{I}$ which is not inductive for $\widehat{T}$, the counterexample to induction cannot in general be transformed to the original model T, as it might depend on the derived relations and the rewritten **assume** commands. An example of this phenomenon appears in Section 4.4.2.

Using the methodology Our description above explains a final successful verification using the proposed methodology. As always, obtaining this involves a series of attempts, where in each attempt the user provides the verification inputs, and gets a counterexample. Each counterexample guides the user to modify the verification inputs, until eventually verification is achieved. As depicted in Figure 4.1, with the EPR verification methodology, the user provides 5 inputs, and could obtain 3 kinds of counterexamples. The inputs are the model, the derived relations, the rewrites, the auxiliary invariant for proving the soundness of the rewrites, and finally the inductive invariant for the resulting model. The possible counterexamples are either a counterexample to induction (CTI) for the auxiliary invariant and the original model, or a counterexample to the soundness of the rewrite itself, or a CTI for the inductive invariant of the transformed model. After obtaining any of the 3 kinds of counterexamples, the user can modify any one of the 5 inputs. For example, a CTI for the inductive invariant of the transformed model may be eliminated by changing the inductive invariant itself, but it may also be overcome by an additional rewrite, which in turn requires an auxiliary invariant for its soundness proof. Indeed, we shall see an example of this in Section 4.4.2.

The task of managing the inter-dependence between the 5 verification inputs may seem daunting, and indeed it requires some expertise and creativity from the user. This is

expected, since the inputs from the user reduce the undecidable problem of safety verification to decidable EPR checks. This burden on the user is eased by the fact that for every input, the user always obtains an answer from the system, either in the form of successful verification, or in the form of a *finite* counterexample, which is displayed graphically and guides the user towards the solution. Furthermore, our experience shows that most of the creative effort is reusable across similar protocols. In the verification of all the variants of Paxos we consider in this chapter, we use the same two derived relations and very similar rewrites (as explained in Sections 4.4 to 4.6).

Incompleteness of EPR verification While the transformation using a given set of derived relations and rewrites results in a bisimilar transition system, the methodology for EPR verification is not complete. This is expected, as there can be no complete proof system for safety in a formalism that is Turing-complete. For the EPR verification methodology, the incompleteness can arise from several sources. It may happen that after applying the transformation, the resulting transition system, while safe, cannot be verified with an inductive invariant that results in EPR verification conditions. Another potential source for incompleteness is our requirement that the rewrites should also be verified in EPR. It can be the case that a certain (sound) rewrite leads to a system that can be verified using EPR, but the soundness of the rewrite itself cannot be verified using EPR. Another potential source of incompleteness can be the inability to express sufficiently powerful axioms about the underlying domain. We note that the three mentioned issues interact with each other, as it may be the case that a certain axiom is expressible in first-order logic, but it happens to introduce a quantifier alternation cycle, when considered together with either the inductive invariant or the verification conditions for the rewrites.

We consider developing a proof-theoretic understanding of which systems can and cannot be verified using EPR to be an intriguing direction for future investigation. We are encouraged by the fact that in practice, the proposed methodology has proven itself to be powerful enough to verify Paxos and its variants considered in this thesis.

Multiple derived relations For simplicity, the description above considered a single derived relation. In practice, we usually add multiple derived relations, where each one captures a different formula. The methodology remains the same, and each derived relation allows us to transform the model and eliminate more quantifier alternations, until the resulting model can be verified in EPR. In this case, the resulting inductive invariant may include properties implied by the representation invariants of several relations and relate

them directly. For example, suppose we add the following derived relations: $r_1(x)$ defined by $\psi_1(x) = \exists y.\, p(x, y)$, and $r_2(x)$ defined by $\psi_2(x) = \exists y, z.\, p(x, y) \wedge p(y, z)$. Then, the inductive invariant may include the property: $\forall x.\, r_2(x) \rightarrow r_1(x)$.

Overapproximating the reachable states Our methodology ensures that the transformed model is bisimilar to the original model. It is however straightforward to generalize our methodology and only require that the modified model simulates the original model, which maintains soundness. This may allow more flexibility both in the update code and in the manual rewrites performed by the user.

4.1.3 Automatic Generation of Update Code

In this subsection, we describe a rather naive scheme for automatic generation of initial condition and update code for derived relations, which suffices for verification of the Paxos variants considered in this thesis. We refer the reader to, e.g., [191, 206], for more advanced techniques for generation of update code for derived relations.

We limit the formula $\psi(\bar{x})$ which defines a derived relation $r(\bar{x})$ to have the following form:

$$\psi(x_1, \ldots, x_n) = \exists y_1, \ldots, y_m.\, \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m) \wedge p(x_{i_1}, \ldots, x_{i_k}, y_1, \ldots, y_m)$$

where φ is a quantifier-free formula, p is a relation symbol and $x_{i_j} \in \{x_1, \ldots, x_n\}$ for every $1 \leq j \leq k$. Note that p occurs positively, and that it depends on *some* (possibly none) of the variables x_i and *all* of the variables y_i. Our scheme further requires that the relations appearing in φ are never modified, and that p is initially empty and only updated by inserting a single tuple at a time. These restrictions can be relaxed, e.g., to support removal of a single tuple or addition of multiple tuples. However, such updates were not needed for verification of the protocols considered in this thesis, so for simplicity of the presentation we do not handle them.

Since p is initially empty, the initial condition for $r(\bar{x})$ is that it is empty as well, i.e.:

$$\forall x_1, \ldots, x_n.\, \neg r(x_1, \ldots, x_n).$$

The only updates allowed for p are insertions of a single tuple by a command of the form:

$$p(x_{i_1}, \ldots, x_{i_k}, y_1, \ldots, y_m) := p(x_{i_1}, \ldots, x_{i_k}, y_1, \ldots, y_m) \vee \left(\bigwedge_{j=1}^{k} x_{i_j} = a_j \wedge \bigwedge_{j=1}^{m} y_j = b_j \right).$$

For such an update, we generate the following update for $r(\bar{x})$:

$$r(x_1, \ldots, x_n) := r(x_1, \ldots, x_n) \vee \left(\varphi(x_1, \ldots, x_n, b_1, \ldots, b_m) \wedge \bigwedge_{j=1}^{k} x_{i_j} = a_j \right).$$

Notice that the update code translates to a purely universally quantified formula, since φ is quantifier-free, so it does not introduce any quantifier alternations.

Lemma 4.5. *The above scheme results in a model that maintains the representation invariant:* $\forall \bar{x}.\, r(\bar{x}) \leftrightarrow \psi(\bar{x})$.

Proof. The representation invariant is an inductive invariant of the resulting model. Initiation is trivial, since both p and r are initially empty. Consecution follows from the following, which is valid in first-order logic:

$$(\forall \bar{x}. r(\bar{x}) \leftrightarrow \psi(\bar{x})) \wedge \left(\forall \bar{w}, \bar{y}. p'(\bar{w}, \bar{y}) \leftrightarrow p(\bar{w}, \bar{y}) \vee (\bar{w} = \bar{a} \wedge \bar{y} = \bar{b})\right) \wedge \left(\forall \bar{x}. r'(\bar{x}) \leftrightarrow r(\bar{x}) \vee \left(\varphi(\bar{x}, \bar{b}) \wedge \bigwedge_{j=1}^{k} x_{i_j} = a_j \right) \right) \models \left(\forall \bar{x}. r'(\bar{x}) \leftrightarrow \psi'(\bar{x})\right)$$

□

4.2 Introduction to Paxos

A popular approach for implementing distributed systems is state-machine replication (SMR) [210], where a (virtual) centralized sequential state machine is replicated across many nodes (processors), providing fault-tolerance and exposing to its clients the familiar semantics of a centralized state machine. SMR can be thought of as repeatedly agreeing on a command to be executed next by the state machine, where each time agreement is obtained by solving a *consensus* problem. In the consensus problem, a set of nodes each propose a value and then reach agreement on a single proposal.

The Paxos family of protocols is widely used in practice for implementing SMR. Its core is the Paxos consensus algorithm [135, 136]. A Paxos-based SMR implementation executes a sequence of Paxos consensus instances, with various optimizations. The rest of this section explains the Paxos consensus algorithm (whose verification in EPR we discuss in Sections 4.3 and 4.4). We return to the broader context of SMR in Sections 4.5 and 4.6, where we discuss verification of Multi-Paxos and various Paxos variants.

The consensus problem In the consensus problem, a set of communicating nodes propose values and are required to decide on a single value that they all agree on. The nodes exchange messages in order to propose values, and to learn which value has been decided. The safety requirements for a consensus algorithm are: (i) only a value that has been proposed may be learned as the decided value, and (ii) any two nodes that learn a decided value, learn the same value. In an asynchronous system with failures, no deterministic consensus algorithm can guarantee termination, even when failures do not occur [77]. The Paxos algorithm achieves *indulgent consensus* [96], which means it guarantees safety, but does not guarantee termination under all conditions. Paxos is widely used in practice because, under good conditions, Paxos is optimal in message and time complexity [139]. In this chapter, we focus on verification of the safety of Paxos, which is maintained unconditionally.

Distributed setting We consider a fixed set of nodes that operate asynchronously and communicate by message passing, where every node can send a message to every node. Messages can be lost, duplicated, and reordered, but they are never corrupted. Nodes can fail by stopping, but otherwise faithfully execute the algorithm. A stop failure of a node can be captured by a loss of all messages to and from this node.

Paxos consensus algorithm We assume that nodes in the Paxos consensus algorithm can all propose values, vote for values, and learn about decisions. (In Paxos lingo, we assume that all nodes act in the roles of proposer, acceptor, and learner.) The algorithm operates in a sequence of numbered rounds (also called ballots) in which nodes can participate. At any given time, different nodes may operate in different rounds, and a node stops participating in a round once it started participating in a higher round. Each round is associated with a single node that is the *owner* of that round. This association from rounds to nodes is static and known to all nodes.

Every round represents a possibility for its owner to propose a value to the other nodes and get it decided on by having a *quorum* of the nodes vote for it in the round. Quorums are sets of nodes such that any two quorums intersect (e.g., sets consisting of a strict majority of the nodes). To avoid the algorithm being blocked by the stop failure of a node that made a proposal, any node can start one of its rounds and make a new proposal in it at any time (in particular, when other rounds are still active). Rounds progress in the following two phases.

Phase 1 The owner p of round r starts the round by communicating with the other nodes to have a majority of them *join* round r, and to determine which values are *choosable*

in lower rounds than r, i.e., values that might have or can still be decided in rounds lower than r.

Phase 2 If a value v is choosable in $r' < r$, in order not to contradict a potential decision in r', node p proposes v in round r. If no value is choosable in any $r' < r$, then p proposes a value of its choice in round r. If a majority of nodes *vote* in round r for p's proposal, then it becomes *decided*.

Note that it is possible for different values to be proposed in different rounds, and also for several decisions to be made in different rounds. Safety is guaranteed by the fact that (by definition of choosable) a value can be decided in a round $r' < r$ only if it is choosable in r', and that if a value v is choosable in round $r' < r$, then a node proposing in r will only propose v. The latter relies on the property that choosable values from prior rounds cannot be missed. Next, we describe in more detail what messages the nodes exchange and how a node makes sure not to miss any choosable value from prior rounds.

Phase 1a The owner p of round r sends a "start-round" message, requesting all nodes to join round r.

Phase 1b Upon receiving a request to join round r, a node will only join if it has not yet joined a higher round. If it agrees to join, it will respond with a "join-acknowledgment" message that will also contain its maximal vote so far, i.e., its vote in the highest round prior to r, or $\perp$ if no such vote exists. By sending the join-acknowledgment message, the node promises that it will not join or vote in any round smaller than r.

Phase 2a After p receives join-acknowledgment messages from a quorum of the nodes, it proposes a value v for round r by sending a "propose" message to all nodes. Node p selects the value v by taking the maximal vote reported by the nodes in their join-acknowledgment messages, i.e., the value that was voted for in the highest round prior to r by any of the nodes whose join-acknowledgment messages formed the quorum. As we will see, only this value can be choosable in any $r' < r$ out of all proposals from lower rounds. If all of these nodes report they have not voted in any prior round, then p may propose any value.

Phase 2b Upon receiving a propose message proposing value v for round r, a node will ignore it if it already joined a round higher than r, and otherwise it will vote for it, by sending a vote message to all nodes. Whenever a quorum of nodes vote for a value in

some round, this value is considered to be decided. Nodes learn this by observing the vote messages.

Note that a node can successfully start a new round or get a value decided only if at least one quorum of nodes is responsive. When quorums are taken to be sets consisting of a strict majority of the nodes, this means Paxos tolerates the failure of at most $\lfloor n/2 \rfloor$ nodes, where n is the total number of nodes. Moreover, Paxos may be caught in a live-lock if nodes keep starting new rounds before any value has a chance to be decided on.

4.3 Paxos in First-Order Logic

The first step of our verification methodology is to model the Paxos consensus algorithm as a transition system in many-sorted first-order logic over uninterpreted domains. This section explains our model of the protocol in first-order logic, as well as its safety proof by an inductive invariant in first-order logic (but not yet in EPR).

4.3.1 Protocol Model

Our model of the Paxos consensus algorithm in first-order logic is listed in Figure 4.6. The model involves some abstraction, as we now explain. Since each round r has a unique owner that will exclusively propose in r, we abstract away the owner node and treat the round itself as the proposer. We also abstract the mechanism by which nodes receive the values up for proposal, and allow them to propose arbitrary values. (In an SMR setting, nodes would propose values based on client requests.)

Additional abstractions are needed as some aspects of the protocol cannot be fully expressed in uninterpreted first-order logic. One such aspect is the fact that round numbers are integers, as arithmetic cannot be fully captured in first-order logic. Another aspect which must be abstracted is the use of sets of nodes as quantification over sets is also beyond first-order logic. We model these aspects in many-sorted first-order logic according to the principles of Chapter 3.

Sorts and axioms We use the following four uninterpreted sorts:

(i) node — to represent nodes of the system,

(ii) value — to represent the values subject to the consensus algorithm,

(iii) round — to model the rounds (ballots) of Paxos, and

```
1  sort node, quorum, round, value
2
3  relation ≤ : round, round
4  axiom Γ_total order[≤]
5  individual ⊥ : round
6
7  relation member : node, quorum
8  axiom ∀q1, q2 : quorum. ∃n : node. member(n, q1) ∧
9                                      member(n, q2)
10
11 relation start_round_msg : round
12 relation join_ack_msg : node, round, round, value
13 relation propose_msg : round, value
14 relation vote_msg : node, round, value
15 relation decision : node, round, value
16
17 init ∀r. ¬start_round_msg(r)
18 init ∀n, r1, r2, v. ¬join_ack_msg(n, r1, r2, v)
19 init ∀r, v. ¬propose_msg(r, v)
20 init ∀n, r, v. ¬vote_msg(n, r, v)
21 init ∀n, r, v. ¬decision(n, r, v)
22
23 action START_ROUND(r : round) {
24  assume r ≠ ⊥
25  start_round_msg(r) := true
26 }
27 action JOIN_ROUND(n : node, r : round) {
28  assume r ≠ ⊥
29  assume start_round_msg(r)
30  assume ¬∃r', r'', v. r' > r ∧ join_ack_msg(n, r', r'', v)
31  # find the maximal round in which n voted, and
32  # the corresponding vote; maxr = ⊥ and v is
33  # arbitrary when n never voted.
34  local maxr, v := max{(r', v') | vote_msg(n, r', v') ∧
35                                  r' < r}
36  join_ack_msg(n, r, maxr, v) := true
37 }
38 action PROPOSE(r : round, q : quorum) {
39  assume r ≠ ⊥
40  assume ∀v. ¬propose_msg(r, v)
41  # 1b from quorum q
42  assume ∀n. member(n, q) → ∃r', v.
43                              join_ack_msg(n, r, r', v)
44  # find the maximal round in which a node in the
45  # quorum reported voting, and the corresponding
46  # vote; v is arbitrary when no such node voted.
47  local maxr, v := max{(r', v') | ∃n. member(n, q) ∧
48                   join_ack_msg(n, r, r', v') ∧ r' ≠ ⊥}
49  propose_msg(r, v) := true   # propose value v
50 }
51 action VOTE(n : node, r : round, v : value) {
52  assume r ≠ ⊥
53  assume propose_msg(r, v)
54  assume ¬∃r', r'', v. r' > r ∧ join_ack_msg(n, r', r'', v)
55  vote_msg(n, r, v) := true
56 }
57 action LEARN(n : node, r : round, v : value, q : quorum) {
58  assume r ≠ ⊥
59  # 2b from quorum q
60  assume ∀n. member(n, q) → vote_msg(n, r, v)
61  decision(n, r, v) := true
62 }
```

Figure 4.6: RML model of Paxos consensus algorithm.

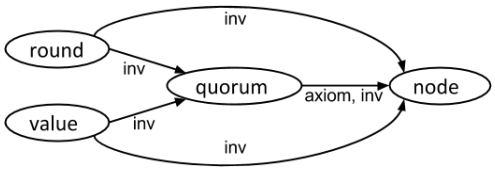


Figure 4.7: Quantifier alternation graph for EPR model of Paxos. The graph is obtained for the model of fig. 4.8, and the inductive invariant given by eqs. (4.10) to (4.16), (4.19), (4.21) and (4.22). The $\forall\exists$ edges come from: (i) the quorum axiom (fig. 4.6 line 8) — edge from quorum to node; (ii) eq. (4.13) — edges from round and value to quorum, and an edge from quorum to node (from the negation of the inductive invariant in the VC); and (iii) eq. (4.19) — edges from round, value and quorum to node.

```
1  relation left_round : node, round
2  relation joined_round : node, round
3  init ∀n, r. ¬left_round(n, r)
4  init ∀n, r. ¬joined_round(n, r)
5
6  action JOIN_ROUND(n : node, r : round) {
7   assume r ≠ ⊥
8   assume start_round_msg(r)
9   assume ¬left_round(n, r) # rewritten
10  local maxr, v := max{(r', v') | vote_msg(n, r', v') ∧
11                                  r' < r}
12  join_ack_msg(n, r, maxr, v) := true
13  # generated update for derived relations:
14  left_round(n, R) := left_round(n, R) ∨ R < r
15  joined_round(n, r) := true
16 }
17 action PROPOSE(r : round, q : quorum) {
18  assume r ≠ ⊥
19  assume ∀v. ¬propose_msg(r, v)
20  # rewritten to aviod quantifier alternation
21  assume ∀n. member(n, q) → joined_round(n, r)
22  # rewritten to use vote_msg instead of join_ack_msg
23  local maxr, v := max{(r', v') | ∃n. member(n, q)
24                   ∧vote_msg(n, r', v') ∧ r' < r}
25  propose_msg(r, v) := true
26 }
27 action VOTE(n : node, r : round, v : value) {
28  assume r ≠ ⊥
29  assume propose_msg(r, v)
30  assume ¬left_round(n, r) # rewritten
31  vote_msg(n, r, v) := true
32 }
```

Figure 4.8: Changes to the Paxos model that allow verification in EPR. Declarations that appear in Figure 4.6 are omitted, as well as the LEARN action which is left unmodified.

(iv) quorum — to model sets of nodes with pairwise intersection in a first-order abstraction.

While nodes and values are naturally uninterpreted, the rounds and the quorums are uninterpreted representations of interpreted concepts: integers and sets of nodes that intersect pairwise, respectively. We express some features that come from the desired interpretation using relations and axioms.

For rounds, we include a binary relation $\leq$, and axiomatize it to be a total order (as in Figure 3.10). Our model also includes a constant $\perp$ of sort round, which represents a special round that is not considered an actual round of the protocol, and instead serves as a special value used in the join-acknowledgment (1b) message when a node has not yet voted for any value. Accordingly, any action assumes that the round it involves is not $\perp$.

The quorum sort is used to represent sets of nodes that contain strictly more than half of the nodes, using the technique presented in Section 3.4.2. We introduce a membership relation *member* between nodes and quorums. An important property for Paxos is that any two quorums intersect. We capture this with an axiom in first-order logic (Figure 4.6 line 8).

State The state of the protocol consists of the set of messages the nodes have sent. We represent these using relations, where each tuple in a relation corresponds to a single message. As explained in Section 3.5, recording messages via relations (i.e., sets) is an abstraction of the network that is consistent with the messaging model we assume here, in which messages may be lost, duplicated, and reordered.[1] The relations *start_round_msg*, *join_ack_msg*, *propose_msg*, *vote_msg* correspond to the 1a, 1b, 2a, 2b phases of the algorithm, respectively. In modeling the algorithm, we assume all messages are sent to all nodes, so the relations do not contain destination fields. The *decision* relation captures the decisions learned by the nodes.

Actions The different atomic steps taken by the nodes in the protocol are modeled using actions. The START_ROUND action models phase 1a of the protocol, sending a start round message to all nodes. The JOIN_ROUND action models the receipt of a start round message and the transmission of a join-acknowledgment (1b) message. The PROPOSE action models the receipt of join-acknowledgment (1b) messages from a quorum of nodes, and the transmission of a propose (2a) message which proposes a value for a round. The VOTE action models the receipt of a propose (2a) message by a node, and voting for a value by sending a vote (2b)

[1] The fact that messages may be lost is captured by the behavior of the model in which a certain message is never received, i.e., the action that receives it is never executed. For safety verification (which is the focus of this chapter) this is straightforward. For liveness verification further bookkeeping is required, along with formalizing fairness assumptions, and this is explained in Chapter 7.

message. Finally, the LEARN action models learning a decision by some node, when a value is voted for by a quorum of nodes.

In these actions, sending a message is expressed by inserting the corresponding tuple to the corresponding relation. Different conditions (e.g., not joining a round if already joined higher round, properly reporting the previous votes, or appropriately selecting the proposed value) are expressed using assume statements. To prepare a join-acknowledgment message in JOIN_ROUND, as well as to propose a value in PROPOSE, a node needs to compute the maximal vote (performed by it or reported to it, respectively). This is done by a max operation (line 34 and line 47) that operates with respect to the order on rounds, and returns the round $\bot$ and an arbitrary value in case the set is empty. The $r, v := \max\{(r', v') \mid \varphi(r', v')\}$ operation is syntactic sugar for an assume of the following formula:

$$(r = \bot \wedge \forall r', v'.\ \neg\varphi(r', v')) \vee (r \neq \bot \wedge \varphi(r, v) \wedge \forall r', v'.\ \varphi(r', v') \rightarrow r' \leq r). \tag{4.9}$$

Note that if φ is a purely existentially quantified formula, then eq. (4.9) is alternation-free.

4.3.2 Inductive Invariant

The key safety property we wish to verify about Paxos is that only a single value can be decided (it can be decided at multiple rounds, as long as it is the same value). This is expressed by the following universally quantified formula:

$$\forall n_1, n_2 : \mathsf{node}, r_1, r_2 : \mathsf{round}, v_1, v_2 : \mathsf{value}.\ \mathit{decision}(n_1, r_1, v_1) \wedge \mathit{decision}(n_2, r_2, v_2) \rightarrow v_1 = v_2 \tag{4.10}$$

While the safety property holds in all the reachable states of the protocol, it is not inductive. That is, assuming that it holds is not sufficient to prove that it still holds after an action is taken. For example, consider a state s in which $\mathit{decision}(n_1, r_1, v_1)$ holds and there is a quorum q of nodes such that, for every node n in q, $\mathit{vote\_msg}(n, r_2, v_2)$ holds, with $v_2 \neq v_1$. Note that the safety property holds in s. However, a LEARN action introduces a transition from state s to a state s' in which both $\mathit{decision}(n_2, r_1, v_1)$ and $\mathit{decision}(n_2, r_2, v_2)$ hold, violating the safety property. This counterexample to induction does not indicate a violation of safety, but it indicates that the safety property needs to be strengthened in order to obtain an inductive invariant. We now describe such an inductive invariant.

Our inductive invariant contains, in addition to the safety property, the following rather

simple statements that are maintained by the protocol and are required for inductiveness:

$$\forall r : \mathsf{round}, v_1, v_2 : \mathsf{value}.\ \mathit{propose\_msg}(r, v_1) \wedge \mathit{propose\_msg}(r, v_2) \rightarrow v_1 = v_2 \tag{4.11}$$

$$\forall n : \mathsf{node}, r : \mathsf{round}, v : \mathsf{value}.\ \mathit{vote\_msg}(n, r, v) \rightarrow \mathit{propose\_msg}(r, v) \tag{4.12}$$

$$\begin{array}{l}\forall r : \mathsf{round}, v : \mathsf{value}.\\ \quad (\exists n : \mathsf{node}.\ \mathit{decision}(n, r, v)) \rightarrow \exists q : \mathsf{quorum}. \forall n : \mathsf{node}.\ \mathit{member}(n, q) \rightarrow \mathit{vote\_msg}(n, r, v)\end{array} \tag{4.13}$$

Equation (4.11) states that there is a unique proposal per round. Equation (4.12) states that a vote for v in round r is cast only when a proposal for v has been made in round r. Equation (4.13) states that a decision for v is made in round r only if a quorum of nodes have voted for v in round r.

In addition, the inductive invariant restricts the join-acknowledgment messages so that they faithfully represent the maximal vote (up to the joined round), or $\bot$ if there are no votes so far, and also asserts that there are no actual votes at round $\bot$:

$$\forall n : \mathsf{node},\ r, r' : \mathsf{round},\ v, v' : \mathsf{value}.\ \mathit{join\_ack\_msg}(n, r, \bot, v) \wedge r' < r \rightarrow \neg \mathit{vote\_msg}(n, r', v') \tag{4.14}$$

$$\forall n : \mathsf{node},\ r, r' : \mathsf{round},\ v : \mathsf{value}.\ \mathit{join\_ack\_msg}(n, r, r', v) \wedge r' \neq \bot \rightarrow r' < r \wedge \mathit{vote\_msg}(n, r', v) \tag{4.15}$$

$$\begin{array}{l}\forall n : \mathsf{node},\ r, r', r'' : \mathsf{round},\ v, v' : \mathsf{value}.\\ \quad \mathit{join\_ack\_msg}(n, r, r', v) \wedge r' \neq \bot \wedge r' < r'' < r \rightarrow \neg \mathit{vote\_msg}(n, r'', v')\end{array} \tag{4.16}$$

$$\forall n : \mathsf{node}, v : \mathsf{value}.\ \neg \mathit{vote\_msg}(n, \bot, v) \tag{4.17}$$

The properties stated so far are rather straightforward, and are usually not even mentioned in paper proofs or explanations of the protocol. The key to the correctness argument of the protocol is the observation that whenever the owner of round r_2 proposes a value in r_2, it cannot miss any value that is choosable at a lower round. Namely, whenever a value v_2 is proposed at round r_2, then in all rounds r_1 prior to r_2, no other value $v_1 \neq v_2$ is choosable. The property that no $v_1 \neq v_2$ is choosable at r_1 is captured in the inductive invariant by the requirement that in any quorum of nodes, there must be at least one node that has already left round r_1 (i.e., joined a higher round), and did not vote for v_1 at r_1 (and hence will also not vote for it in the future). Formally, this is:

$$\begin{array}{l}\forall r_1, r_2 : \mathsf{round}, v_1, v_2 : \mathsf{value}, q : \mathsf{quorum}.\\ \quad \mathit{propose\_msg}(r_2, v_2) \wedge r_1 < r_2 \wedge v_1 \neq v_2 \rightarrow\ \exists n : \mathsf{node}, r', r'' : \mathsf{round}, v : \mathsf{value}.\\ \qquad \mathit{member}(n, q) \wedge \neg \mathit{vote\_msg}(n, r_1, v_1) \wedge r' > r_1 \wedge \mathit{join\_ack\_msg}(n, r', r'', v)\end{array} \tag{4.18}$$

The fact that this property is maintained by the protocol is obtained by the proposal

mechanism and the interaction between phase 1 and phase 2 (see [188, Appendix B] for a detailed explanation).

Equations (4.10) to (4.18) define an inductive invariant that proves the safety of the Paxos model of Figure 4.6. However, the verification condition for this inductive invariant contains cyclic quantifier alternations, and is therefore outside of EPR. We now review the quantifier alternations in the verification condition, which originate both from the model and from the inductive invariant.

In the model, the axiomatization of quorums (Figure 4.6 line 8) introduces a $\forall\exists$-edge from quorum to node. In addition, the assumption in the PROPOSE action that join-acknowledgment messages were received from a quorum of nodes (line 42) introduces $\forall\exists$-edges from node to round and from node to value.

In the inductive invariant, only eqs. (4.13) and (4.18) include quantifier alternations (the rest are universally quantified). Equation (4.13) has quantifier structure[2] $\forall$ round, value $\exists$ quorum $\forall$ node. Recall that the inductive invariant appears both positively and negatively in the verification conditions, so eq. (4.13) adds $\forall\exists$-edges from round to quorum and from value to quorum (from the positive occurrence), as well as an edge from quorum to node (from the negative occurrence). While the latter coincides with the edge that comes from the quorum axiomatization (line 8), the former edges closes a cycle in the quantifier alternation graph. Equation (4.18) has quantifier prefix $\forall$ round, value, quorum $\exists$ node, round, value. Thus, it introduces 9 edges in the quantifier alternation graph, including self-loops at round and value. In conclusion, while the presented model in first-order logic has an inductive invariant in first-order logic, the resulting verification condition is outside of EPR.

4.4 Paxos in EPR

The quantifier alternation graph of the model of Paxos described in Section 4.3 contains cycles. To obtain a safety proof of Paxos in EPR, we apply the methodology described in Section 4.1 to transform this model in a way that eliminates the cycles from the quantifier alternation graph. The resulting changes to the model are presented in Figure 4.8, and the rest of this section explains them step by step.

[2] Note that the local existential quantifier in $(\exists n : \mathsf{node}.\ \mathit{decision}(n, r, v))$ does not affect the quantifier alternation graph.

4.4.1 Derived Relation for Left Rounds

We start by addressing the quantifier alternation that appears in eq. (4.18) as part of the inductive invariant. We observe that the following existentially quantified formula appears both as a subformula there, and in the conditions of the JOIN_ROUND and the VOTE actions (Figure 4.6 lines 30 and 54):

$$\psi_1(n, r) = \exists r', r'' : \mathsf{round}, v : \mathsf{value}.\ r' > r \wedge \mathit{join\_ack\_msg}(n, r', r'', v)$$

This formula captures the fact that node n has joined a round higher than r, which means it has promised to never participate in round r in any way, i.e., it will neither join nor vote in round r. We add a derived relation called *left_round* to capture ψ_1, so that $\mathit{left\_round}(n, r)$ captures the fact that node n has left round r. The formula ψ_1 is in the class of formulas handled by the scheme described in Section 4.1.3, and thus we obtain the initial condition and update code for *left_round*. The result appears in Figure 4.8 lines 3 and 14.

Rewriting (steps (3) and (4)) Using the *left_round* relation, we rewrite the conditions of the JOIN_ROUND and VOTE actions (Figure 4.8 lines 9 and 30). These rewrites are trivially sound as explained in Section 4.1.2 (with a trivial rewrite condition). We also rewrite eq. (4.18) as:

$$\begin{aligned} &\forall r_1, r_2 : \mathsf{round}, v_1, v_2 : \mathsf{value}, q : \mathsf{quorum}.\ \mathit{propose\_msg}(r_2, v_2) \wedge r_1 < r_2 \wedge v_1 \neq v_2 \rightarrow \\ &\quad \exists n : \mathsf{node},\ \mathit{member}(n, q) \wedge \neg \mathit{vote\_msg}(n, r_1, v_1) \wedge \mathit{left\_round}(n, r_1) \end{aligned} \tag{4.19}$$

Equation (4.19) contains less quantifer alternations than eq. (4.18), and it will be part of the inductive invariant that will eventually be used to prove safety.

4.4.2 Derived Relation for Joined Rounds

After the previous transformation, the verification condition is still not stratified. The reason is the combination of eq. (4.19) and the condition of the PROPOSE action (Figure 4.6 line 42):

$$\forall n : \mathsf{node}.\ \mathit{member}(n, q) \rightarrow \exists r' : \mathsf{round},\ v : \mathsf{value}.\ \mathit{join\_ack\_msg}(n, r, r', v).$$

While each of these introduces quantifier alternations that are stratified when viewed separately, together they form cycles. Equation (4.19) introduces $\forall\exists$-edges from round and value to node, while the PROPOSE condition introduces edges from node to round and value. The PROPOSE condition expresses the fact that every node in the quorum q has joined

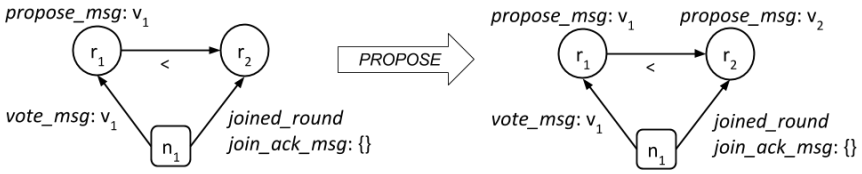


Figure 4.20: Counterexample to induction for EPR model of Paxos after the first attempt. The counterexample contains one node n_1 (depicted as square), two rounds $r_1 < r_2$ (depicted as circles), two values v_1, v_2, and a single quorum (that contains n_1). The figure displays the the *join_ack_msg*, *joined_round*, *propose_msg*, *vote_msg* relations, as well as the $<$ relation (derived from $\leq$). The action occurring in the counterexample is PROPOSE, where an arbitrary value is proposed for r_2, even though node n_1 voted for v_1 in r_1. This erroneous behavior occurs due to the fact that the representation invariant of *joined_round* is violated in the pre-state of the counterexample: $joined\_round(n_1, r_2)$ holds, eventhough there is no corresponding *join_ack_msg* entry. This allows a PROPOSE action to erroneously propose v_2 for r_2, in spite of the fact that v_1 is choosable at r_1.

round r by sending a join-acknowledgment (1b) message to round r. However, because the join-acknowledgment message contains two more fields (representing the node's maximal vote so far), the condition existentially quantifies over them. To eliminate this existential quantification and remove the cycles, we add a derived relation, called *joined_round*, that captures the projection of *join_ack_msg* over its first two components, given by the formula:

$$\psi_2(n, r) = \exists r' : \mathsf{round}, v : \mathsf{value}.\ join\_ack\_msg(n, r, r', v)$$

This binary relation over nodes and rounds records the sending of join-acknowledgment messages, ignoring the maximal vote so far reported in the message. Thus, $joined\_round(n, r)$ captures the fact that node n has agreed to join round r. The formula ψ_2 is in the class of formulas handled by the scheme of Section 4.1.3, and thus we obtain the initial condition and update code for *joined_round*, as it appears in Figure 4.8 lines 4 and 15.

Rewriting (steps (3) and (4)): first attempt We rewrite the condition of the PROPOSE action to use *joined_round* instead of ψ_2. (The rewrite condition is again trivial, ensuring soundness.) The result appears in Figure 4.8 line 21, and is purely universally quantified. When considering the transformed model, and the candidate inductive invariant given by the conjunction of eqs. (4.10) to (4.17) and (4.19), the resulting quantifier alternation graph is acyclic. This means that the verification condition is in EPR and hence decidable to check. However, it turns out that this candidate invariant is not inductive, and the check yields a counterexample to induction.

The counterexample is depicted in Figure 4.20. The counterexample shows a PROPOSE action that leads to a violation of eq. (4.19). The example contains a single node n_1 (which

forms a quorum) that has voted for value v_1 in round r_1, and yet a different value v_2 is proposed for a later round r_2 (based on the quorum composed only of n_1) which leads to a violation of eq. (4.19). The PROPOSE action is enabled since $\mathit{joined\_round}(n_1, r_2)$ holds. However, an arbitrary value is proposed since *join_ack_msg* is empty. The root cause for the counterexample is that the inductive invariant does not capture the connection between *joined_round* and *join_ack_msg*, so it allows a state in which a node n_1 has joined round r_2 according to *joined_round* (i.e., $\mathit{joined\_round}(n_1, r_2)$ holds), but it has not joined it according to *join_ack_msg* (i.e., $\exists r', v.\ \mathit{join\_ack\_msg}(n_1, r_2, r', v)$ does not hold). Note that the counterexample is spurious, in the sense that it does not represent a reachable state. However, for a proof by an inductive invariant, we must eliminate this counterexample nonetheless.

Rewriting (steps (3) and (4)): second attempt One obvious way to eliminate the counterexample discussed above is to add the representation invariant of *joined_round* to the inductive invariant. However, this will result in a cyclic quantifier alternation, causing the verification condition to be outside of EPR. Instead, we will eliminate this counterexample by rewriting the code of the PROPOSE action, relying on an auxiliary invariant to verify the rewrite, as explained in Section 4.1.2. We observe that the mismatch between *joined_round* and *join_ack_msg* is only problematic in this example because node n_1 voted in $r_1 < r_2$. While the condition of the PROPOSE action is supposed to ensure that the max operation considers past votes of all nodes in the quorum, such a scenario where the *joined_round* is inconsistent with *join_ack_msg* makes it possible for the PROPOSE action to overlook past votes, which is the case in this counterexample. Our remedy is therefore to rewrite the max operation (which is implemented by an assume command, as explained before) to consider the votes directly by referring to the *vote* messages instead of the *join-acknowledgment* messages that report them. We first formally state the rewrite and then justify its correctness.

As before, we rewrite the condition of the PROPOSE action to use *joined_round* (Figure 4.8 line 21). In addition, we rewrite the max operation in Figure 4.6 line 47, i.e., $\max\{(r', v') \mid \varphi_1(r', v')\}$ where

$$\varphi_1(r', v') = \exists n.\ \mathit{member}(n, q) \wedge \mathit{join\_ack\_msg}(n, r, r', v') \wedge r' \neq \bot$$

to the max operation in Figure 4.8 line 24, i.e., $\max\{(r', v') \mid \varphi_2(r', v')\}$ where

$$\varphi_2(r', v') = \exists n.\ \mathit{member}(n, q) \wedge \mathit{vote\_msg}(n, r', v') \wedge r' < r$$

The key to the correctness of this change is that a join-acknowledgment message from node n to round r contains its maximal vote prior to round r, and once the node sent this message, it will never vote in rounds smaller than r. Therefore, while the original PROPOSE action considers the maximum over votes reflected by join-acknowledgment messages from a quorum, looking at the *actual votes* from the quorum in rounds prior to r (as captured by the *vote_msg* relation) yields the same maximum.

Formally, we establish the rewrite condition of step (3) given by eq. (4.3) using an auxiliary invariant I_{aux}, defined as the conjunction of eqs. (4.11), (4.12) and (4.14) to (4.17). This invariant captures the connection between *join_ack_msg* and *vote_msg* explained above. The invariant I_{aux} is inductive for the original model of Figure 4.6, and its verification conditions are in EPR (the resulting quantifier alternation graph is acyclic). Second, we prove that under the assumption of I_{aux} and the condition $\forall n.\, member(n, q) \rightarrow \exists r', v.join\_ack\_msg(n, r, r', v)$ (Figure 4.6 line 42), the operation $\max\{(r', v') \mid \varphi_1(r', v')\}$ is equivalent to the operation $\max\{(r', v') \mid \varphi_2(r', v')\}$ (recall that both translate to assume's according to eq. (4.9)). This check is also in EPR. In conclusion, we are able to establish the rewrite condition using EPR checks: both for proving I_{aux}, and for proving eq. (4.3).

Inductive invariant After the above rewrite, the conjunction of eqs. (4.10) to (4.17) and (4.19) is still not an inductive invariant, due to a counterexample to induction in which a node has joined a higher round according to *joined_round*, but has not left a lower round according to *left_round*. As before, the counterexample is inconsistent with the representation invariants. However, this time the counterexample (and another similar one) can be eliminated by strengthening the inductive invariant with the following facts, which are implied by the representation invariants of *joined_round* and *left_round*:

$$\forall n : \mathsf{node},\, r_1, r_2 : \mathsf{round}.\; r_1 < r_2 \wedge joined\_round(n, r_2) \rightarrow left\_round(n, r_1) \tag{4.21}$$

$$\forall n : \mathsf{node},\, r, r' : \mathsf{round},\, v : \mathsf{value}.\; join\_ack\_msg(n, r, r', v) \rightarrow joined\_round(n, r) \tag{4.22}$$

Both are purely universally quantified and therefore do not affect the quantifier alternation graph.

Finally, the invariant given by the conjunction of eqs. (4.10) to (4.17), (4.19), (4.21) and (4.22) is indeed an inductive invariant for the transformed model. Figure 4.7 depicts the quantifier alternation graph of the resulting verification conditions. This graph is acyclic, and so the invariant can be verified in EPR. This invariant proves the safety of the transformed model (Figure 4.8). Using Theorem 4.4, the safety of the original Paxos model (Figure 4.6)

follows.

4.5 Multi-Paxos

In this section we describe our verification of Multi-Paxos. Multi-Paxos is an implementation of state-machine replication (SMR): nodes run a sequence of instances of the Paxos algorithm, where the i^{th} instance is used to decide on the i^{th} command in the sequence of commands executed by the machine. For efficiency, when starting a round, a node uses a single message to simultaneously do so in all instances. In response, each node sends a join-acknowledgment message that reports its maximal vote in each instance; this message has finite size as there are only finitely many instances in which the node ever voted. The key advantage over using one isolated incarnation of Paxos per instance is that when a unique node takes on the responsibility of starting a round and proposing commands, only phase 2 of Paxos has to be executed for every proposal.

Below we provide a description of the EPR verification of Multi-Paxos. The main change compared to the Paxos consensus algorithm is in the modeling of the algorithm in first-order logic, where we use the technique of Section 3.4.1 to model higher-order concepts. The transformation to EPR is essentially the same as in Section 4.4, using the same derived relations and rewrites.

4.5.1 Protocol Model in First-Order Logic

Multi-Paxos uses the same message types as the basic Paxos algorithm, but when a node joins a round, it sends a join-acknowledgment message (1a) with its maximal vote for each instance (there will only be a finite set of instances for which it actually voted). Upon receipt of join-acknowledgment messages from a quorum of nodes that join round r, the owner of round r determines the instances for which it is obliged to propose a value (because it may be choosable in a prior round) and proposes values accordingly for these instances. Other instances are considered *available*, and in these instances the round owner can propose any value. Next, the owner of round r can propose commands in available instances (in response to client requests, which are abstracted in our model). This means that the PROPOSE action of Paxos is split into two actions in Multi-Paxos: one that processes the join-acknowledgment messages from a quorum, and another that proposes new values.

Our model of Multi-Paxos in first-order logic appears in Figure 4.23. We explain the key differences compared to the Paxos model from Figure 4.6.

```
1  sort node
2  sort quorum
3  sort round
4  sort value
5  sort instance
6  sort votemap
7
8  relation ≤ : round, round
9  axiom Γ_{total order}[≤]
10 individual ⊥ : round
11 relation member : node, quorum
12 axiom ∀q1, q2 : quorum. ∃n : node. member(n, q1) ∧ member(n, q2)
13
14 relation start_round_msg : round
15 relation join_ack_msg : node, round, votemap
16 relation propose_msg : instance, round, value
17 relation active : round
18 relation vote_msg : node, instance, round, value
19 relation decision : node, instance, round, value
20 function roundof : votemap, instance → round
21 function valueof : votemap, instance → value
22
23 init ∀r. ¬start_round_msg(r)
24 init ∀n, r, m. ¬join_ack_msg(n, r, m)
25 init ∀i, r, v. ¬propose_msg(i, r, v)
26 init ∀r. ¬active(r)
27 init ∀n, i, r, v. ¬vote_msg(n, i, r, v)
28 init ∀r, v. ¬decision(r, v)
29
30 action START_ROUND(r : round) {
31     assume r ≠ ⊥
32     start_round_msg(r) := true
33 }
34 action JOIN_ROUND(n : node, r : round) {
35     assume r ≠ ⊥ ∧ start_round_msg(r) ∧ ¬∃r', m. r' > r ∧ join_ack_msg(n, r', m)
36     local m : votemap := *
37     assume ∀i. (roundof(m, i), valueof(m, i)) = max {(r', v') | vote_msg(n, i, r', v') ∧ r' < r}
38     join_ack_msg(n, r, m) := true
39 }
40 action INSTATE_ROUND(r : round, q : quorum) {
41     assume r ≠ ⊥
42     assume ¬active(r)
43     assume ∀n. member(n, q) → ∃m. join_ack_msg(n, r, m)
44     local m : votemap := *
45     assume ∀i. (roundof(m, i), valueof(m, i)) = max {(r', v') | ∃n, m'. member(n, q) ∧ join_ack_msg(n, r, m') ∧
46                                                                  r' = roundof(m', i) ∧ v' = valueof(m', i) ∧ r' ≠ ⊥ }
47     active(r) := true
48     propose_msg(I, r, V) := propose_msg(I, r, V) ∨ (roundof(m, I) ≠ ⊥ ∧ V = valueof(m, I))
49 }
50 action PROPOSE_NEW_VALUE(r : round, i : instance, v : value) {
51     assume r ≠ ⊥
52     assume active(r) ∧ ∀v. ¬propose_msg(i, r, v)
53     propose_msg(i, r, v) := true
54 }
55 action VOTE(n : node, i : instance, r : round, v : value) {
56     assume r ≠ ⊥ ∧ propose_msg(i, r, v) ∧ ¬∃r', m. r' > r ∧ join_ack_msg(n, r', m)
57     vote_msg(n, i, r, v) := true
58 }
59 action LEARN(n : node, i : instance, r : round, v : value, q : quorum) {
60     assume r ≠ ⊥
61     assume ∀n. member(n, q) → vote_msg(n, i, r, v)
62     decision(n, i, r, v) := true
63 }
```

Figure 4.23: RML model of Multi-Paxos.

State We extend the vocabulary of the Paxos model with two new sorts: instance and votemap. The instance sort represents instances, and the *propose_msg*, *vote_msg* and *decision* relations are extended to include an instance in each tuple. In practice, instances may be natural numbers that give the order in which commands of the state machine must be executed by each replica. However, we are only interested in proving consistency (i.e., that decisions are unique per instance), and the consistency proof does not depend on the instances being ordered. Therefore, our model does not include a total order over instances.

The votemap sort models maps from instances to (round, value) pairs, which are passed in the join-acknowledgment messages (captured by the relation *join_ack_msg* : node, round, votemap). We use the encoding explained in Section 3.4.1, and add two functions that allow access to the content of a votemap: *roundof* : votemap, instance → round and *valueof* : votemap, instance → value.

We also add a new relation *active* : round, to support the splitting of the PROPOSE action into two actions. A round is considered *active* once the round owner has received a quorum of join-acknowledgment messages, and then it can start proposing new values for available instances.

Actions The START_ROUND action is identical to Paxos, and the VOTE and LEARN actions are identical except they are now parameterized by an instance. The JOIN_ROUND action is identical in principle, except now it must obtain and deliver a votemap that maps each instance to the maximal vote of the node (and $\bot$ for instances in which the node did not vote). To express the computation of the maximal vote of every instance i, we use an assume statement in line 37 of the form $\forall i.\ (\mathit{roundof}(m,i), \mathit{valueof}(m,i)) = \max\{(r',v') \mid \varphi(i,r',v')\}$ which follows a non-deterministic choice of m : votemap. The assume statement is realized by the following formula in the transition relation (which is an adaptation of eq. (4.9) to account for multiple instances):

$$\begin{aligned}\forall i.\ &\big(\mathit{roundof}(m,i) = \bot \wedge \forall r', v'.\ \neg\varphi(i,r',v')\big)\ \vee \\ &\big(\mathit{roundof}(m,i) \neq \bot \wedge \varphi(i, \mathit{roundof}(m,i), \mathit{valueof}(m,i)) \wedge \\ &\quad \forall r', v'.\ \varphi(i,r',v') \rightarrow r' \leq \mathit{roundof}(m,i)\big)\end{aligned} \tag{4.24}$$

Note that for line 37, φ is quantifier free, so eq. (4.24) is purely universally quantified.

The most notable difference in the actions is that, in Multi-Paxos, the PROPOSE action is split into two actions: INSTATE_ROUND and PROPOSE_NEW_VALUE. INSTATE_ROUND processes the join-acknowledgment messages from a quorum, and PROPOSE_NEW_VALUE

proposes new values, modeling the fact that only phase 2 is repeated for every instance.

The INSTATE_ROUND action takes place when the owner of a round received join-acknowledgment messages from a quorum of nodes. The owner then finds the maximal vote reported in the messages for each instance, which is done in line 45. This is realized using eq. (4.24). Observe that φ here contains existential quantifiers over node and votemap. This introduces quantifier alternations that result in $\forall\exists$ edges from instance to both node and votemap. Fortunately, these edges do not create cycles in the quantifier alternation graph. Next, in line 47, the round is marked as active, and in line 48, all obligatory proposals are made.

The PROPOSE_NEW_VALUE action models the proposal of a new value in an available instance, after the INSTATE_ROUND action took place and the round is made active. This action occurs due to client requests, which are abstracted in our model. Therefore, the only preconditions of this action in our model are that the INSTATE_ROUND action took place (captured by the *active* relation), and that the instance is still available, i.e., that the round owner has not proposed any other value for it. These preconditions are expressed in line 52. In practice, a round owner will choose the next available instance (according to some total order). However, since this is not necessarily for correctness, our model completely abstracts the total order over instances, and allows a new value to be proposed in any available instance.

4.5.2 Inductive Invariant

The safety property we wish to prove for Multi-Paxos is that each Paxos instance is safe. Formally:

$$\begin{aligned}&\forall i : \mathsf{instance}, n_1, n_2 : \mathsf{node}, r_1, r_2 : \mathsf{round}, v_1, v_2 : \mathsf{value}.\\ &\qquad\qquad decision(n_1, i, r_1, v_1) \wedge decision(n_2, i, r_2, v_2) \rightarrow v_1 = v_2\end{aligned} \tag{4.25}$$

Equation (4.25) generalizes eq. (4.10) by universally quantifying over all instances. The inductive invariant that proves safety contains similarly generalized versions of eqs. (4.11) to (4.17), where the *join_ack_msg* relation is adjusted to contain a votemap element instead of a (round, value) pair as the message content. In addition, the inductive invariant asserts that proposals are only made for active rounds:

$$\forall i : \mathsf{instance}, r : \mathsf{round}, v : \mathsf{value}.\ propose\_msg(i, r, v) \rightarrow active(r) \tag{4.26}$$

Finally, the inductive invariant for Multi-Paxos contains a generalized version of eq. (4.18), that states that if round r is active, then any value not proposed in r is also not choosable for lower rounds. Formally:

$$\begin{aligned}\forall i : \mathsf{instance}, r_1, r_2 : \mathsf{round}, v : \mathsf{value}, q : \mathsf{quorum}.\\ \mathit{active}(r_2) \wedge r_1 < r_2 \wedge \neg \mathit{propose\_msg}(i, r_2, v) \rightarrow \exists n : \mathsf{node}, r' : \mathsf{round}, m : \mathsf{votemap}.\\ \mathit{member}(n, q) \wedge \neg \mathit{vote\_msg}(n, r_1, v) \wedge r' > r_1 \wedge \mathit{join\_ack\_msg}(n, r', m)\end{aligned} \tag{4.27}$$

This generalizes eq. (4.18), in which the condition is that another value is proposed, since proposals are unique by eq. (4.11). Equations (4.25) to (4.27) together with eqs. (4.11) to (4.17) (generalized by universally quantifying over all instances) provide an inductive invariant for the model of Figure 4.23.

4.5.3 Transformation to EPR

As with the Paxos model of Figure 4.6, the resulting verification conditions for the Multi-Paxos model are outside of EPR, and the model must be transformed to allow EPR verification. The required transformations are essentially identical to the Paxos model (the new sorts do not appear in any cycles in the quantifier alternation graph), where we define the *left_round* relation by:

$$\psi_1(n, r) = \exists r', m.\ r' > r \wedge \mathit{join\_ack\_msg}(n, r', m)$$

and the *joined_round* relation by:

$$\psi_2(n, r) = \exists m.\ \mathit{join\_ack\_msg}(n, r, m)$$

The *left_round* relation is used to rewrite eq. (4.27) in precisely the same way it was used to rewrite eq. (4.18). In addition, we rewrite the max operation in line 45 of Figure 4.23 to use *vote_msg* instead of *join_ack_msg*, which is exactly the same change that was required to verify the Paxos model. Thus, the transformations to EPR are in this case completely reusable, and allow EPR verification of Multi-Paxos.

4.6 Paxos Variants

In this section we briefly describe our verification in EPR of several variants of Paxos. More elaborate explanations are provided in Appendix A. In all cases, the transformations to

EPR of Section 4.4 were employed (with slight modifications), demonstrating the reusability of the derived relations and rewrites across different Paxos variants. Interestingly, some variants require different quorum axiomatizations that are slightly more complex than the simple intersection property used by Paxos and Multi-Paxos. However, in all cases the axiomatization required for the proof was expressible in EPR.

4.6.1 Vertical Paxos

Vertical Paxos [141] is a variant of Paxos whose set of participating nodes and quorums (called the configuration) can be changed dynamically by an external reconfiguration master. By using reconfiguration to replace failed nodes, Vertical Paxos makes Paxos reliable in the long-term. The reconfiguration master dynamically assigns configurations to rounds, which means that each round uses a different set of quorums. A significant algorithmic complication is that old configurations must be eventually retired in practice. This is achieved by having the nodes inform the master when a round r becomes *complete*, meaning that r holds all the necessary information about choosable values in lower rounds. The master tracks the highest complete round and passes it on to each new configuration to indicate that lower rounds need not be accessed. Rounds below the highest complete round can then be retired safely.

We model configurations in first-order logic by introducing a new sort config that represents a set of quorums, with a suitable member relation called *quorum_in*. Moreover, we change the axiomatization of quorums to only require that quorums of the same configuration intersect:

$$\forall c : \mathsf{config}, q_1, q_2 : \mathsf{quorum}.\ \mathit{quorum\_in}(q_1, c) \land \mathit{quorum\_in}(q_2, c) \rightarrow$$
$$\exists n : \mathsf{node}.\ \mathit{member}(n, q_1) \land \mathit{member}(n, q_2)$$

To model the complete round associated to each configuration, we introduce a function symbol $\mathit{complete\_of} : \mathsf{config} \rightarrow \mathsf{round}$. This function symbol introduces additional cycles to the quantifier alternation graph. The transformation to EPR replaces the *complete_of* function by a derived relation defined by the formula $\mathit{complete\_of}(c) = r$. With this derived relation, we can rewrite the model and invariant so that the function no longer appears in the verification condition, and hence the cycles that it introduced are eliminated. Other than that, the transformation to EPR uses the same derived relations and rewrites of Section 4.4 (in fact, it only requires the *left_round* derived relation). A full description appears in Appendix A.1.

4.6.2 Fast Paxos

Fast Paxos [138] is a variant of Multi-Paxos that improves its latency. The key idea is to mark some of the rounds as *fast* and allow any node to directly propose values in these rounds without going through the round owner. As a result, multiple values can be proposed, as well as voted for, in the same (fast) round. In order to maintain consistency, Fast Paxos uses two kinds of quorums: *classic* quorums and *fast* quorums. The quorums have the property that any two classic quorums intersect, and any classic quorum and *two* fast quorums intersect. Now, in a propose action receiving join-acknowledgment messages from the classic quorum q with a maximal vote reported in a fast round *maxr*, multiple different values may be reported by nodes in q for *maxr*. To determine which one may be choosable in *maxr*, a node will check whether there exists a *fast* quorum f such that all the nodes in $q \cap f$ reported voting v in round *maxr*. If yes, by the intersection property of quorums, only this value v may be choosable in *maxr*, and hence must be proposed.

We model fast quorums with an additional sort $\mathsf{quorum_f}$ (and relation f_member), and axiomatize its intersection property as:

$$\forall q : \mathsf{quorum}, f_1, f_2 : \mathsf{quorum_f}.\ \exists n : \mathsf{node}.\ member(n, q) \wedge f\_member(n, f_1) \wedge f\_member(n, f_2)$$

The rest of the details of the model appear in Appendix A.2. The transformation to EPR is similar to Section 4.4. This includes the rewrite of the new condition for proposing a value. Interestingly, in this case, the verification of the latter rewrite is not in EPR when considering the formulas as a whole, but it is in EPR when we consider only the subformulas that change, as supported by the methodology presented in Section 4.1.2.

4.6.3 Flexible Paxos

Flexible Paxos [107] extends Paxos based on the observation that it is only necessary for safety that every phase 1 quorum intersects with every phase 2 quorum (quorums of the same phase do not have to intersect). This allows greater flexibility and introduces an adjustable trade-off between the cost of deciding on new values and the cost of starting a new round. For example, in a system with 10 nodes, one may use sets of 8 nodes as phase 1 quorums, and sets of 3 nodes phase 2 quorums. EPR verification of Flexible Paxos is essentially the same as for normal Paxos, except we introduce two quorum sorts (for phase 1 and phase 2),

and adapt the intersection axiom to:

$$\forall q_1 : \mathsf{quorum}_1, q_2 : \mathsf{quorum}_2.\ \exists n : \mathsf{node}.\ member_1(n, q_1) \wedge member_2(n, q_2)$$

The detailed model and the adjusted invariant appear in Appendix A.3.

4.6.4 Stoppable Paxos

Stoppable Paxos [140] extends Multi-Paxos with the ability for a node to propose a special stop command in order to stop the algorithm, with the guarantee that if the stop command is decided in instance i, then no command is ever decided at an instance $j > i$. Stoppable Paxos therefore enables Virtually Synchronous system reconfiguration [28, 47]: Stoppable Paxos stops in a state known to all participants, which can then start a new instance of Stoppable Paxos in a new configuration (e.g., in which participants have been added or removed); moreover, no pending commands can leak from a configuration to the next, as only the final state of the command sequence is transfered from one configuration to the next.

Stoppable Paxos may be the most intricate algorithm in the Paxos family: as acknowledged by Lamport et al. [140], "getting the details right was not easy". The main algorithmic difficulty in Stoppable Paxos is to ensure that no command may be decided after a stop command while at the same time allowing a node to propose new commands without waiting, when earlier commands are still in flight (which is important for performance). In contrast, in the traditional approach to reconfigurable SMR [142], a node that has c outstanding command proposals may cause up to c commands to be decided after a stop command is decided; those commands need to be passed-on to the next configuration and may contain other stop commands, adding to the complexity of the reconfiguration system.

Before proposing a command in an instance in Stoppable Paxos, a node must check if other instances have seen stop commands proposed and in which round. This creates a non-trivial dependency between rounds and instances, which are mostly orthogonal concepts in other variants of Paxos. This manifest as the instance sort having no incoming edge in the quantifier alternation graph in other variants, while such edges appear in Stoppable Paxos. Interestingly, the rule given by Lamport et al. to propose commands results in verification conditions that are not in EPR, and rewriting seems difficult. However, we found an alternative rule which results in EPR verification conditions. This alternative rule soundly overapproximates the original rule (and introduces new behaviors), and, as we have verified (in EPR), it also maintains safety. The details of the modified rule and its verification appear

	EPR		$\mathbf{I}_{\text{aux}}$		**RW**		**FOL** $-$ 2			**FOL** $-$ 4			**FOL** $-$ 8			**FOL** $-$ 16		
Protocol	μ	σ	μ	σ	μ	σ	μ	σ	τ	μ	σ	τ	μ	σ	τ	μ	σ	τ
Paxos	1.0	0.1	0.7	0.004	0.5	0.002	1.2	0.2	0	2.5	1.2	0	86	112	2	278	65	9
Multi-Paxos	1.2	0.1	0.8	0.004	0.6	0.01	1.2	0.1	0	1.8	0.4	0	107	129	3	229	110	7
Vertical Paxos	2.2	0.2	–	–	–	–	27	10	0	47	32	0	209	104	5	274	78	9
Fast Paxos	4.7	1.6	0.9	0.004	0.6	0.002	3.7	0.9	0	19	25	0	127	97	2	300	0	10
Flexible Paxos	1.0	0.01	0.7	0.003	0.5	0.01	1.1	0.1	0	2.7	2.1	0	100	120	2	275	75	9
Stoppable Paxos	3.8	0.9	1.0	0.04	0.6	0.01	186	123	5	300	0	10	300	0	10	300	0	10

Figure 4.28: Run times (in seconds) of checking verification conditions using Ivy and Z3. Each experiment was repeated 10 times (with random seeds used for Z3's heuristics). μ reports the mean time, σ reports the standard deviation, and τ reports the number of runs that timed out at 300 seconds (where this occurred). **EPR** is the verification of the EPR model. $\mathbf{I}_{\text{aux}}$ is the verification of the auxiliary invariant. **RW** is the verification of the rewrite condition. **FOL** $-$ N is the run time of semi-bounded verification of the first-order model, with bound 2 for values and bound N for rounds (in all variants, bounding the number of values and rounds eliminates cycles from the quantifier alternation graph).

in Appendix A.4.

For Stoppable Paxos, liveness also requires a careful argument, since one must show that nodes cannot be stuck in a situation where no command can be decided due to a stop command that is perceived to be choosable, while at the same time the stop command never gets decided either. While this chapter is focused on safety proofs, in Chapter 7 we develop a technique for proving liveness, and also prove liveness of Stoppable Paxos in EPR. Our liveness proof in Chapter 7 leverages and reuses the techniques developed here for modeling in EPR.

4.7 Evaluation

We have implemented our methodology using the Ivy deductive verification system [171, 186], which uses the Z3 theorem prover [61] for checking verification conditions. Figure 4.28 lists the run times for the automated checks performed when verifying the different Paxos variants. The experiments were performed on a laptop running Linux, with a Core-i7 1.8 GHz CPU. Z3 version 4.5.0 was used, along with the latest version of Ivy at the time (commit `7ce6738`). Z3 uses heuristics that employ randomness. Therefore, each experiment was repeated 10 times using random seeds. We report the mean times, as well as the standard deviation and the number of experiments which timed out at 300 seconds (these are included in the mean). The Ivy files used for these experiments are available at the supplementary web page

of [187][3].

For each Paxos protocol, Figure 4.28 reports the time for checking the inductive invariant that proves the safety of the EPR model, as well as the times required to verify auxiliary invariants and rewrite conditions (see Section 4.1.2). We also report on the times required to check the inductive invariant for the original first-order logic models, using semi-bounded verification. In all variants, quantifier alternation cycles can be eliminated by bounding the number of values and rounds. We bound the number of values to 2, and vary the bound on the number of rounds between 2 and 16 (where most runs time out).

As Figure 4.28 demonstrates, using our methodology for EPR verification results in verification conditions that are solved by Z3 in a few seconds, with no timeouts, and with negligible variance among runs with different random seeds. In contrast, when using semi-bounded verification, the run time quickly increases as we attempt to increase the number of rounds. Moreover, the variance in run time increases significantly, causing an unpredictable experience for verification users. We have also attempted to use unbounded verification for the first-order logic models, but Z3 diverged in this case for all variants. This did not change when we increased the timeout from 300 seconds to a few hours.

Our empirical results demonstrate that Z3 is stable on EPR formulas, as the performance has negligible variance when running with different random seeds. This point is one of the key advantages of EPR for verification, compared to other (undecidable) logics. During the verification process, one typically runs the verification many times with wrong models or invariants, as part of the development process. In this process, an unstable solver may fail unpredictably and without a clear reason. In contrast, our experience with the verification projects described in this thesis, which is also quantified by the variance data of Figure 4.28, is that for all practical purposes Z3 performs reliably on EPR formulas. That is, whenever the verification conditions are in EPR, the solver returns with either a proof or a counterexample. The solver run time is usually a few seconds, and has a very low sensitivity to random seeds or other minor perturbations.

Overall, our evaluation demonstrates the practical value of our methodology for EPR verification, as it allows to transform models whose verification condition cannot be handled by Z3 (and demonstrate poor scalability and predictability for bounded verification), into models that can be reliably verified by Z3 in a few seconds.

[3] http://www.cs.tau.ac.il/~odedp/paxos-made-epr.html

4.8 Related Work for Chapter 4

Automated verification of distributed protocols Here we review several works that developed techniques for automated verification of distributed protocols, and compare them with our approach.

The Consensus Verification Logic $\mathbb{CL}$ [69] is a logic tailored to verify consensus algorithms in the partially synchronous Heard-Of Model [44], with a decidable fragment that can be used for verification. PSync [70] is a domain-specific programming language and runtime for developing formally verified implementations of consensus algorithms based on $\mathbb{CL}$ and the Heard-Of Model. Once the user provides inductive invariants and ranking functions in $\mathbb{CL}$, safety and liveness can be automatically verified. PSync's verified implementation of LastVoting (Paxos in the Heard-Of Model) is comparable in performance with state-of-the-art unverified systems. Compared to these works, our approach tackles a more general setting. The Heard-Of Model is partially synchronous, while our methodology allowed us to verify Paxos protocols in the more general asynchronous setting.

Many interesting theoretical decidability results, as well as the ByMC verification tool, have been developed based on the formalism of Threshold Automata [30, 120–122, 143]. This formalism allows to express a restricted class of distributed algorithms operating in a partially synchronous communication mode. This restriction allows decidability results based on cutoff theorems, for both safety and liveness.

A decidable fragment of Presburger arithmetic with function symbols and cardinality constraint over interpreted sets is developed in [14]. Their work is motivated by applications to the verification of fault-tolerant distributed algorithms, and they demonstrate automatic safety verification of some fault-tolerant distributed algorithms expressed in a partially synchronous round-by-round model similar to PSync.

#Π [231] present a logic that combines reasoning about set cardinalities and universal quantifiers, along with an invariant synthesis method. The logic is not decidable, so a sound and incomplete reasoning method is used to check inductive invariants. Inductive invariants are automatically synthesized by method of Horn constraint solving. The technique is applied to automatically verify a collection of parameterized systems, including mutual exclusion, consensus, and garbage collection. However, Paxos-like algorithms are beyond the reach of this verification methodology since they require more complicated inductive invariants.

Recently, [165] presented a cutoff result for consensus algorithms. They define *ConsL*, a domain specific language for consensus algorithms, whose semantics is based on the Heard-Of Model. *ConsL* admits a cutoff theorem, i.e., a parameterized algorithm expressed in *ConsL*

is correct (for any number of processors) if and only if it is correct up to some finite bounded number of processors (e.g., for Paxos the bound is 5) . This theoretical result shows that for algorithms expressible in *ConsL*, verification is decidable. However, *ConsL* is focused on algorithms for the core consensus problem, and does not support the infinite-state per process that is needed, e.g., to model Multi-Paxos and SMR.

The above mentioned works obtain automation (and some decidability) by restricting the programming model. We note that our approach takes a different path to decidability compared to these works. We axiomatize arithmetic, set cardinalities, and other higher-order concepts in an uninterpreted first-order abstraction. This is in contrast to the above works, in which these concepts are baked into specially designed logics and formalisms. Furthermore, we start with a Turing-complete modeling language and invariants with unrestricted quantifier alternation, and provide a methodology to reduce quantifier alternation to obtain decidability. This allows us to employ a general-purpose decidable logic to verify asynchronous Paxos, Multi-Paxos, and their variants, which are beyond the reach of all of the above works.

Deductive verification in undecidable logic IronFleet [100] is a verified implementation of SMR, using the Dafny [148] program verifier, with verified safety and liveness properties. Compared to our work, this system implementation is considerably more detailed. The verification using Dafny employs Z3 to check verification conditions expressed in undecidable logics that combine multiple theories and quantifier alternations. This leads to great difficulties due to the unpredictability of the solver. To mitigate some of this unpredictability, IronFleet adopted a style they call *invariant quantifier hiding*. This style attempts to specify the invariants in a way that reduces the quantifiers that are explicitly exposed to the solver. Our work is motivated by the IronFleet experience and observations. The methodology we develop provides a more systematic treatment of quantifier alternations, and reduces the verification conditions to a decidable logic.

Verification using interactive theorem provers Recently, the Coq [26] proof assistant has been used to develop verified implementations of systems, such as a file system [46], and shared memory data structures [214]. Closer to our work is Verdi [235, 237], which presents a verified implementation of an SMR system based on Raft [183], a Paxos-like algorithm. This approach requires great effort, due to the manual process of the proof; developing a verified SMR system requires many months of work by verification experts, and proofs measure in tens of thousands of lines.

In [204, 209] safety of implementations of consensus and SMR algorithms is verified in

the EventML programming language. EventML interfaces with the Nuprl theorem prover, in which proofs are conducted, and uses Nuprl's code generation facilities.

Other works applied interactive theorem proving to verify Paxos protocols at the algorithmic level, without an executable implementation. In [115] correctness of the Disk Paxos algorithm is proved in Isabelle/HOL, in about 6,500 lines of proof script. Recently, [40] presented safety proofs of Paxos and Multi-Paxos using the TLA+ [137] specification language and the TLA Proof System TLAPS [45]. TLA+ has also been used in Amazon to model check distributed algorithms [180]. However, they did not spend the effort required to obtain formal proofs, and only used the TLA+ models for bug finding via the TLA+ model checker [238].

Compared to our approach, using interactive theorem provers requires more user expertise and is more labor intensive. We note that part of the difficulty in using an interactive theorem prover lies in the unpredictability of the automated proof methods available and the considerable experience needed to write proofs in an idiomatic style that facilitates automation. An interesting direction of research is to integrate our methodology in an interactive theorem prover to achieve predictable automation in a style that is natural to systems designers.

Works based on EPR In [111–113] it was shown that EPR can express a limited form of transitive closure, in the context of linked lists manipulations, and this was also presented in Section 3.3. We notice that in the context of the methodology presented in this chapter, the treatment of transitive closure can be considered as adding a derived relation. Both works show that EPR is surprisingly powerful, when augmented with derived relations.

In [74], bounded quantifier instantiation is explored as a possible solution to the undecidability caused by quantifier alternations. This work shares some of the motivation and challenges with our work, but proposes an alternative solution. The context we consider here is also wider, since we deal not only with quantifier alternations in the inductive invariant, but also with quantifier alternations in the transition relation. In [74] an interesting connection is also shown between derived relations and quantifier instantiation, and these insights may apply to our methodology as well. An appealing future research direction is to combine user provided derived relations and rewrites together with heuristically generated quantifier instantiation.

In [219], Ivy is extended beyond what is presented in this thesis to also allow extraction of verified executable implementations, and verified implementations are developed for both Multi-Paxos and Raft [183]. The work presented in this thesis has been an enabler for [219],

which shows that the techniques we present are general enough to be applied to other protocols, and to be extended for verifying implementations.

Part II

Invariant Inference

Chapter 5

Decidability of Invariant Inference

This chapter is based on the results published in [185].

When using a decidable fragment for modeling transition systems and their invariants, checking if an invariant is inductive is decidable. It is therefore natural to investigate the decidability of the problem of invariant *inference*, i.e., the problem of finding a suitable inductive invariant for proving that a given system satisfies a given safety property. This problem is parameterized by a language (i.e., infinite set of first-order sentences) L, the search space of potential inductive invariants. This chapter explores this problem in a general context, develops restrictions under which this problem is decidable, and also obtains undecidability results that show the restrictions are necessary.

Many techniques attempt to mechanize the search for inductive invariants. Such tools are only able to infer inductive invariants in a certain language, and are hence necessarily incomplete in verifying safety. Their output may be: (i) program is safe (found inductive invariant); or (ii) program is unsafe (found a concrete counterexample); or (iii) don't know or diverge. Since the safety verification problem is undecidable, this incompleteness is expected and therefore mostly accepted by users of such techniques. However, since actual tools search for inductive invariants in a certain language, the underlying decision problem they address is in fact "is there an inductive invariant in a certain language?" A key observation is that this problem is different from the safety verification problem, and hence might be decidable even in cases where safety verification is not.

This chapter investigates the decidability of the problem of inferring inductive invariants in a *given* language. Here, the expected outcome is not "safe/unsafe", but rather "inductive invariant exists/does not exist in the given language". Investigating the decidability of this problem is important to better understand the foundation of existing methods for invariant

inference (e.g., abstract interpretation [56, 57], IC3/PDR [34, 116]): whenever the problem is undecidable, no tool will be able to be complete even for the language in which it is searching; in contrast, when the problem is decidable, tool developers can aim to have complete algorithms for a restricted language.

This chapter formulates the general problem of inferring inductive invariants in a restricted language L (Section 5.2), and applies the technique of [5–7, 9, 76] based on well-quasi-orders (wqo's) to get sufficient conditions for decidability of invariant inference (Section 5.3). The formalization is parametric in the language L, and associates with each language L a quasi-order $\sqsubseteq_L$ on the state space, such that if $\sqsubseteq_L$ is a wqo, then invariant inference in L is decidable. This leads to a (parametric) connection between transition systems specified in first-order logic, first-order languages, and the decidability technique based on wqo's (Section 5.4). This connection is the basis for the following main results of this chapter.

Decidability of universal invariants for deterministic paths We prove (Section 5.5) that for programs manipulating graphs with outdegree one, modeled in EPR as discussed in Section 3.3, and restricting to universally quantified inductive invariants, the invariant inference problem is decidable (while safety is still undecidable). This class includes many programs manipulating singly-linked-lists [112]. The technical proof builds on Kruskal's Tree Theorem [129] to show that the suitable $\sqsubseteq_L$ is a wqo, as it corresponds to homeomorphic embedding of graphs. Being formulated in logic, this result naturally extends to capture programs with additional structure beyond graph reachability (e.g., sorting algorithms for linked lists). The complexity of inferring universally quantified invariants even for linked-list programs is shown to be non-elementary (Section 5.7.4).

Undecidability of alternation-free invariants for deterministic paths We show (Section 5.7.2) that in the same setting as above, inferring alternation-free invariants is undecidable. Namely, mixing universal and existential information even without alternation makes invariant inference undecidable. This demonstrates that the invariant inference problem is theoretically harder than invariant checking, since in this setting checking inductiveness of an alternation-free invariant is decidable. This also shows that in this setting the restriction to universal invariants is necessary for decidability of invariant inference.

Undecidability of universal invariants for general systems Modeling systems beyond deterministic paths (or linked-lists) requires additional unrestricted relations. However, we show (Section 5.7.3) that in the presence of a single unrestricted binary relation, invariant

```
x := 1
y := 2
while *:
    assert x > 0
    x := x + y
    y := y + 1
```

Figure 5.1: A simple loop example.

inference is undecidable even when restricting to universal invariants. This is done by constructing a safe transition system that has a universal inductive invariant if and only if a given counter machine halts.

New decidability from old: systematic constructions for decidability To overcome the general undecidability while allowing unrestricted relations, we provide (Section 5.6) systematic ways to construct classes of systems and languages for invariants for which invariant inference is decidable. These constructions start with some established wqo, (e.g., the deterministic paths class with universal invariants) and gradually extend it to construct new systems with suitable wqo's. This process results in systems richer than the original one, while decidability is maintained by further restricting the language of potential invariants. We demonstrate the constructions by obtaining a decidable fragment of the invariant inference problem that captures a nontrivial example of a network learning switch.

5.1 Overview

This section provides a short overview of the problem addressed in this chapter and the main results.

5.1.1 Motivation and Background

Inductive invariants can be difficult to find both manually and using automatic program analysis. This was discussed in Section 1.2 and identified as Challenge 2, and here we elaborate further to motivate the investigation of this chapter.

Consider the example listed in Figure 5.1, of a simple loop, with the safety property $P = x > 0$.[1] The program executes the loop for an unbounded number of iterations, starting from a state where $x = 1$ and $y = 2$. Clearly, $x < 1000$ for example is not an invariant at all in this program, since it is violated after 500 loop iterations. Interestingly, the required

[1] We note that this work does not consider numeric domains, but it is useful to illustrate the notion of inductive invariants using a simple numeric program.

safety property $P = x > 0$ is invariant in this program but it is not an *inductive* invariant. For example, if we execute the loop body in a state in which y is negative, it will be violated. In order to come up with an inductive invariant, we need to also prove something about y. For example, $x > 0 \wedge y \geq 2$ and $x > 0 \wedge y > 0$ are both inductive invariants that prove P for this program. However, if we restrict the language of inductive invariants to consider x only, i.e., to a language whose formulas are not allowed to mention y, then no inductive invariant for P exists for this program.

Inferring inductive invariants in a restricted language This chapter addresses the decision problem that corresponds to inductive invariant inference, defined as follows:

> Given a transition system $T = (\Sigma, \Gamma, \iota, \tau)$, a safety property P, and a (usually infinite) family of candidate invariants L: does there exist a formula in L that is an inductive invariant for (T, P)?

Notice that this problem is relevant for abstract interpretation [56, 57], since L can be viewed as an abstract domain, in which case this question amounts to asking if the abstract domain is precise enough for the given program and property. Also notice that in reality one is interested in algorithms for this problem, which are not considered in this thesis. Finally, we note that if L is a finite set and if invariant checking is decidable, then the problem of inferring inductive invariants is trivially decidable.

Working with a restricted language of potential invariants L is beneficial in many situations, as it may lead to decidability both of invariant checking, and of invariant inference. Additionally, we note that a negative answer to the invariant inference problem of the form: "There exists no inductive invariant in L which can be used to verify that your program satisfies P" can be useful for programmers. The programmer may decide to simplify their program such that there will be an inductive invariant in L. To facilitate this, the verifier should also provide a clear explanation as to why there is no inductive invariant in the restricted language. In this case, such an answer is significantly better than an inconclusive alarm of the form: "Your program may not satisfy P", which even the most sophisticated static analysis tools sometimes provide.

Decidability via well-quasi-orders To address the decidability of inferring inductive invariants in a restricted language L, we apply the technique of [5–7] based on well-quasi-orders (wqo's).

The set of potential inductive invariants L naturally defines the following quasi-order on

states (models):

$$s \sqsubseteq_L s' \quad \text{iff} \quad \forall \varphi \in L.\ s' \models \varphi \ \Rightarrow\ s \models \varphi \tag{5.2}$$

Thus, smaller or lower states according to $\sqsubseteq_L$ satisfy more formulas from L. Notice that this is a quasi-order, i.e., it is reflexive and transitive. Whenever $\sqsubseteq_L$ is a well-quasi-order (wqo), that is, any infinite sequence $s_0, s_1, \ldots$, contains an increasing pair $s_i \sqsubseteq_L s_j$ with $i < j$, then invariant inference in L is decidable (provided some natural effectiveness assumptions). Intuitively, invariant inference in L can be done by backward reachability analysis (which can be seen as iterative application of Dijkstra's weakest precondition). The fact that $\sqsubseteq_L$ is a wqo guarantees the termination of this process.

Universal invariants A natural and useful language for inductive invariants is the set of universally quantified sentences, or universal invariants. Universal invariants can be used to prove properties of many infinite-state systems, e.g., parameterized systems and programs with unbounded heap allocation or unbounded arrays. The universal quantification is usually over all nodes in a network, or all the elements in the heap or the array (e.g., [208] for heaps and [59, 193] for arrays). Furthermore, linked-lists can be formulated using a theory of list reachability which allows deciding inductiveness of universal invariants using effectively proportional logic (EPR) [112]. When L is the set of all universal sentences, we denote the quasi-order of Equation (5.2) by $\sqsubseteq_{\forall^*}$. We note that $\sqsubseteq_{\forall^*}$ is the substructure relation known from model theory (e.g., [42]). That is, $s \sqsubseteq_{\forall^*} s'$ iff s is isomorphic to a substructure of s'.

5.1.2 Decidability of Inferring Universal Invariants for Deterministic Paths

When using the encoding presented in Section 3.3 to model reachability in graphs of outdegree one, we prove that the structures that arise are well-quasi-ordered by the substructure relation. This leads to decidability of inferring universal invariants for programs manipulating such graphs, including many programs manipulating singly-linked-lists [112]. The proof utilizes Kruskal's Tree Theorem [129], and relates homeomorphic embedding of trees with the substructure relation. The proof using Kruskal's Tree Theorem naturally supports adding any finite number of unary relations, which denote sets of individuals, while still maintaining decidability of invariant inference. This allows adding unary instrumentation relations, which is useful for example in verifying sorting implementations [151].

5.1.3 Undecidability and Complexity of Invariant Inference

A natural way to extend universal invariants is by allowing Boolean combinations of universal invariants and existential invariants. These are called *alternation-free* invariants since they forbid alternating universal and existential quantifiers. In [112], it was shown that this class of invariants suffices to prove partial correctness of many linked-list manipulating programs. Moreover, checking inductiveness of alternation-free invariants is still decidable in EPR.

While checking inductiveness is still decidable, we show that the invariant inference problem is undecidable for alternation-free invariants. This is proven by a reduction from the halting problem of counter machines: given a counter machine we construct a program and property such that the counter machine terminates in k steps if and only if there exists an inductive invariant with $O(k)$ quantifiers for the program. While this result is somehow expected given the unbounded nature of quantified invariants and the complexity of inductive reasoning, we note that it differs from the standard reductions (e.g., [32]) that show that the *safety* problem for certain classes of programs is undecidable, as our constructed program is safe whether the counter machine halts or not. This result shows an interesting case where invariant inference is undecidable while invariant checking is decidable.

The undecidability result implies that for linked-list manipulating programs, restricting to universal invariants is necessary for the decidability of invariant inference. By using a similar reduction, we also show that inferring universal invariants for linked-list programs has non-primitive complexity. This reduction is from the safety problem of lossy counter machines: given a lossy counter machine, we construct a program such that the program has a universal inductive invariant iff the lossy counter machine is safe.

For modeling systems beyond linked-lists, additional unrestricted relations are needed. However, we show that even when restricting to universal invariants, invariant inference becomes undecidable in the presence of a single unrestricted binary relation. We do this by a similar reduction from the halting problem of counter machines. As before, we construct a system that has a universal invariant with $O(k)$ quantifiers iff a given counter machine halts in k steps. This shows that for many general systems beyond linked-lists, inferring universal invariants is undecidable.

5.1.4 Systematic Constructions for Decidability

The most exciting part of the work presented in this chapter is the ability to show that it is decidable to infer restricted universal invariants in many parametric systems. We note that despite their inherent limitations, universal invariants can be used to model many aspects

of systems by using uninterpreted relations. However, as noted above, for the language of all universal invariants, even one unrestricted binary relation makes invariant inference undecidable. To obtain decidability, we further restrict the language of potential universal invariants. To do so, we start from an already established "base" wqo (e.g., the deterministic paths class with universal invariants), and present systematic constructions that extend classes of systems and languages for invariants in a limited way to construct new systems for which $\sqsubseteq_L$ is a wqo by construction, and thus invariant inference is decidable (see Section 5.6). The resulting languages are subsets of the set of all universal invariants.

Symmetric lifting We show that a decidability result for some class of programs and language which relies on an underlying theory (e.g., deterministic paths) can be lifted to systems that are modeled by an unbounded number of instances of the theory, by creating a language of restricted universal sentences that do not correlate the different instances. We call this operation *symmetric lifting*, and it can be used to model systems beyond linked-lists and deterministic paths by using high-arity relations. For example, the routing tables in a network of switches may be modeled by a ternary relation *route*, where $route(d, m, n)$ denotes that messages sent to destination d which arrive at switch m will be forwarded to n. Essentially, the routing tables can be viewed as an unbounded number of graphs with outdegree one — one graph for each destination d. In many cases the invariants do not need to correlate the different instances of the original theory (e.g., it is unnecessary to relate the routing tables for different destinations). For such cases, we show that an established wqo (e.g., that of deterministic paths) can be lifted to obtain a wqo for a system where the relations have an increased arity, which corresponds to an unbounded number of instances of the base theory. The fact that wqo's are preserved by symmetric lifting is proved using Higman's Lemma [104].

Bounded occurrences of unrestricted relations It is sometimes necessary for the invariant to mention unrestricted relations (i.e., relations that do not obey any background theory). For example, a model of a distributed protocol such as a learning switch may use a relation *msg* of arity 4, where $msg(s, d, m, n)$ denotes that a packet with source s and destination d is pending on the link from n to m. Moreover, the invariant might need to relate the *msg* relation to reachability information such as forwarding paths stored in the *route* relation mentioned above. To handle such cases, we show that wqo and thus decidability of invariant inference is preserved by extending a language of universal invariants to include unrestricted relations, as long as only a *bounded* number of occurrences of these

relations appear in each universal clause. For the case of the learning switch, adding one such occurrence suffices to obtain an inductive invariant.

This extension is not trivial since the occurrences of the unrestricted relations are allowed to create correlations with relations that may appear an unbounded number of times under an unbounded number of quantifiers (e.g., to correlate *msg* and *route*). For this reason, this result cannot be obtained as a straightforward cartesian product with a finite domain, and the proof also relies on Higman's Lemma to show that the wqo is preserved.

5.2 The Inductive Invariant Inference Problem

In this section we formalize the inductive invariant decision problem parameterized by the language of invariants, and contrast it with the classical safety decision problem. The basic definitions used here are those of Section 2.3 for a transition system $T = (S, S_0, R)$, a safety property $P \subseteq S$, an inductive invariant $I \subseteq S$, and a CTI (counterexample to induction) $s \in S$. In this chapter, a transition system will always have an associated safety property, so by overloading of terminology, we will also use *transition system* to mean a transition system and a safety property: (S, S_0, R, P).

Classes of transition systems and properties We are interested in formulating decision problems where the input consists of a transition system and a safety property. In the following sections we investigate decidability of these problems for various *classes* of transition systems and safety properties, where the class is understood to define an effective way to represent its transition systems and safety properties. We use $\mathcal{C}$ to denote such a class. For example, one could define the class of digital circuits, where the state space is given by the valuation of Boolean variables, and the set of initial states, the transition relation, and the safety property are described using propositional formulas. Another example is that of programs in some programming language, where the safety property might be defined via assertions within the program. Our focus will be of course classes of first-order transition systems, as defined in Section 2.4, but we start with a general mindset. We write $(T, P) \in \mathcal{C}$ to denote that (T, P) is an instance of class $\mathcal{C}$.

The safety decision problem The classical problem of determining the safety of a transition system taken from a given class $\mathcal{C}$ of transition systems can be formulated as the

following decision problem:

$$\texttt{SAFE}[\mathcal{C}] = \{(T, P) \in \mathcal{C} \mid T \models P\}$$

Recall that for many classes of infinite-state transition systems, $\texttt{SAFE}[\mathcal{C}]$ is undecidable (as T can encode a Turing machine).

Languages for expressing inductive invariants We are interested in deciding whether a given transition system (including a safety property) has an inductive invariant which is *expressible in a given language*. Therefore, the input to the inductive invariant inference problem is a transition system with a safety property, (T, P), and a language $L \subseteq \mathcal{P}(S)$ (where S is the set of states of T) that determines the inductive invariants of interest. As such, the inductive invariant problem is defined not only with respect to a class $\mathcal{C}$ of transition systems (and properties), but also with respect to a class $\mathcal{L}$ of languages.

For example, if $\mathcal{C}$ is the class of digital circuits represented by propositional formulas over Boolean variables, then $\mathcal{L}$ might restrict to languages L that contain sets expressible by propositional formulas over some of the variables. In this case, if the formulas can mention all the propositional variables, then every set of states is in L. On the other hand, if $\mathcal{C}$ is the class of transition systems specified in first-order logic, and $\mathcal{L}$ restricts to languages L that contain sets expressible by quantifier free formulas (or universal formulas), then some sets of states might not be in L. Since the definition of $\mathcal{L}$ might depend on $\mathcal{C}$, we consider them as a pair $(\mathcal{C}, \mathcal{L})$. We write $(T, P, L) \in (\mathcal{C}, \mathcal{L})$ to denote that (T, P, L) is an instance of $(\mathcal{C}, \mathcal{L})$. We say that a set $A \subseteq S$ is expressible in L if $A \in L$.

The inductive invariant inference problem Given $\mathcal{C}$ and $\mathcal{L}$, we define the decision problem $\texttt{INV}[\mathcal{C}, \mathcal{L}]$ as follows: given a transition system, $T = (S, S_0, R)$, a safety property, P, and a language, $L \subseteq \mathcal{P}(S)$, such that $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, is there an inductive invariant for (T, P) which is expressible in L. Formally:

$$\texttt{INV}[\mathcal{C}, \mathcal{L}] = \{(T, P, L) \in (\mathcal{C}, \mathcal{L}) \mid \exists I \in L \text{ s.t. } I \text{ is an inductive invariant for } (T, P)\}$$

Note that for every $\mathcal{C}$ and $\mathcal{L}$, if $(T, P, L) \in \texttt{INV}[\mathcal{C}, \mathcal{L}]$ then $(T, P) \in \texttt{SAFE}[\mathcal{C}]$. That is, if there exists an inductive invariant (in L) for a transition system then it is safe. The converse does not necessarily hold, and there could be cases where $(T, P) \in \texttt{SAFE}[\mathcal{C}]$ but $(T, P, L) \notin \texttt{INV}[\mathcal{C}, \mathcal{L}]$. This can happen if the language L is not expressive enough to describe

an inductive invariant for (T, P). This suggests that $\texttt{INV}[\mathcal{C}, \mathcal{L}]$ may be decidable even if $\texttt{SAFE}[\mathcal{C}]$ is undecidable.

Also note that decidability issues are of interest when $\mathcal{C}$ allows the definition of an infinite set of states and $\mathcal{L}$ allows the definition of infinitely many sets, since in the finite case both $\texttt{SAFE}[\mathcal{C}]$ and $\texttt{INV}[\mathcal{C}, \mathcal{L}]$ can be decided by a naive enumeration. This is the case, for example, for the class of digital circuits with invariants expressed as propositional formulas.

Effectiveness assumptions In this chapter we restrict our attention to classes $\mathcal{C}$ and $\mathcal{L}$ such that the following *effectiveness assumptions* hold:

(i) there is a decision procedure that, given $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, determines membership in S, S_0, R, P, and L,

(ii) there is a decision procedure that checks, given $(T, P, L) \in (\mathcal{C}, \mathcal{L})$ and a set $I \in L$, whether I is an inductive invariant for (T, P), and provides $s \in S$, a CTI (counterexample to induction) for I, if it is not (see Section 2.3 for the definition of a CTI).

Note that as $\mathcal{C}$ and $\mathcal{L}$ are used to define decision problems, they come with a finite encoding of their instances, and the set of instances of $(\mathcal{C}, \mathcal{L})$ is also assumed to be decidable.

5.3 Sufficient Conditions for Decidability of $\texttt{INV}[\mathcal{C}, \mathcal{L}]$

In this section, we apply the technique of [5–7, 9] based on well-quasi-orders (wqo's) to obtain sufficient conditions for the decidability of $\texttt{INV}[\mathcal{C}, \mathcal{L}]$. To do so, for $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, we define a quasi-order, denoted $\sqsubseteq_L$, on the states of the transition system T. The quasi-order $\sqsubseteq_L$ has the property that if it is a well-quasi-order for any $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, and if $(\mathcal{C}, \mathcal{L})$ has several other simple properties, then $\texttt{INV}[\mathcal{C}, \mathcal{L}]$ is decidable.

Well-founded sets and well-quasi-orders We first recall the definitions of a well-quasi-order (wqo) and a well-founded set. Let $(X, \leq)$ be a quasi-order, i.e., $\leq$ is reflexive and transitive. We say that $(X, \leq)$ is *well-founded* if it does not contain any infinite strictly decreasing chain $x_0 > x_1 > \ldots$. We say that the infinite sequence $x_0, x_1, \ldots$ is an *antichain* if every two elements in it are incomparable, i.e., $x_i \not\leq x_j$ for all $i \neq j$. We say that $(X, \leq)$ is a *well-quasi-order (wqo)* [130] if for every infinite sequence of elements $x_0, x_1, \ldots$ there exists $i < j$ such that $x_i \leq x_j$. Equivalently, $(X, \leq)$ is a wqo if $(X, \leq)$ is well-founded and does not contain antichains.

5.3.1 Quasi-Order and Exclusion Operator for L

We now present several definitions, parameterized by a language for potential inductive invariants.

Definition 5.3 ($\sqsubseteq_L$)**.** For any language $L \subseteq \mathcal{P}(S)$, we define a quasi-order $\sqsubseteq_L$ on S given by

$$s_1 \sqsubseteq_L s_2 \text{ iff for all } A \in L : s_2 \in A \text{ implies } s_1 \in A$$

Thinking of A as an L-property, $s_1 \sqsubseteq_L s_2$ says that every L-property satisfied by s_2 is satisfied by s_1. That is, s_1 satisfies more (or the same) L-properties than s_2.

Definition 5.4 (Avoid_L)**.** Let $s \in S$ be a state and let $A \in L$ be a set such that $s \notin A$, and for every $A' \in L$, if $s \notin A'$ then $A' \subseteq A$. Then we denote A by $\text{Avoid}_L(s)$.

That is, $\text{Avoid}_L(s)$ is the maximum (w.r.t. set inclusion) over all sets in L that do not include s. Note that $\text{Avoid}_L(s)$ need not exist. However, if it exists, it is unique and equal to the union of all sets in L that do not include s. $\text{Avoid}_L(s)$ and $\sqsubseteq_L$ are strongly related:

Lemma 5.5. *For all s, s' such that $\text{Avoid}_L(s)$ exists, $s \sqsubseteq_L s'$ iff $s' \notin \text{Avoid}_L(s)$.*

Proof. The "only-if" direction directly follows from the definitions. For the "if" direction, assume $s \not\sqsubseteq_L s'$. Then there exists $A \in L$ such that $s' \in A$, but $s \notin A$. The latter implies $A \subseteq \text{Avoid}_L(s)$ and thus, $s' \in \text{Avoid}_L(s)$. □

5.3.2 L-Relaxed Transition System & Decidability of $\texttt{INV}[\mathcal{C}, \mathcal{L}]$

In this section, we formulate sufficient conditions for decidability of $\texttt{INV}[\mathcal{C}, \mathcal{L}]$. Given $(\mathcal{C}, \mathcal{L})$, we say that Avoid_L is *computable in* $(\mathcal{C}, \mathcal{L})$ if there exists a procedure that, given $(T, P, L) \in (\mathcal{C}, \mathcal{L})$ and $s \in S$, computes $\text{Avoid}_L(s)$. (In particular, this implies that $\text{Avoid}_L(s)$ exists for every $s \in S$.)

Theorem 5.6. *Let $(\mathcal{C}, \mathcal{L})$ be such that:*

(i) for every $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, L is closed under finite intersections;

(ii) Avoid_L is computable in $(\mathcal{C}, \mathcal{L})$; and

(iii) for every $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, where $T = (S, S_0, R)$, $(S, \sqsubseteq_L)$ is a wqo.

Then $\texttt{INV}[\mathcal{C}, \mathcal{L}]$ is decidable. Furthermore, Algorithm 5.1 is a decision procedure for it.

Algorithm 5.1: Invariant inference by backward reachability analysis

```
1 I := S
2 while I is not an inductive invariant for (T, P) do
3     if S_0 ⊈ I then return no inductive invariant in L
4     let s be a counterexample to induction for I
5     I := I ∩ Avoid_L(s)
6 return I is an inductive invariant in L
```

We prove Theorem 5.6 by proving partial correctness and termination of Algorithm 5.1. Intuitively, Algorithm 5.1 simultaneously searches for an inductive invariant in L and for an indication that such an inductive invariant does not exist. The algorithm uses counterexamples to induction to iteratively strengthen the candidate invariant. Strengthening is performed by excluding the counterexamples using Avoid_L (Line 5). An indication that an inductive invariant in L does not exist comes in the form of a trace of a *relaxed transition system*, which we define below, from an initial state to a "bad" state (violating P). Such a trace does not imply that the original transition system is unsafe, but as we show next, it implies that no inductive invariant exists in L, hence partial correctness of the algorithm follows. Finally, the wqo property is used to rule out an infinite sequence of strengthenings, thus ensuring termination of Algorithm 5.1 and proving Theorem 5.6.

Definition 5.7 (L-Relaxed Transition System)**.** Given a transition system $T = (S, S_0, R)$ and a language $L \subseteq \mathcal{P}(S)$, we define the *$L$-relaxed transition system* $T^L = (S, S_0, R^L)$ by

$$(s, s') \in R^L \text{ iff } (s, s') \in R \text{ or } s' \sqsubseteq_L s.$$

We say that a trace of T^L is an *L-relaxed trace* of T.

Lemma 5.8. *Let $(T, P, L) \in (\mathcal{C}, \mathcal{L})$ be a transition system and language. Then $(T, P, L) \in \texttt{INV}[\mathcal{C}, \mathcal{L}]$ implies $T^L \models P$.*

Proof. If $(T, P, L) \in \texttt{INV}[\mathcal{C}, \mathcal{L}]$, there is $I \in L$ that is an inductive invariant for (T, P). Let $s_0, \ldots, s_N$ be an L-relaxed trace. We prove by induction that for all $i \leq N$, $s_i \in I$. Indeed, since $s_0 \in S_0$ and $S_0 \subseteq I$, we have $s_0 \in I$. For the induction step, assume $s_i \in I$ and consider the L-relaxed transition step $(s_i, s_{i+1}) \in R^L$. If $(s_i, s_{i+1}) \in R$, then $s_{i+1} \in I$ by inductiveness of I. If $s_{i+1} \sqsubseteq_L s_i$, then $s_{i+1} \in I$ since $I \in L$. Since $I \subseteq P$, we conclude that any reachable state of T^L is in P, and thus $T^L \models P$. □

Lemma 5.9 (Partial Correctness of Algorithm 5.1)**.** *If Algorithm 5.1 terminates, then its output is correct.*

Proof. If Algorithm 5.1 determines that I is an inductive invariant, correctness follows from the loop condition. For the case where Algorithm 5.1 determines that no inductive invariant exists in L, we prove by induction on the loop iterations of Algorithm 5.1 that for every state $s' \notin I$, there exists an L-relaxed trace of T leading from s' to some state $\notin P$. Hence, if $S_0 \not\subseteq I$, it follows that $T^L \not\models P$, and by Lemma 5.8, $(T, P, L) \notin \texttt{INV}[\mathcal{C}, \mathcal{L}]$. The base case of the induction is trivial (initially $I = S$). For the induction step, let s' be a state that is removed from I since it is not in $\text{Avoid}_L(s)$. By the definition of a CTI (Section 2.3), either $s \notin P$ or there exists $s'' \notin I$ such that $(s, s'') \in R$. By the induction hypothesis for I, we get that in both cases, s itself has an L-relaxed trace leading to some state $\notin P$. By Lemma 5.5, $s \sqsubseteq_L s'$, so s' also has an L-relaxed trace leading to a state $\notin P$. □

In the general case, Algorithm 5.1 is not guaranteed to terminate. However, the following lemma gives a natural condition for its termination.

Lemma 5.10 (Termination of Algorithm 5.1)**.** *If $(L, \subseteq)$ is well-founded, then Algorithm 5.1 always terminates.*

Proof. For $i \geq 0$, let I_i denote the set I at the i'th loop iteration of Algorithm 5.1. The sequence $I_0, I_1, \ldots$ is strictly decreasing with respect to set inclusion, so it must be finite by well-foundedness. □

The proof Theorem 5.6 is completed by the following lemma.

Lemma 5.11. *Let $L \subseteq \mathcal{P}(S)$ be a language such that $(S, \sqsubseteq_L)$ is a wqo, then $(L, \subseteq)$ is well-founded.*

Proof. Assume to the contrary that there exists a strictly decreasing infinite sequence $A_0 \supset A_1 \supset A_2 \ldots$ of sets in L. For every $i \geq 0$ let s_i be a state in $A_i \setminus A_{i+1}$. For every $i < j$, $A_j \subseteq A_{i+1}$, and therefore $s_i \notin A_j$. Hence, for every $i < j$, $A_j \in L$ and $s_j \in A_j$ but $s_i \notin A_j$. This implies that $s_i \not\sqsubseteq_L s_j$ for every $i < j$, in contradiction to the fact that $(S, \sqsubseteq_L)$ is a wqo. □

In fact, the correctness proof of Algorithm 5.1 also provides the following corollary, which is the converse of Lemma 5.8 under the conditions of Theorem 5.6. This will become useful in Section 5.7.4.

Corollary 5.12. *Let $(\mathcal{C}, \mathcal{L})$ be such that: (i) for every $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, L is closed under finite intersections; (ii) Avoid_L is computable in $(\mathcal{C}, \mathcal{L})$; and (iii) for every $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, where $T = (S, S_0, R)$, $(S, \sqsubseteq_L)$ is a wqo. Then $(T, P) \in \texttt{INV}[\mathcal{C}, \mathcal{L}]$ if and only if $T^L \models P$.*

Proof. If $T^L \models P$, then a L-relaxed trace leading from an initial state to a state $\notin P$ does not exist. By the proof of Lemma 5.9, this implies that Algorithm 5.1 does not terminate in Line 3. Thus, it necessarily finds an inductive invariant. □

5.4 EPR Classes of Transition Systems and Invariants

In the rest of this chapter we focus on classes of transition systems, safety properties, and languages of inductive invariants that are expressed in first-order logic, and specifically in EPR. This section provides the necessary formal definitions of these classes, building on the formalism of Section 2.4 to represent transition systems, safety properties, and inductive invariants using formulas in first-order logic.

5.4.1 Classes of Transition Systems and Properties

For the rest of this chapter, we will be interested in classes of transition systems and properties that are subsets of the class of *EPR transition systems*, defined below. For simplicity of the presentation, in this chapter we use the relational *unsorted* EPR fragment, i.e., the vocabulary contains a single sort and no function symbols.[2]

Definition 5.13 ($\mathcal{C}_{\text{EPR}}$)**.** The class $\mathcal{C}_{\text{EPR}}$, of *EPR transition systems* contains transition systems and properties (T, P), where $T = (\Sigma, \Gamma, \iota, \tau)$, such that Σ contains only relation and constant symbols (i.e., no function symbols), Γ is a finite set of $\exists^*\forall^*$ formulas (i.e., formulas whose prenex normal form has quantifier prefix $\exists^*\forall^*$) over Σ, and the formulas ι, τ, and $\neg P$ are also $\exists^*\forall^*$ formulas.

As mentioned in Section 5.2, a class is expected to provide an effective encoding of sets of states (e.g., initial states, transitions), and for $\mathcal{C}_{\text{EPR}}$ such sets are represented by formulas. A class is also expected to provide an efficient encoding of individual states in the state space of its transition systems. To accomplish this for $\mathcal{C}_{\text{EPR}}$, in this chapter we use the *finite structure semantics*, defined in Section 2.4.3. This means that the states of a transition system are only the *finite* structures over Σ that satisfy Γ. Finite structures can clearly be effectively encoded. Note that in this chapter we are only interested in safety properties, and only under restrictions that ensure that all verification conditions are in EPR, which has the finite model property. Thus, as also explained in Section 2.4.3, the distinction between the finite structure semantics and the first-order semantics is immaterial, and the results hold in both semantics.

[2] This restriction in not detrimental since functions can be represented as relations of increased arity. However, such an encoding loses the fact that relations representing functions are total.

In the rest of this chapter, we use STRUCT[Σ] to denote the set of *finite* structures over Σ, and STRUCT[Σ, Γ] to denote the set of finite structures over Σ that satisfy Γ. Thus, for an EPR transition system $T = (\Sigma, \Gamma, \iota, \tau)$, the state space is STRUCT[Σ, Γ].

5.4.2 Languages of Inductive Invariants

As we consider classes of transition systems expressed in EPR, we also wish to express inductive invariants as formulas in first-order logic, such that the verification conditions for checking inductiveness (see Section 2.4.2) are in EPR. We focus on the two classes defined below (and subclasses thereof to be defined later).

Definition 5.14 ($\mathcal{L}_{\forall^*}$)**.** The $\mathcal{L}_{\forall^*}$ class of languages restricts invariants to closed universal formulas (i.e., formulas with a $\forall^*$ quantifier prefix). Each vocabulary Σ induces another language of class $\mathcal{L}_{\forall^*}$. Since Σ is typically clear from the context, we omit it from the notation, and simply write $\forall^*$ to denote a language from $\mathcal{L}_{\forall^*}$, i.e., the set of universally quantified formulas over some vocabulary.

Definition 5.15 ($\mathcal{L}_{\mathrm{AF}}$)**.** The $\mathcal{L}_{\mathrm{AF}}$ class of languages restricts invariants to alternation-free formulas. These are formulas obtained by conjunctions and disjunctions of closed universal formulas (with $\forall^*$ quantifier prefix) and closed existential formulas (with $\exists^*$ quantifier prefix).

Effectiveness assumptions in EPR For classes that are subclasses of $\mathcal{C}_{\mathrm{EPR}}$ and $\mathcal{L}_{\mathrm{AF}}$ (in particular, including $\mathcal{L}_{\forall^*} \subseteq \mathcal{L}_{\mathrm{AF}}$), effectiveness assumption (i) formulated in Section 5.2 corresponds to decidability of checking whether a given structure satisfies a formula, and effectiveness assumption (ii) corresponds to decidability of satisfiability of the verification conditions (as formalized in Section 2.4.2). Since all verification conditions are in EPR, checking their satisfiability is decidable, and moreover, produces a finite structure as a CTI if the invariant is not inductive. Therefore, all the effectiveness requirements are satisfied for any $\mathcal{C}' \subseteq \mathcal{C}_{\mathrm{EPR}}$ and $\mathcal{L}' \subseteq \mathcal{L}_{\mathrm{AF}}$. Also observe that the finite model property of the verification conditions means that the set $\texttt{INV}[\mathcal{C}', \mathcal{L}']$ is the same regardless of whether we interpret its definition under the finite structure semantics or the first-order semantics.

5.4.3 $\sqsubseteq_L$ and Avoid$_L$ in EPR

Finally, we state the first-order logic and EPR realization of the definitions of $\sqsubseteq_L$ and Avoid$_L(s)$, which are used to formulate the sufficient conditions for decidability in Theorem 5.6.

In the first-order formalism, the state space is STRUCT[Σ, Γ], and a language L of potential inductive invariants is a set of first-order formulas (each of which represents a set of states). In this setting, $s_1 \sqsubseteq_L s_2$ iff for all $\varphi \in L$, $s_2 \models \varphi$ implies $s_1 \models \varphi$. Similarly, $\text{Avoid}_L(s)$ is the weakest formula $\varphi \in L$ such that $s \not\models \varphi$. That is, $s \not\models \text{Avoid}_L(s)$, and for every $\varphi \in L$, if $s \not\models \varphi$ then $\Gamma, \varphi \models \text{Avoid}_L(s)$.

We note that for first-order languages, requirement i of Theorem 5.6 corresponds to closure of L under (finite) conjunctions. This property holds for all languages we consider. The theorem also requires computability of Avoid_L (requirement ii). Next, we show that this requirement holds for $(\mathcal{C}_{\text{EPR}}, \mathcal{L}_{\forall^*})$ and subclasses thereof.

Computability of $\text{Avoid}_{\forall^*}$ for EPR classes To show that $\text{Avoid}_{\forall^*}$ is computable, we show that $\sqsubseteq_{\forall^*}$ and $\text{Avoid}_{\forall^*}$ are closely related to the model theoretic notions of *substructure* and *diagram*, which are themselves closely related (see e.g., [42]).

Definition 5.16 (Diagram)**.** Let $s = (\mathcal{D}, \mathcal{I})$ be a finite structure over Σ and let $\mathcal{D} = \{e_1, \ldots, e_{|\mathcal{D}|}\}$. The *diagram* of s, denoted by $\mathit{Diag}(s)$, is the following formula over Σ:

$$\exists x_1 \ldots x_{|\mathcal{D}|}. \texttt{distinct}(x_1, \ldots, x_{|\mathcal{D}|}) \wedge \psi$$

where $\texttt{distinct}(x_1, \ldots, x_n) = \bigwedge_{1 \le i < j \le n} x_i \neq x_j$, and ψ is the conjunction of:

- $x_i = c$ for every constant symbol c such that $\mathcal{I}(c) = e_i$;
- $r(x_{i_1}, \ldots, x_{i_k})$ for any relation r of arity k in Σ and any $i_1, \ldots, i_k$ s.t. $(e_{i_1}, \ldots, e_{i_k}) \in \mathcal{I}(r)$; and
- $\neg r(x_{i_1}, \ldots, x_{i_k})$ for any relation r of arity k in Σ and any $i_1, \ldots, i_k$ s.t. $(e_{i_1}, \ldots, e_{i_k}) \notin \mathcal{I}(r)$.

Intuitively, one can think of $\mathit{Diag}(s)$ as the formula produced by treating individuals in $\mathcal{D}$ as existentially quantified variables and explicitly encoding the interpretation of every constant and every relation symbol. Clearly, $s \models \mathit{Diag}(s)$. Recall that $s_1 = (\mathcal{D}_1, \mathcal{I}_1)$ is a substructure of $s_2 = (\mathcal{D}_2, \mathcal{I}_2)$ if $\mathcal{D}_1 \subseteq \mathcal{D}_2$ and for every $a \in \Sigma$, $\mathcal{I}_1(a)$ is the restriction of $\mathcal{I}_2(a)$ to $\mathcal{D}_1$. It is well known (e.g., [42]) that $s_2 \models \mathit{Diag}(s_1)$ iff s_1 is isomorphic to a substructure of s_2. Moreover, for every closed existential formula φ over Σ, $s \models \varphi$ iff $\mathit{Diag}(s) \models \varphi$. This immediately implies the following two lemmas.

Lemma 5.17. *$s_1 \sqsubseteq_{\forall^*} s_2$ iff s_1 is isomorphic to a substructure of s_2.*

Proof. Suppose $s_1 \sqsubseteq_{\forall^*} s_2$. Recall that $s_1 \sqsubseteq_{\forall^*} s_2$ iff s_1 satisfies all the universal formulas that are satisfied by s_2. $s_1 \models \mathit{Diag}(s_1)$ so $s_1 \not\models \neg\mathit{Diag}(s_1)$, hence $s_2 \not\models \neg\mathit{Diag}(s_1)$ (as $s_1 \sqsubseteq_{\forall^*} s_2$ and $\neg\mathit{Diag}(s_1) \in \forall^*$), so, $s_2 \models \mathit{Diag}(s_1)$, and s_1 is isomorphic to a substructure of s_2.

For the converse, suppose s_1 is isomorphic to a substructure of s_2. Then $s_2 \models \mathit{Diag}(s_1)$, and therefore $\mathit{Diag}(s_2) \models \mathit{Diag}(s_1)$. To see that $s_1 \sqsubseteq_{\forall^*} s_2$, consider $\varphi \in \forall^*$ such that $s_2 \models \varphi$. Assume to the contrary that $s_1 \not\models \varphi$, i.e., $s_1 \models \neg\varphi$, hence $\mathit{Diag}(s_1) \models \neg\varphi$, which implies that $\mathit{Diag}(s_2) \models \neg\varphi$, i.e. $s_2 \models \neg\varphi$, which provides a contradiction. □

So, for finite structures, $\sqsubseteq_{\forall^*}$ is the same as the substructure relation (up to isomorphism).

Lemma 5.18. *For every $s = (\mathcal{D}, \mathcal{I})$, $\mathit{Avoid}_{\forall^*}(s)$ is given by the prenex normal form of $\neg\mathit{Diag}(s)$. In particular, it is computable.*

Proof. First, $s \not\models \neg\mathit{Diag}(s)$ since $s \models \mathit{Diag}(s)$. In addition, let $\varphi \in \forall^*$ be such that $s \not\models \varphi$. Then $s \models \neg\varphi$ (where $\neg\varphi$ is existential), hence $\mathit{Diag}(s) \models \neg\varphi$, so $\varphi \models \neg\mathit{Diag}(s)$ as needed. □

With this, we have shown that requirements i and ii of Theorem 5.6 are easily satisfied by $(\mathcal{C}_{\mathrm{EPR}}, \mathcal{L}_{\forall^*})$ and subclasses thereof. The more difficult and interesting requirement is of course requirement iii, namely that $\sqsubseteq_L$ is a wqo. This requirement will be our focus in Sections 5.5 and 5.6.

5.5 Decidability of Inferring Universal Invariants for Deterministic Paths

In this section we consider the class of *deterministic paths transition systems* — transition systems that manipulate a graph with outdegree one, modeled in first-order logic using the reachability encoding presented in Section 3.3. In addition to the *path relation* representing paths in the graph (namely $\leq^2$, $\preceq^2$, btw^3, or p^3, as in Section 3.3), deterministic paths transition systems may include an unbounded number of constant symbols and unary relations. We use Theorem 5.6 to prove that inferring universal invariants for deterministic paths transition systems is decidable. We note that this result already covers many programs manipulating singly-linked-lists [112]. Furthermore, Section 5.6 will present constructions that allow to build on this class and extend it, while maintaining decidability of invariant inference. We first formalize this class, and then continue to show that it satisfies requirement iii of Theorem 5.6, namely, that $\sqsubseteq_{\forall^*}$ is always a well-quasi-order (wqo) over the state space.

5.5.1 Transition System Class for Deterministic Paths

Recall that Section 3.3 presented four encodings for paths in various graphs with outdegree one: line graphs, forests, rings, and general graphs with outdegree one (see Figure 3.11). Each encoding uses a *path relation* to capture paths in the graph: binary relation $\leq$ for lines, binary relation $\preceq$ for forests, ternary relation *btw* for rings, and ternary relation p for general graphs with outdegree one. Each encoding also provides a finite set of universally quantified axioms that serve as a theory for transition systems manipulating graphs of the appropriate class, as well as a first-order formula $\varphi_s(x, y)$ that checks if there is an edge from x to y.

We now define a class of transition systems that manipulate such graphs, and in addition to the path relation may also manipulate constant symbols and unary relations, but no function symbols nor other non-unary relations.

Definition 5.19 ($\mathcal{C}_{\text{DP}}$)**.** The class of *deterministic paths transition systems*, denoted $\mathcal{C}_{\text{DP}}$, is the set of EPR transition systems $(T, P) \in \mathcal{C}_{\text{EPR}}$, where $T = (\Sigma, \Gamma, \iota, \tau)$, such that Σ consists only of a finite number of unary relation symbols $u_1, ..., u_k$, a finite number of constant symbols $c_1, ..., c_m$, and *one* of the following relation symbols:

1. a binary relation $\leq$, in which case Γ must contain the axioms of Figure 3.12 for axiomatizing line graphs;
2. a binary relation $\preceq$, in which case Γ must contain the axioms of Figure 3.15 for axiomatizing forest graphs;
3. a ternary relation *btw*, in which case Γ must contain the axioms of Figure 3.18 for axiomatizing ring graphs; or
4. a ternary relation p, in which case Γ must contain the axioms of Figure 3.21 for axiomatizing general graphs with outdegree one.

Our main result for deterministic paths transition systems is the following theorem.

Theorem 5.20. *$\mathit{INV}[\mathcal{C}_{DP}, \mathcal{L}_{\forall^*}]$ is decidable.*

Namely, inferring universally quantified inductive invariants for deterministic paths transition systems is decidable. We note that this is in contrast to the fact that $\texttt{SAFE}[\mathcal{C}_{\text{DP}}]$ is undecidable, since this class includes programs manipulating unbounded linked-lists, which is Turing-complete. Theorem 5.48 presented in Section 5.7 shows that restricting to universal invariants is necessary for decidability, since inferring alternation-free invariants for $\mathcal{C}_{\text{DP}}$ is

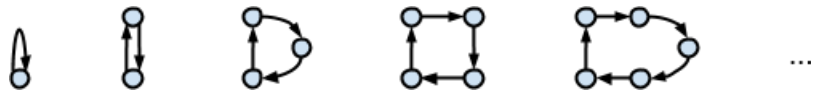


Figure 5.22: Infinite sequence of incomparable structures (antichain) w.r.t. $\sqsubseteq_{\forall^*}$. Edges represent a binary relation (i.e., not using any paths encoding).

undecidable. Section 5.7 also shows that inferring universal invariants for a more general class of EPR transition systems is undecidable (Theorem 5.51).

We prove Theorem 5.20 by applying Theorem 5.6. Requirements i and ii of Theorem 5.6 were established in the previous section, and the following subsection establishes requirement iii, namely, that the state space of any transition system in $\mathcal{C}_{\mathrm{DP}}$ is well-quasi-ordered by $\sqsubseteq_{\forall^*}$ (Theorem 5.21).

5.5.2 Deterministic Paths are Well-Quasi-Ordered by $\sqsubseteq_{\forall^*}$

The proof of Theorem 5.20 is completed by the following theorem:

Theorem 5.21. *For* $(\Sigma, \Gamma, \iota, \tau) \in \mathcal{C}_{DP}$, $(STRUCT[\Sigma, \Gamma], \sqsubseteq_{\forall^*})$ *is a wqo.*

The above theorem contains four cases, namely lines, forests, rings, and general graphs with outdegree one. The case of lines and the case of rings are easier and less interesting, and they also follow from the cases of forests and general graphs with outdegree one. For the latter cases, we present a proof based on Kruskal's Tree Theorem (which we explain next). The case of forests is also implied by the case of general graphs with outdegree one, but in the interest of presentation we present the simpler case of forests first (in which the graphs are acyclic), and then explain how to adjust the proof to handle cyclic graphs.

We also note that in general, $(\mathrm{STRUCT}[\Sigma], \sqsubseteq_{\forall^*})$ is not a wqo if Σ contains a binary relation symbol (mind the absence of a theory). For example, the infinite sequence of structures depicted in Figure 5.22 is an antichain for $\sqsubseteq_{\forall^*}$, i.e., the structures are all incomparable w.r.t. $\sqsubseteq_{\forall^*}$, since none of them can be isomorphically embedded into another (note that edges represent the binary relation).

Kruskal's Tree Theorem Let X be a set. A *labeled graph over* X is a finite *undirected* graph $G = (V, E, \ell)$ that includes a vertex labeling function $\ell : V \to X$. If G is a tree (undirected connected acyclic graph), then it is called a *labeled tree over* X.

Definition 5.23 (Tree homeomorphic embedding)**.** Suppose that $(X, \leq)$ is a quasi-ordered set. Let $(\mathcal{T}(X), \trianglelefteq)$ be the set of all labeled trees over X, with the following ordering: $t_1 \trianglelefteq t_2$ iff t_1 can be obtained from t_2 by a finite number of the following operations:

- Removing a vertex of degree 1 (and the corresponding edge).
- Removing a vertex of degree 2 (and the corresponding edges), and adding an edge between its two neighbors.
- Changing the label of a vertex to a lower value.

Note that the degree of a vertex is the number of adjacent edges (and not the number of children—since we consider unrooted trees, the notion of children of a vertex is not well defined).

Fact 5.24 (Kruskal's Tree Theorem, [129, 177].)**.** *If* $(X, \leq)$ *is a wqo, then so is* $(\mathcal{T}(X), \trianglelefteq)$.

We now prove Theorem 5.21 for the case of forests (acyclic graphs with outdegree one), and then adjust the proof to handle general (possibly cyclic) graphs with outdegree one.

Proof of Theorem 5.21 for forests. Let $\Gamma_\preceq$ denote a theory that contains the axioms of Figure 3.15 for representing *directed* acyclic graphs (forests) using the binary relation $\preceq$, and let Σ be a vocabulary that, as in Definition 5.19, consists only of a finite number of unary relation symbols $u_1, ..., u_k$, a finite number of constant symbols $c_1, ..., c_m$, and a binary relation symbol $\preceq$.

To show that $(\text{STRUCT}[\Sigma, \Gamma_\preceq], \sqsubseteq_{\forall^*})$ is a wqo, we will encode the directed forests of $\text{STRUCT}[\Sigma, \Gamma_\preceq]$ into the undirected trees of $(\mathcal{T}(X), \leq)$ where X is a certain finite set under the trivial wqo $(X, =)$. We will then apply Fact 5.24 to obtain that $(\mathcal{T}(X), \leq)$ is a wqo. The properties of the encoding will guarantee that this implies that $(\text{STRUCT}[\Sigma, \Gamma_\preceq], \sqsubseteq_{\forall^*})$ is a wqo. Namely, the encoding will be such that $\trianglelefteq$ of the encodings entails $\sqsubseteq_{\forall^*}$ of the structures of $\text{STRUCT}[\Sigma, \Gamma_\preceq]$.

We first define a function $f\colon \text{STRUCT}[\Sigma, \Gamma_\preceq] \to \mathcal{T}(X)$ that encodes each directed forest s as an undirected tree $f(s)$. The mapping f adds a special new root, v_{root}, connects v_{root} to each root of s, and makes all the directed edges determined by $\preceq$, undirected.

Let $X = \mathcal{P}(\{u_1, ..., u_k, c_1, ..., c_m\}) \cup \{l_{\text{root}}\}$. We use this finite set X to label each vertex in $f(s)$ according to whether the corresponding vertex from s satisfies each unary predicate, u_i and whether it is equal to the constant c_j. The new vertex v_{root} is labeled l_{root}. The mapping f is illustrated in Figure 5.25.

The following equations explicitly define the mapping f. Given $s = (\mathcal{D}, \mathcal{I}) \in$

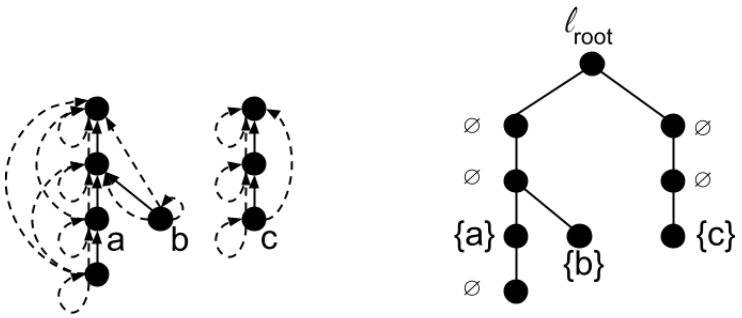


Figure 5.25: The transformation between STRUCT$[\Sigma, \Gamma_{\preceq}]$ and $\mathcal{T}(X)$. Left: a structure with three constants: a, b, c; dashed edges depict $\preceq$. Right: an undirected labeled tree corresponding to the structure on the left.

STRUCT$[\Sigma, \Gamma_{\preceq}]$, we put $f(s) = (V, E, \ell)$ where:

$$V = \mathcal{D} \cup \{v_{\text{root}}\} \text{ where } v_{\text{root}} \notin \mathcal{D}$$

$$E = \{\{a, b\} \mid a, b \in \mathcal{D} \text{ and } s \models \varphi_s(a, b)\} \cup \{\{a, v_{\text{root}}\} \mid a \in \mathcal{D} \text{ and } s \models \forall x.\, \neg\varphi_s(a, x)\}$$

where φ_s is as in Figure 3.15

$$\ell(a) = \begin{cases} \{u_i \mid a \in \mathcal{I}(u_i)\} \cup \{c_i \mid a = \mathcal{I}(c_i)\} & a \neq v_{\text{root}} \\ l_{\text{root}} & a = v_{\text{root}} \end{cases}$$

The labeling clearly maintains all the information contained in the structure about the constants and the unary predicates, and also maintains the distinction of the new vertex v_{root} (via the label l_{root}). It is easy to regain the directed forest s from the labeled undirected tree, $f(s)$. To formalize this, we define $\mathcal{T}_{\preceq} \subseteq \mathcal{T}(X)$ to be the set of all labeled trees (V, E, ℓ) over X such that there is exactly one vertex labeled l_{root}, and for every constant symbol c_i there is exactly one vertex $a \in V$ such that $c_i \in l(a)$. Then, f is one-to-one and onto (up to isomorphism) from STRUCT$[\Sigma, \Gamma_{\preceq}]$ to $\mathcal{T}_{\preceq}$.

Define the inverse mapping $g\colon \mathcal{T}_{\preceq} \to$ STRUCT$[\Sigma, \Gamma_{\preceq}]$ as follows. Given $t = (V, E, \ell) \in \mathcal{T}_{\preceq}$, let v_{root} be the unique vertex in V labeled l_{root}, and let $g(t) = (\mathcal{D}, \mathcal{I}) \in$ STRUCT$[\Sigma, \Gamma_{\preceq}]$, where:

$$\mathcal{D} = V \setminus \{v_{\text{root}}\}$$

$$\mathcal{I}(c_i) = v_{c_i} \text{ such that } c_i \in \ell(v_{c_i})$$

$$\mathcal{I}(u_i) = \{v \in \mathcal{D} \mid u_i \in \ell(v)\}$$

$$\mathcal{I}(\preceq) = \{(u, v) \in \mathcal{D}^2 \mid u = v \text{ or the path in } t \text{ from } u \text{ to } v_{\text{root}} \text{ contains } v\}$$

Note that for any $t \in \mathcal{T}_{\preceq}$, $g(t) \models \Gamma_{\preceq}$, and $g = f^{-1}$.

To complete the proof that $(\text{STRUCT}[\Sigma, \Gamma_{\preceq}], \sqsubseteq_{\forall^*})$ is a wqo, let $s_1, s_i, \ldots$ be an infinite sequence of structures in $\text{STRUCT}[\Sigma, \Gamma_{\preceq}]$. Consider the infinite sequence $f(s_1), f(s_2), \ldots$ of labeled trees. Since $(\mathcal{T}(X), \trianglelefteq)$ is a wqo, there exists $i < j$ such that $f(s_i) \trianglelefteq f(s_j)$, i.e., $f(s_i)$ can be obtained from $f(s_j)$ by the three operations of Definition 5.23. Since the labels are ordered by equality, there are only two operations to consider: removing a vertex of degree 1, and replacing a vertex of degree 2 by an edge. Note that since $f(s_i), f(s_j) \in \mathcal{T}_{\preceq}$, the vertex labeled by l_{root} and vertices representing constants cannot be removed. Now, consider any $t \in \mathcal{T}_{\preceq}$. Observe that if t' is obtained from t by removing a vertex v of degree 1 that is not labeled by l_{root} and does not represent a constant, then $g(t')$ is isomorphic to a substructure of $g(t)$, obtained by removing v from the domain of $g(t)$. This is also true for removing a vertex of degree 2 and replacing it by an edge, since this operation preserves $\preceq$ (representing the reflexive transitive closure of edges) between all remaining vertices. Since $f(s_i)$ can be obtained from $f(s_j)$ by a finite sequence of such operations, we conclude that s_i is isomorphic to a substructure of s_j, i.e., $s_i \sqsubseteq_{\forall^*} s_j$. Thus, $(\text{STRUCT}[\Sigma, \Gamma_{\preceq}], \sqsubseteq_{\forall^*})$ is a wqo. □

For the case of general graphs with outdegree one, we use the same proof strategy, and the crux of the argument is still Kruskal's Tree Theorem. However, we must adapt the function f to encode general directed graphs with outdegree one (including cyclic graphs) into undirected labeled trees, while maintaining the connection between substructures and homeomorphic embedding. The key idea for this is to break the cycles arbitrarily, and then remember which vertices where part of cycles using the labels; this allows to recover the original (cyclic) graph, while ensuring the required connection to homeomorphic embedding.

Proof of Theorem 5.21 for general graphs with outdegree one. Let Γ_p denote a theory that contains the axioms of Figure 3.21 for representing general *directed* graphs with outdegree one using the ternary relation p, and let Σ be a vocabulary that, as in Definition 5.19, consists only of a finite number of unary relation symbols $u_1, ..., u_k$, a finite number of constant symbols $c_1, ..., c_m$, and a ternary relation symbol p.

Define a function $f\colon \text{STRUCT}[\Sigma, \Gamma_p] \to \mathcal{T}(X)$ that encodes each graph $s \in \text{STRUCT}[\Sigma, \Gamma_p]$ as an undirected tree $f(s)$. As before, the mapping f adds a special new root, v_{root}, connects v_{root} to each "root" of s (see below), and makes all the directed edges determined by p (exluding some edges in order to break cycles, see below), undirected. Here we also add a new label l_{cycle}, acting as a unary predicate that encodes whether a vertex is part of a cycle (i.e., is the vertex reachable from itself by a path with at least one edge).

Let $X = \mathcal{P}(\{u_1, ..., u_k, c_1, ..., c_m, l_{cycle}\}) \cup \{l_{\text{root}}\}$. Given $s = (\mathcal{D}, \mathcal{I}) \in \text{STRUCT}[\Sigma, \Gamma_p]$, recall that it defines a *directed* graph with outdegree one, which may contain cycles, as per Theorem 3.24. Denote this directed graph by $(\mathcal{D}, E')$, where $E' \subseteq \mathcal{D} \times \mathcal{D}$. Let $E'' \subseteq E'$ be a maximal subset of E' such that $(\mathcal{D}, E'')$ is acyclic[3]. Let $C' = \{v \in \mathcal{D} \mid s \models p(v, v, v)\}$ be the set of vertices that are part of a cycle in $(\mathcal{D}, E')$ (see Section 3.3.4, specifically Figure 3.21). Set $f(s) = (V, E, \ell)$ where:

$$
\begin{aligned}
V &= \mathcal{D} \cup \{v_{\text{root}}\} \text{ where } v_{\text{root}} \notin \mathcal{D} \\
E &= \left\{\{a, b\} \mid (a, b) \in E''\right\} \cup \left\{\{a, v_{\text{root}}\} \mid a \in \mathcal{D} \text{ and } \forall b \in \mathcal{D}.\, (a, b) \notin E''\right\} \\
\ell(a) &= \begin{cases}
\{u_i \mid a \in \mathcal{I}(u_i)\} \cup \{c_i \mid a = \mathcal{I}(c_i)\} & a \neq v_{\text{root}} \text{ and } a \notin C' \\
\{u_i \mid a \in \mathcal{I}(u_i)\} \cup \{c_i \mid a = \mathcal{I}(c_i)\} \cup \{l_{cycle}\} & a \neq v_{\text{root}} \text{ and } a \in C' \\
l_{\text{root}} & a = v_{\text{root}}
\end{cases}
\end{aligned}
$$

Observe that the labeling maintains enough information to reconstruct the directed graph s from the labeled undirected tree, $f(s)$. The cycles can be reconstructed from the l_{cycle} labels, by adding an edge from any vertex u that is both adjacent to v_{root} and has $l_{cycle} \in \ell(u)$ to the "first" v whose path to v_{root} contains u and has $l_{cycle} \in \ell(u)$. Here, the "first" v is the v that maximizes the length of the path from v to v_{root} in $f(s)$, while still having $l_{cycle} \in \ell(v)$. This v is unique since each cycle in s becomes a line in $f(s)$.

To formalize this, we define $\mathcal{T}_p \subseteq \mathcal{T}(X)$ to be the set of all labeled trees (V, E, ℓ) over X such that there is exactly one vertex v_{root} labeled l_{root}, for every constant symbol c_i there is exactly one vertex $a \in V$ such that $c_i \in l(a)$, and the vertices $C = \{v \in V \mid l_{cycle} \in \ell(v)\}$ whose label includes l_{cycle} have the following two properties: (i) if $v \in C$, then $P(v) \subseteq C$, where $P(v)$ is the set of vertices on the path from v to v_{root} (including v, excluding v_{root}); and (ii) if $u, v \in C$, then either $u \in P(v)$, $v \in P(u)$, or $P(u) \cap P(v) = \emptyset$. The condition on C ensures that the vertices whose label includes l_{cycle} form lines ending in vertices adjacent to v_{root}, rather than allowing them to form arbitrary sub trees. With this definition, f is one-to-one and onto (up to isomorphism) from $\text{STRUCT}[\Sigma, \Gamma_p]$ to $\mathcal{T}_p$.

Define the inverse mapping $g \colon \mathcal{T}_p \to \text{STRUCT}[\Sigma, \Gamma_p]$ as follows. Given $t = (V, E, \ell) \in \mathcal{T}_p$, let v_{root} be the unique vertex in V labeled l_{root}, and $C = \{v \in V \mid l_{cycle} \in \ell(v)\}$ as above.

[3]Such a set E'' must exist, and it can be consistently chosen.

We extract the directed (possibly cyclic) graph $(\mathcal{D}, E')$ as follows:

$$\mathcal{D} = V \setminus \{v_{\text{root}}\}$$
$$E' = \left\{(u,v) \in \mathcal{D}^2 \mid P(u) = \{u\} \cup P(v)\right\} \cup \{(u, c(u)) \mid u \in C \text{ and } \{u, v_{\text{root}}\} \in E\}$$
$$\text{where } c(u) = \underset{v \in C \text{ s.t. } u \in P(v)}{\arg\max} |P(v)|$$

The function $c(u)$ defined above is well-defined due to the properties of $\mathcal{T}_p$ (i.e., the arg max is unique). From $(\mathcal{D}, E')$, one can obtain an interpretation for p (as defined in Section 3.3.4), and the interpretation of unary relations and constants can be obtained as in the proof for forests.

To complete the proof that $(\text{STRUCT}[\Sigma, \Gamma_p], \sqsubseteq_{\forall^*})$ is a wqo, observe that removing a vertex of degree 1, or removing a vertex of degree 2 and adding an edge (as in Definition 5.23), in the undirected tree (V, E) corresponds to removing a vertex (and possibly adding an edge) in the directed graph $(\mathcal{D}, E')$, while the valuation of p is preserved on all remaining vertices. This means that for $s_i, s_j \in \text{STRUCT}[\Sigma, \Gamma_p]$, if $f(s_i) \trianglelefteq f(s_j)$ then $s_i \sqsubseteq_{\forall^*} s_j$, and the proof is completed by the same argument used for the case of forests. Thus, $(\text{STRUCT}[\Sigma, \Gamma_p], \sqsubseteq_{\forall^*})$ is a wqo. □

5.6 Systematic Constructions of Decidable Classes

The result of Section 5.5 implies that inference of universal invariants is decidable for programs manipulating linked-lists, as well as other programs and systems that can be modeled with a graph that has outdegree one and unary relations. However, as we have seen, e.g., in Chapter 4, to model distributed protocols and other interesting systems one usually has to use relations of higher arity, and/or relations unrestricted by any background theory (e.g., to model messages, states of nodes, etc.). Unfortunately, in Section 5.7.3 we show that this quickly leads to undecidability of inferring universal invariants (Theorem 5.51). To mitigate this gap, in this section we develop constructions that allow to include higher arity relations and relations unrestricted by background theories, and maintain decidability of invariant inference by imposing further syntactic restrictions on the potential inductive invariants. When combined, the constructions we develop are quite powerful, and can capture the interesting example of the "Learning Switch" network routing protocol.

Motivating example: network learning switch As a motivating example, we consider the network learning switch protocol. Learning switches maintain routing tables, and learn

```
1  sort node
2
3  relation msg : node, node, node, node
4  relation learned : node, node
5  relation route : node, node, node
6  relation route* : node, node, node
7
8  axiom Γ_route*
9
10 init ∀w, x, y, z. ¬msg(w, x, y, z)
11 init ∀x, y. ¬learned(x, y)
12 init ∀x, y, z. ¬route(x, y, z)
13 init ∀x, y, z. route*(x, y, z) ↔ y = z
14
15 action NEW_PACKET(s : node, d : node) {
16     # generat new packet by making it self−pending at its source, so it will be forwarded
17     assume s ≠ d
18     msg(s, d, s, s) := true
19 }
20
21 action FORWARD(s : node, d : node, n : node, m : node) {
22     # forward a packet, and learn a route to its source
23     assume msg(s, d, n, m)
24     msg(s, d, n, m) := *
25     # when s is unknown at m, learn a new route to s
26     if ¬learned(s, m) ∧ s ≠ m then {
27         assert ¬route*(s, n, m) # safety property: check that a cycle is not created
28         # update the routing table
29         learned(s, m) := true
30         route(s, m, n) := true
31         route*(s, X, Y) := route*(s, X, Y) ∨ (route*(s, X, m) ∧ route*(s, n, Y))
32     }
33     # when d = m consume the packet, otherwise forward it
34     if d ≠ m then {
35         if ¬learned(d, m) then {
36             # when no route to d is known, flood the packet
37             msg(s, d, m, X) := msg(s, d, m, X) ∨ (link(m, X) ∧ X ≠ n)
38         } else {
39             # when a route is knonw, forward according to route
40             msg(s, d, m, X) := msg(s, d, m, X) ∨ route(d, m, X)
41         }
42 }
```

Figure 5.26: RML model of the learning switch network routing algorithm. The sort node represents switches in the network. The theory Γ_{route^*} is obtained from applying the axioms of Figure 3.15 to $route^*(d, \cdot, \cdot)$ for all d, as explained in Section 5.6.4.

routes as they receive packets. When a packet first arrives from an unknown source, the switch learns a route to its source through the incoming link. It then checks if it has a route to the packet's destination, and either forwards the packet to a known route or floods it to all but the incoming link. For this protocol, we consider the safety property that the created routing tables do not contain forwarding loops. The learning switch is a parameterized distributed system with infinite-state, as there is an unbounded number of switches, and the routing table of each switch contains an unbounded number of entries. The system and safety property can be modeled as an EPR transition system, using the techniques of Chapter 3. Specifically, transitive closure must be encoded in order to state the safety property that forwarding graphs are acyclic. Figure 5.26 provides a model of the learning switch protocol in RML.

We now describe the relations used by the model. The relation $link^2$ describes the links in the network. The relation msg^4 describes pending packets, and $msg(s, d, sw_1, sw_2)$ denotes the fact that a packet with source field s and destination field d is pending on the link from switch sw_1 to switch sw_2. The relations $learned^2, route^3, route^{*3}$ store information about the current routing tables of the switches. $learned(d, sw)$ denotes that switch sw has learned a route to destination d. $route(d, sw_1, sw_2)$ denotes that a packet with destination field d will be routed by switch sw_1 to switch sw_2. Thus for any d, $route(d, \cdot, \cdot)$ holds the *forwarding graph* for d, which describes how packets with destination d will be forwarded in the network. We wish to verify that this graph is acyclic for all d. To this end, the relation $route^*$ describes paths in this graph: $route^*(d, sw_1, sw_2)$ holds if $route(d, \cdot, \cdot)$ contains a path from sw_1 to sw_2. Formally, $route^*(d, \cdot, \cdot)$ is the reflexive transitive closure of $route(d, \cdot, \cdot)$ for any d. The model maintains this fact by the standard technique of updating transitive closure (e.g., [112], also presented in Section 3.3). The **assert** statement asserts that whenever a switch learns a new route, it does not introduce a cycle in the forwarding graph.

A key point that we later exploit is that for each d, the forwarding graph for d has outdegree one. Note however that here we use a relaxed version of the encoding of Section 3.3, since we do not replace the edges by the path relation, and instead keep both $route$ and $route^*$. This is still sound, and as we shall see, precise enough for verifying the learning switch protocol.

Gradual extensions While the model of Figure 5.26 represents an EPR transition system, it is not apparent how the results of Sections 5.3 to 5.5 can be applied to obtain decidability of invariant inference for it. In the rest of this section we develop constructions that obtain this by limiting the invariants to universal sentences that satisfy further syntactic restrictions.

We do so by gradual extensions of classes of transition systems and languages. The extensions start from an established decidable class, and each step extends the expressive power of the transition system or the language for invariants in a limited way that preserves the fact that $\sqsubseteq_L$ is a wqo, and Avoid_L is computable (L is the language of invariants).

Formally, each extension starts with an EPR class paired with a language class of universal invariants $(\mathcal{C}, \mathcal{L})$ which satisfy the conditions of Theorem 5.6, and constructs a new EPR class and a corresponding language class $(\mathcal{C}', \mathcal{L}')$ that also satisfy these conditions, ensuring that $\texttt{INV}[\mathcal{C}', \mathcal{L}']$ is decidable. We define $(\mathcal{C}', \mathcal{L}')$ by describing the vocabulary Σ', the theory Γ', and the language L' used in instances of $(\mathcal{C}', \mathcal{L}')$. For each extension, we show that $\sqsubseteq_{L'}$ is a wqo and $\text{Avoid}_{L'}$ is computable (note that since this section only considers EPR classes and languages of universal sentences, all the effectiveness assumptions are trivially satisfied).

Notation. For the remainder of this section, let us fix a vocabulary Σ, a theory Γ consisting of universal sentences (note that this is the case for theories used for encoding deterministic paths as in Section 3.3), and a base language L of universal sentences, taken from $(\mathcal{C}, \mathcal{L})$ that satisfy the conditions of Theorem 5.6. Let $cl(L)$ be the closure of L under conjunction, disjunction, and rewriting into an equivalent formula. Note that $\sqsubseteq_L = \sqsubseteq_{cl(L)}$. Thus, we assume w.l.o.g. that L is closed, i.e., $L = cl(L)$.

Every universal sentence can be written as a conjunction of closed *universal clauses*, each of which has the form $\forall x_1 \dots x_r. \beta$ where β is the *body*, consisting of a disjunction of literals over the variables $x_1 \dots x_r$. Observe that a language L of universal formulas (assuming $L = cl(L)$) is determined by the set of *its bodies*, i.e., the set $\{\beta \mid \beta \text{ is a disjunction of literals over } x_1 \dots x_r \text{ and } (\forall x_1 \dots x_r. \beta) \in L\}$.

5.6.1 Basic Extensions

The following basic extensions of wqo's are immediate: if $(\text{STRUCT}[\Sigma, \Gamma], \sqsubseteq_L)$ is a wqo then it remains a wqo if we strengthen the background theory, restrict the language, or extend the vocabulary (while keeping the language the same set of formulas).

Proposition 5.27. *If $(STRUCT[\Sigma, \Gamma], \sqsubseteq_L)$ is a wqo then so are:*

1. $(STRUCT[\Sigma, \Gamma'], \sqsubseteq_L)$, *if* $\Gamma' \models \Gamma$
2. $(STRUCT[\Sigma, \Gamma], \sqsubseteq_{L'})$, *if* $L' \subseteq L$
3. $(STRUCT[\Sigma', \Gamma], \sqsubseteq_L)$, *if* $\Sigma \subseteq \Sigma'$

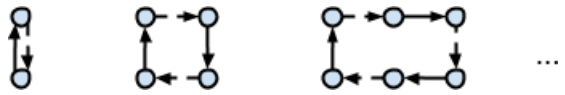


Figure 5.31: Infinite sequence of incomparable models w.r.t. $\sqsubseteq_{\forall^*}^{\preceq_1,\preceq_2}$. Solid arrows are edges determined by $\preceq_1$, and dashed arrows are edges determined by $\preceq_2$.

Proof. If $\Gamma' \models \Gamma$, then $\text{STRUCT}[\Sigma,\Gamma] \subseteq \text{STRUCT}[\Sigma,\Gamma']$, and case 1 follows. If $L' \subseteq L$, then $\sqsubseteq_L \subseteq \sqsubseteq_{L'}$, and case 2 follows. Finally, if $\Sigma \subseteq \Sigma'$, let $(s'_i)_{i=1}^{\infty}$ be an infinite sequence of structures in $\text{STRUCT}[\Sigma',\Gamma]$. By projecting each state s'_i to a state s_i over Σ, we obtain the sequence $(s_i)_{i=1}^{\infty}$, for which there exist $i < j$ such that $s_i \sqsubseteq_L s_j$. Since L is defined over Σ, for every $\varphi \in L$, $\varphi \models s_i$ iff $\varphi \models s'_i$. Therefore, $s'_i \sqsubseteq_L s'_j$ as well, and case 3 follows. □

Remark 5.28. In the context of Proposition 5.27, if there is a procedure to compute Avoid_L, then the same procedure will work for cases 1 and 3, but not necessarily for case 2.

The wqo property is also preserved under unions of languages:

Proposition 5.29. *If* $(\mathit{STRUCT}[\Sigma,\Gamma],\sqsubseteq_{L_1})$, $(\mathit{STRUCT}[\Sigma,\Gamma],\sqsubseteq_{L_2})$ *are wqo's then so is* $(\mathit{STRUCT}[\Sigma,\Gamma],\sqsubseteq_L)$ *where* $L = cl(L_1 \cup L_2)$.

Proof. We first prove the claim for $L = L_1 \cup L_2$. As explained above, $\sqsubseteq_L = \sqsubseteq_{cl(L)}$, hence the claim follows for $cl(L_1 \cup L_2)$ as well. Suppose that $s_1, s_2, \ldots$ is an infinite sequence of structures from $(\text{STRUCT}[\Sigma,\Gamma],\sqsubseteq_L)$. Since $(\text{STRUCT}[\Sigma,\Gamma],\sqsubseteq_{L_1})$ is a wqo, there must exist an infinite increasing subsequence under $\sqsubseteq_{L_1}$, $s_{i_1}, s_{i_2}, \ldots$. Since $(\text{STRUCT}[\Sigma,\Gamma],\sqsubseteq_{L_2})$ is a wqo, there exist $j < k$ such that $s_{i_j} \sqsubseteq_{L_2} s_{i_k}$ Thus, every formula in $L_1 \cup L_2$ satisfied by s_{i_k} is satisfied by s_{i_j}. That is, $s_{i_j} \sqsubseteq_L s_{i_k}$, as desired. □

Remark 5.30. If there are procedures to compute Avoid_{L_1} and Avoid_{L_2}, then Avoid_L for $L = cl(L_1 \cup L_2)$ is also computable, and given by:

$$\text{Avoid}_L(s) = \text{Avoid}_{L_1}(s) \vee \text{Avoid}_{L_2}(s)$$

Remark 5.32. Proposition 5.29 considers $L = cl(L_1 \cup L_2)$, which contains conjunctions and disjunctions of *closed* universal clauses from L_1 and L_2. Let $L' \supseteq L$ be the universal language containing also clauses whose bodies are obtained by disjunctions of bodies of L_1 with bodies of L_2 (within the scope of the quantifier prefix). It is tempting to try to prove that $\sqsubseteq_{L'}$ is a wqo. However, in general this is not the case. Consider a language $\sqsubseteq_{\forall^*}^{\preceq_1,\preceq_2}$ of universal formulas for structures in the signature $\{\preceq_1,\preceq_2\}$ and the background theory $\Gamma_{\preceq_1} \cup \Gamma_{\preceq_2}$, that is, a union of two copies of the theory of forests (as in Figure 3.15), but for two separate

relations. Then we can construct an infinite antichain using cycles of even length formed by interchanging $\preceq_1$ and $\preceq_2$ edges, as depicted in Figure 5.31. This shows that $\sqsubseteq_{\forall^*}^{\preceq_1,\preceq_2}$ is not a wqo.

We can also extend L by adding any ground atom, g, to L (this is applicable for example if case 3 of Proposition 5.27 was applied to add the predicate and/or constants in g to Σ without extending L).

Proposition 5.33. *If* $(STRUCT[\Sigma,\Gamma],\sqsubseteq_L)$ *is a wqo and* g *is a ground atom of* Σ*, then* $(STRUCT[\Sigma,\Gamma],\sqsubseteq_{L'})$ *is a wqo where* L' *has the bodies* $\beta, g\vee\beta, \neg g\vee\beta$ *for each body* β *of* L*.*

Proof. Let $(s_i)_{i=1}^{\infty}$ be an infinite sequence of structures in $\mathrm{STRUCT}[\Sigma,\Gamma]$. Since g is a ground atom, every structure gives it a valuation of true or false, so $(s_i)_{i=1}^{\infty}$ contains an infinite subsequence where all structures give g the same valuation. Therefore there exist $i<j$ such that $s_i\sqsubseteq_L s_j$ and s_i, s_j give g the same valuation. It follows that $s_i\sqsubseteq_{L'} s_j$, and thus $(\mathrm{STRUCT}[\Sigma,\Gamma],\sqsubseteq_{L'})$ is a wqo. □

Remark 5.34. If there is a procedure to compute Avoid_L, then for L' of Proposition 5.33, $\mathrm{Avoid}_{L'}$ is also computable, and given by

$$\mathrm{Avoid}_{L'}(s)=\begin{cases}\mathrm{Avoid}_L(s)\vee\neg g & s\models g\\ \mathrm{Avoid}_L(s)\vee g & s\not\models g\end{cases}$$

By combining Propositions and Remarks 5.27, 5.28, 5.33, and 5.34 we get the following corollary:

Corollary 5.35. *Extending the vocabulary* Σ *and the language* L *by adding to* Σ *any number of new relations and adding to any body of* L *any number of disjunctions of ground literals constructed from the new relations maintains wqo and computability of* $Avoid_L$*.*

An operation needed for the constructions that follow later is extension by a new constant symbol. This requires some uniformity from the base language, formalized in the following definition:

Definition 5.36 (Constant-extendable)**.** Let $(\mathrm{STRUCT}[\Sigma,\Gamma],\sqsubseteq_L)$ be a wqo with Avoid_L computable. We say that L is *constant-extendable* (in the context of Σ and Γ), if for any finite set of fresh constant symbols C:

- $(\mathrm{STRUCT}[\Sigma\cup C,\Gamma],\sqsubseteq_{L'})$ is also a wqo, where the bodies of L' are obtained from the bodies of L by any number of substitutions of variables by constants from C.

- $\text{Avoid}_{L'}$ is computable over $\text{STRUCT}[\Sigma \cup C, \Gamma]$.

Remark 5.37. Since Theorem 5.21 allows an arbitrary number of constants to begin with, all languages in $(\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*})$ are constant-extendable. Also, all constructions presented so far result in constant-extendable languages, assuming that the base languages are constant-extendable. Obviously, in the context of Definition 5.36, if L is constant-extendable then so is L'.

5.6.2 Symmetric Lifting

In this section, we show that a decidability result for some vocabulary, theory and language can be lifted to a vocabulary which describes an unbounded number of instances of the original theory, by parameterizing the theory and creating a language of *symmetric* sentences, that do not correlate the different instances. As an example for this, consider the routing tables of learning switches. For each destination d, each switch has a single "next" pointer for packets destined to d, which is described by the *route* relation. Thus, the routing tables can be seen as an unbounded number of graphs with outdegree one (actually forests, since they are also acyclic as asserted by the safety property), parameterized by the destination of packets. The extension developed in this section can be used to lift the results of Section 5.5 to capture the ternary relation $route^*$, and allow invariants to refer to paths in the forwarding graphs with unbounded quantification, as long as they do not correlate forwarding graphs of different destinations.

The basis for lifting a wqo from a theory to an unbounded number of instances of the theory relies on the following corollary of Higman's Lemma [104]:

Fact 5.38 (Higman's Lemma for finite sets [104])**.** *If $(X, \leq)$ is a wqo, then so is $(\mathcal{P}_{fin}(X), \preceq)$ where $\mathcal{P}_{fin}(X)$ is the set of finite subsets of X, and $A \preceq B$ iff $\forall s \in A.\, \exists t \in B.\, s \leq t$.*

We start by using Fact 5.38 to show that we can perform symmetric lifting and preserve wqo's and computability of Avoid_L. We define *symmetric lifting* as the removal of a constant symbol a from the vocabulary Σ, while replacing a by a new universally quantified variable v, both in the theory Γ and in the formulas of L. The latter operation is denoted ρ_a (e.g., $\rho_a(\forall x.\, P(a) \vee Q(a, x)) = \forall v, x.\, P(v) \vee Q(v, x)$). We also extend ρ_a to sets of formulas, i.e., $\rho_a(A) = \{\rho_a(\varphi) \mid \varphi \in A\}$. The next proposition shows that symmetric lifting preserves wqo's and computability of Avoid_L.

Proposition 5.39 (Symmetric lifting)**.** *If $(STRUCT[\Sigma, \Gamma], \sqsubseteq_L)$ is a wqo and a is a constant*

symbol in Σ*, then* $(STRUCT[\Sigma', \Gamma'], \sqsubseteq_{L'})$ *is a wqo where*

$$\Sigma' = \Sigma \setminus \{a\}, \quad \Gamma' = \rho_a(\Gamma), \quad L' = \rho_a(L)$$

Proof. Define the function $A_a \colon \mathrm{STRUCT}[\Sigma'] \to \mathcal{P}_{fin}(\mathrm{STRUCT}[\Sigma])$ by

$$A_a((\mathcal{D}, \mathcal{I})) = \{(\mathcal{D}, \mathcal{I}[a \mapsto d]) \mid d \in \mathcal{D}\}$$

That is, A_a maps a structure of Σ' to the set of structures of Σ in which we interpret the new constant symbol, a, in all possible ways. Note that since all our structures are finite, there are only finitely many ways to interpret a in any given structure.

The semantics of a universal quantifier tells us that for any structure $s' \in \mathrm{STRUCT}[\Sigma']$ and any formula φ of vocabulary Σ,

$$s' \models \rho_a(\varphi) \quad \text{iff} \quad s \models \varphi \ \text{ for each } s \in A_a(s'). \tag{5.40}$$

Thus, if $s' \models \Gamma'$ then for any $s \in A_a(s')$ we have $s \models \Gamma$. Thus, A_a maps $\mathrm{STRUCT}[\Sigma', \Gamma']$ to $\mathcal{P}_{fin}(\mathrm{STRUCT}[\Sigma, \Gamma])$.

To prove that $(\mathrm{STRUCT}[\Sigma', \Gamma'], \sqsubseteq_{L'})$ is a wqo, let $(s'_i)_{i=1}^{\infty}$ be an infinite sequence of structures in $\mathrm{STRUCT}[\Sigma', \Gamma']$. Now, consider the infinite sequence $(A_a(s'_i))_{i=1}^{\infty}$. This is an infinite sequence of elements of $\mathcal{P}_{fin}(\mathrm{STRUCT}[\Sigma, \Gamma])$, which is a wqo by Fact 5.38 and the fact that $(\mathrm{STRUCT}[\Sigma, \Gamma], \sqsubseteq_L)$ is a wqo. Thus, we have $i < j$ such that:

$$\forall s_1 \in A_a(s'_i)\ \exists s_2 \in A_a(s'_j)\ s_1 \sqsubseteq_L s_2 \tag{5.41}$$

To show that $s'_i \sqsubseteq_{L'} s'_j$, let φ' be an arbitrary formula from L' such that $s'_j \models \varphi'$. Thus $\varphi' = \rho_a(\varphi)$ for some $\varphi \in L$. By eq. (5.40), $\forall s_2 \in A_a(s'_j)\ s_2 \models \varphi$. Thus, by eq. (5.41), $\forall s_1 \in A_a(s'_i)\ s_1 \models \varphi$. Thus, again by eq. (5.40), $s'_i \models \varphi$. □

Remark 5.42. In the setting of Proposition 5.39, if Avoid_L is computable for structures of $\mathrm{STRUCT}[\Sigma, \Gamma]$, then $\mathrm{Avoid}_{L'}$ is computable for structures of $\mathrm{STRUCT}[\Sigma', \Gamma']$ and is given by:

$$\mathrm{Avoid}_{L'}(s') = \bigvee_{s \in A_a(s')} \rho_a(\mathrm{Avoid}_L(s))$$

The correctness of this definition follows from the definitions of L' and $A_a(s')$, the properties of Avoid_L and Equation (5.40). Furthermore, if L is constant-extendable (in the context of Σ and Γ), then L' is constant-extendable (in the context of Σ' and Γ').

We now show that Proposition 5.39 can be used to increase the arity of relations and maintain wqo's (e.g., to go from $\preceq$ to $route^*$). To do this, we start from a constant-extendable language L, and extend it by a fresh constant symbol a. We use a to replace an n-ary relation r with a relation r' of arity $n+1$. Define $e_a^{r\mapsto r'}$ to denote the substitution of $r(t_1,\ldots,t_n)$ by $r'(a,t_1,\ldots,t_n)$, where $t_1,\ldots,t_n$ are terms (e.g., $e_a^{\preceq\mapsto route^*}(\preceq(x,y)) = route^*(a,x,y)$). We also extend $e_a^{r\mapsto r'}$ to sets of formulas, i.e., $e_a^{r\mapsto r'}(A) = \{e_a^{r\mapsto r'}(\varphi) \mid \varphi \in A\}$. Then, the next proposition is straightforward:

Proposition 5.43 (Arity extension)**.** *If $(STRUCT[\Sigma,\Gamma],\sqsubseteq_L)$ is a wqo, $r \in \Sigma$ is a relation symbol of arity n, $r' \notin \Sigma$ is a new relation symbol of arity $n+1$, and $a \in \Sigma$ is a constant symbol, then $(STRUCT[\Sigma',\Gamma'],\sqsubseteq_{L'})$ is a wqo where*

$$\Sigma' = \Sigma \setminus \{r\} \cup \{r'\}, \quad \Gamma' = e_a^{r\mapsto r'}(\Gamma) \quad L' = e_a^{r\mapsto r'}(L)$$

Remark 5.44. In the setting of Proposition 5.43, if Avoid_L is computable for structures of $\text{STRUCT}[\Sigma,\Gamma]$, then $\text{Avoid}_{L'}$ is computable for structures of $\text{STRUCT}[\Sigma',\Gamma']$, and is given by:

$$\text{Avoid}_{L'}(s') = e_a^{r\mapsto r'}(\text{Avoid}_L(s))$$

where, for $s' = (\mathcal{D}',\mathcal{I}')$, we define $s = (\mathcal{D}',\mathcal{I})$ where $\mathcal{I}$ (defined over $\Sigma' \setminus \{r'\} \cup \{r\}$) is the same as $\mathcal{I}'$, except for $\mathcal{I}(r)$ which is obtained from $\mathcal{I}'(r')$ by truncating the first element in each tuple. Furthermore, if L is constant-extendable (in the context of Σ and Γ), then L' is constant-extendable (in the context of Σ' and Γ').

Extending L by a constant and using Proposition 5.43 followed by Proposition 5.39 results in a vocabulary, theory and language where a relation r has been replaced by a relation r' with increased arity (e.g., replacing $\preceq$ by $route^*$). The obtained language L' contains universal sentences, where the occurrences of r' are *symmetric* in their first argument: every universal clause in L' can only use one universally quantified variable as the first argument of r' in all its appearances. Therefore, formulas in L' cannot correlate values of r' for different elements as the first argument. Note however that the variable used for the first argument of r' can appear elsewhere in the clause, and can be correlated to other relations, including other occurrences of r' (see Section 5.6.4 for a concrete example).

5.6.3 Adding Occurrences of Arbitrary Relation Symbols

It is sometimes necessary for the invariant to mention relations that do not obey any background theory (e.g., the *msg* relation in the learning switch example). This section

shows that such relations can be allowed in the language for potential invariants while maintaining decidability, as long as only a *bounded* number of occurrences of these relations appear in each universal clause.

We note that the clauses of the original language may contain unbounded quantification, and the bounded occurrences of the new relations can correlate to other literals in these clauses (the new relations are added to the bodies, i.e. within the scope of the universal quantifiers). For this reason, this result requires the use of Higman's Lemma, and cannot be obtained as a straightforward cartesian product with a finite domain. We prove this result more concisely by building on the operation of symmetric lifting and Proposition 5.39 (which was proven using Higman's Lemma).

Let $r \in \Sigma$ be a relation symbol of arity n. We are interested in extending L by adding one occurrence of r to any body, β, of L. Let $A_r(\beta)$ be the set of bodies of the form β, $r(t_1, \ldots, t_n) \vee \beta$, or $\neg r(t_1, \ldots, t_n) \vee \beta$, where $t_1, \ldots, t_n$ are terms (including free variables that appear in β). Let $A_r(L)$ have exactly the bodies $A_r(\beta)$ for β a body of L.

Proposition 5.45. *If $(STRUCT[\Sigma, \Gamma], \sqsubseteq_L)$ is a wqo, $Avoid_L$ is computable, L is constant-extendable, and $r \in \Sigma$ is a relation symbol, then:*

- *$(STRUCT[\Sigma, \Gamma], \sqsubseteq_{A_r(L)})$ is a wqo,*
- *$Avoid_{A_r(L)}$ is computable, and*
- *$A_r(L)$ is constant-extendable.*

Proof. Let n be the arity of r. Let $c_1, \ldots, c_n$ be n fresh constant symbols. Let $\Sigma_1 = \Sigma \cup \{c_1, \ldots, c_n\}$ and let L_1 be L extended by the constant symbols $c_1, \ldots, c_n$ as in Definition 5.36. Then, since L is constant-extendable, we get that $(\text{STRUCT}[\Sigma_1, \Gamma], \sqsubseteq_{L_1})$ is a wqo, Avoid_{L_1} is computable, and L_1 is constant-extendable.

Let L_2 have the bodies, β, of L_1 plus $\beta \vee g$ and $\beta \vee \neg g$ for the ground atom $g = r(c_1, \ldots, c_n)$ Then by Proposition 5.33, $(\text{STRUCT}[\Sigma_1, \Gamma], \sqsubseteq_{L_2})$ is a wqo, Avoid_{L_2} is computable, and L_2 is constant-extendable.

Finally, let L_3 be the language obtained by applying Proposition 5.39 and Remark 5.42 n times to remove the constants $c_1, \ldots, c_n$, from the vocabulary (replacing them by universally quantified variables). By Proposition 5.39 and Remark 5.42 $(\text{STRUCT}[\Sigma, \Gamma], \sqsubseteq_{L_3})$ is a wqo, Avoid_{L_3} is computable, and L_3 is constant-extendable. By the constructions of L_1, L_2, L_3, we get that $L_3 = A_r(L)$, which completes the proof. □

It immediately follows from Proposition 5.45 that:

Corollary 5.46. *Extending the vocabulary* Σ *by adding an* arbitrary *relation (i.e., with* any arity*) and extending* L *by adding to the bodies of* L *any number* $\leq k$ *of occurrences of the new relation symbol, for some* fixed $k \geq 0$*, maintains the wqo and computability of* $Avoid_L$*.*

5.6.4 Putting It All Together: Application to Learning Switch

We now illustrate the results of this section by applying it to the learning switch model of Figure 5.26, and obtaining a decidable class for invariant inference that captures it. The class we obtain contains an inductive invariant that proves the absence of forwarding loops.

Recall that for any d, $route^*(d, \cdot, \cdot)$ describes the reflexive transitive closure of $route(d, \cdot, \cdot)$, which is a functional relation (as explained above), and thus $route^*(d, \cdot, \cdot)$ obeys $\Gamma_{\preceq}$ when replaced for $\preceq$. Thus, we start with the result of Section 5.5, and apply the construction of Section 5.6.2 to lift $\preceq$ to $route^*$. We denote the resulting theory Γ_{route^*} and the resulting language L_0. L_0 contains universal clauses with any number of occurrences of $route^*$ that are symmetric with respect to its first argument. For the vocabulary $\Sigma_0 = \{route^*\}$, we have that $(\text{STRUCT}[\Sigma_0, \Gamma_{route^*}], \sqsubseteq_{L_0})$ forms a wqo with Avoid_{L_0} computable.

For any $k > 0$, we obtain L_k by applying Proposition 5.45 $4k$ times (starting with L_0) to allow at most k occurrences of any of the relations *link*, *learned*, *msg* and *route*. Thus, for $\Sigma_{\texttt{ls}} = \{link, msg, learned, route, route^*\}$, we have that $(\text{STRUCT}[\Sigma_{\texttt{ls}}, \Gamma_{route^*}], \sqsubseteq_{L_k})$ is a wqo with Avoid_{L_k} computable.

For $k = 1$, L_k contains an inductive invariant for the learning switch. This inductive invariant contains clauses such as:

$$\forall x, y, z.\, route^*(x, y, z) \wedge y \neq z \rightarrow learned(x, y)$$
$$\forall w, x, y, z.\, msg(w, x, y, z) \wedge w \neq y \rightarrow learned(w, y)$$

Note that these clauses create correlations between the first argument of $route^*$ and the other relations, and also between the other relations and themselves. These correlations are allowed by the constructions of this section. An example for a clause that would not be allowed is: $\forall x, y, z.\, route^*(x, y, z) \rightarrow route^*(z, y, x)$, as it creates correlations of $route^*$ with different first arguments.

5.7 Undecidability and Complexity of INV[$\mathcal{C}, \mathcal{L}$]

In this section, we present several hardness results for INV[$\mathcal{C}, \mathcal{L}$] for cases of $\mathcal{C}$ and $\mathcal{L}$. We first present a general scheme for proving undecidability by an interesting reduction from

the halting problem of counter machines. We use this scheme to prove that allowing alternation-free invariants for $\mathcal{C}_{\text{DP}}$ leads to undecidability, i.e., $\texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\text{AF}}]$ is undecidable (Theorem 5.48). We then use similar arguments to show that even if we allow only universal invariants, but consider $\mathcal{C}$ that allows a single "unrestricted" binary relation (unlike for example the $\preceq$ relation restricted by the background theory $\Gamma_{\preceq}$ in $\mathcal{C}_{\text{DP}}$) then $\texttt{INV}[\mathcal{C}, \mathcal{L}_{\forall^*}]$ is undecidable. In both cases, this undecidability is in contrast with the fact that verifying inductiveness of a given invariant *is* decidable.

We conclude this section by adapting the reduction from counter machines to a reduction from lossy counter machines, and applying it to $\mathcal{C}_{\text{DP}}$ and $\mathcal{L}_{\forall^*}$ to prove that the decidable problem $\texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*}]$ has non-elementary complexity.

5.7.1 Reduction from Counter Machines to $\texttt{INV}[\mathcal{C}, \mathcal{L}]$

This subsection presents a reduction scheme from the halting problem of Minsky (2-counter) machines to $\texttt{INV}[\mathcal{C}, \mathcal{L}]$. The scheme is later instantiated to obtain two undecidability results. The reduction is parameterized by an encoding for counters, denoted by $\mathcal{E}$. We first present the reduction, and then the conditions on $\mathcal{E}$ needed for it to be correct.

Input We are given an arbitrary Minsky machine, $M = (Q, c_1, c_2)$, where c_1, c_2 are counters, both initially 0, and $Q = q_1, \ldots, q_n$ is a finite sequence of instructions, where q_1 is the first instruction, and q_n is the halting instruction. The possible instructions are:

i_k: increment counter c_k

d_k: decrement counter c_k

$t_k(j)$: if counter c_k is 0 go to instruction j

After every instruction, control is passed to the next instruction, except for $t_k(j)$ when $c_k = 0$, in which case the branch is taken and control is passed to instruction j.

Idea The reduction constructs $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, such that

$$(T, P, L) \in \texttt{INV}[\mathcal{C}, \mathcal{L}] \quad \text{iff} \quad M \text{ halts}$$

The idea is for T to simulate M, and in parallel simulate a third counter c_3 that is initially 0, and will always contain an *even* number. Specifically, each transition of T will simulate one step of M, and will either (non-deterministically) increment or decrement c_3 by 2. The safety property P will assert that c_3 does not contain the value 1. Notice that $T \models P$ regardless of

whether M halts. Both T and P will be encoded in first-order logic, and the encoding will be constructed such that two *correctness conditions* hold:

1. If M halts, T will have finitely many reachable configurations, and there will be an inductive invariant in L, constructed by a disjunction over formulas in L representing these configurations.

2. If M does not halt, there will be no inductive invariant in L, since L will not be able to express the fact that the value of c_3 is even, which is needed for inductiveness.

Construction To have $T = (\Sigma, \Gamma, \iota, \tau)$ simulate M, we use nullary relations $q_1, \ldots, q_n$ to keep track of M's current instruction (we overload the q_i's for instructions and nullary relation symbols). We also need to encode the value of the three counters, which have infinitely many possible values. To do so, we will use an encoding $\mathcal{E}$ of the counters over vocabulary $\Sigma_{\mathcal{E}}$, which will be provided by each instantiation of the reduction (see Section 5.7.2 and Section 5.7.3). The encoding $\mathcal{E}$ provides a vocabulary $\Sigma_{\mathcal{E}}$, a theory $\Gamma_{\mathcal{E}}$ (over $\Sigma_{\mathcal{E}}$) and formulas for manipulating the encoded counter, which we use to construct ι, τ, and P. Specifically, the encoding $\mathcal{E}$ provides the following formulas (for $i = 1, 2, 3$):

- inc_i a transition formula for increasing the value of c_i by 1
- dec_i a transition formula for decreasing the value of c_i by 1
- id_i a transition formula for keeping the value of c_i unchanged
- $zero_i$ a formula for testing if the value of c_i is 0
- $init$ a formula for the initial state s.t. $init \Rightarrow \bigwedge_{i=1}^{3} zero_i$

Given the Minsky machine M, the output of the reduction is $(T, P, L) \in (\mathcal{C}, \mathcal{L})$, where $T = (\Sigma, \Gamma, \iota, \tau)$. The vocabulary is given by $\Sigma = \Sigma_{\mathcal{E}} \cup \{q_1, \ldots, q_n\}$. The theory is provided by the encoding, i.e., $\Gamma = \Gamma_{\mathcal{E}}$. The formulas ι, τ, and P can be easily constructed from the formulas $\mathcal{E}$ provides listed above. Note that if $\mathcal{E}$ provides these formulas in $\exists^*\forall^*$ form, then T can be constructed to be an EPR transition system.

Correctness conditions For the reduction to be correct, $\mathcal{E}$ must guarantee both correctness conditions defined above. For the first correctness condition, assuming M halts, there are finitely many reachable configuration, each defined by the current instruction of M and the values of c_1, c_2, c_3. Denote by $\mathcal{R}_M \subseteq Q \times \mathbb{N}^3$ the finite set of reachable configurations of

M. An inductive invariant $I \in L$ for T can be defined as follows:

$$I = \bigvee_{(q_i,\ell_1,\ell_2,\ell_3)\in\mathcal{R}_M} q_i \wedge \psi_{\mathcal{E}}(\ell_1,\ell_2,\ell_3),$$

where for any $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, $\psi_{\mathcal{E}}(\ell_1,\ell_2,\ell_3)$ is a "witness" formula *in the language* L that is specific to $\mathcal{E}$. Intuitively, $\psi_{\mathcal{E}}(\ell_1,\ell_2,\ell_3)$ provides an approximation in L to the statement "$c_1 = \ell_1 \wedge c_2 = \ell_2 \wedge c_3 = \ell_3$". I will be inductive if $\mathcal{E}$ guarantees that for any $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$ the following holds:

$$\begin{gathered}
\Gamma_{\mathcal{E}}, \bigwedge_{i=1}^{3} zero_i \models \psi_{\mathcal{E}}(0,0,0) \\
\Gamma_{\mathcal{E}},\ \psi_{\mathcal{E}}(\ell_1,\ell_2,\ell_3) \wedge inc_1 \wedge id_2 \wedge id_3 \models \psi_{\mathcal{E}}(\ell_1+1,\ell_2,\ell_3)' \\
\Gamma_{\mathcal{E}},\ \psi_{\mathcal{E}}(\ell_1,\ell_2,\ell_3) \wedge dec_1 \wedge id_2 \wedge id_3 \models \psi_{\mathcal{E}}(\ell_1-1,\ell_2,\ell_3)' \text{ for } \ell_1 \geq 1 \\
\ldots \text{ similarly for } inc_2,\ dec_2,\ inc_3,\ dec_3 \ldots \\
\Gamma_{\mathcal{E}},\ \psi_{\mathcal{E}}(\ell_1,\ell_2,\ell_3) \models \neg dec_i \text{ for } \ell_i = 0 \\
\Gamma_{\mathcal{E}},\ \psi_{\mathcal{E}}(\ell_1,\ell_2,\ell_3) \models \neg zero_i \text{ for } \ell_i \neq 0
\end{gathered} \tag{5.47}$$

The first requirement guarantees that the initial states will satisfy I, and the others guarantee that I will be inductive (i.e., closed under transitions of τ), and imply the safety property $safe$.

5.7.2 Undecidability of $\texttt{INV}[\mathcal{C}_{\mathbf{DP}}, \mathcal{L}_{\mathbf{AF}}]$

Recall that $\mathcal{L}_{\text{AF}}$ allows only alternation-free invariants. In this subsection we instantiate the reduction scheme of Section 5.7.1 to prove the following theorem:

Theorem 5.48. *$\texttt{INV}[\mathcal{C}_{DP}, \mathcal{L}_{AF}]$ is undecidable.*

To instantiate the reduction scheme of Section 5.7.1 for $\texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\text{AF}}]$, we present an encoding $\mathcal{E}_{\preceq}$ of the counters using the relation $\preceq$ and $\Gamma_{\preceq}$ (as per $\mathcal{C}_{\text{DP}}$), and show witness formulas $\psi_{\mathcal{E}}$ that are *alternation-free* (as per $\mathcal{L}_{\text{AF}}$) and satisfy the conditions ensuring the correctness of the reduction. The intuitive idea is to represent counters by linked-lists, where the value of the counter is the length of the list. We note that a similar encoding can also be created for any of the other four cases that make up $\mathcal{C}_{\text{DP}}$.

Encoding We encode the 3 counters c_1, c_2, c_3 using 3 disjoint linked-lists, where the length of list i encodes the value of c_i. The vocabulary $\Sigma_{\mathcal{E}_{\preceq}}$ contains the binary relation $\preceq$ and 3

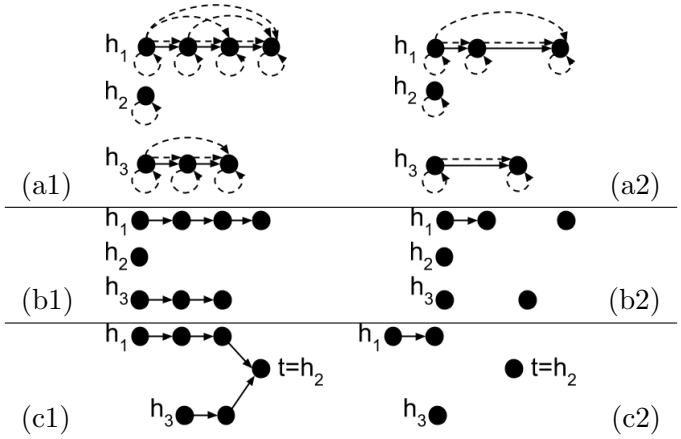


Figure 5.49: Encodings used in the reduction from counter machines. Depicted are the structures obtained by different encodings of counters for $c_1 = 3, c_2 = 0, c_3 = 2$: a structure of $\mathcal{E}_{\preceq}$ (a1) and its substructure (a2) (see Section 5.7.2); a structure of an unsuccessful attempt for $\mathcal{E}_r$ (b1) and its substructure (b2) (see Section 5.7.3); a structure of a successful $\mathcal{E}_r$ (c1) and its substructure (c2) (see Section 5.7.3).

constant symbols h_1, h_2, h_3 for the heads of the lists. The theory is $\Gamma_{\mathcal{E}_{\preceq}} = \Gamma_{\preceq}$ given by the axioms of Figure 3.15. Figure 5.49(a1) provides an example of a structure that arises in this encoding for $c_1 = 3, c_2 = 0$ and $c_3 = 2$. The encoding formulas will be:

- inc_i will prepend a new node to list i
- dec_i will remove a node from the start of list i, assuming there is an edge from h_i (otherwise dec_i is not satisfied)
- id_i will keep h_i unchanged
- $zero_i$ will test if there is no edge from h_i
- $init$ will assert $\preceq$ is the identity relation and the h_i's are distinct

These formulas are all easy to write in EPR using $\preceq$ (see Figure 3.15 for explicit forms of all the necessary ingredients).

Correctness For the second correctness condition of the reduction, we need to show that if M does not halt, there is no alternation-free inductive invariant. Intuitively, this is true for this encoding, due to the fact that it is not expressible in first-order logic to say that the length of a list is even using $\preceq$.

More formally, any inductive invariant must be true of all the reachable states. If M does not halt, then these states include segments corresponding to c_3 that are line graphs of unbounded even length. Suppose that our inductive invariant, I, has quantifier depth

k. Let s be any state satisfying I such that the length of the list encoding c_3 is $\ell > 2^k$. If we modify s to s' only by adding one more segment to c_3's list leaving everything else the same, then it is easy to show using Ehrenfeucht-Fraïssé games [109] that $s \equiv_k s'$, i.e., that they agree on all first-order formulas of quantifier depth k. Thus $s' \models I$. But this leads to a contradiction because s' is not safe. That is, from s' we can construct a trace that decreases c_3 until it becomes 1, which violates P.

Witness formulas For the first correctness condition (namely that if M halts there is an alternation-free inductive invariant), we need to present the existence of alternation-free witness formulas that meet the conditions of Equation (5.47). For any $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, the witness formula $\psi_{\mathcal{E}}(\ell_1, \ell_2, \ell_3)$ will be the conjunction of the following:

- A universal formula asserting the h_i's point to disjoint lists:

$$\bigwedge_{1 \le i < j \le 3} \forall x. \neg(h_i \preceq x \wedge h_j \preceq x)$$

- An existential formula asserting the length of list i is at least ℓ_i:

$$\exists x_1 \dots x_{\ell_i}. \texttt{distinct}(h_i, x_1, \dots, x_{\ell_i}) \wedge \bigwedge_{j=1}^{\ell_i} h_i \preceq x_j$$

- A universal formula asserting the length of list i is at most ℓ_i:

$$\forall x_0 \dots x_{\ell_i}. \neg \left(\texttt{distinct}(h_i, x_0, \dots, x_{\ell_i}) \wedge \bigwedge_{j=0}^{\ell_i} h_i \preceq x_j \right)$$

The conjunction is clearly alternation-free. These witness formulas guarantee the conditions of Equation (5.47). Intuitively, this is because $\psi_{\mathcal{E}}(\ell_1, \ell_2, \ell_3)$ is strong enough to guarantee that in any model of it, the h_i's points to 3 disjoint linked-lists of exactly the correct lengths.

5.7.3 Undecidability of $\texttt{INV}[\mathcal{C}_r, \mathcal{L}_{\forall^*}]$

For the second undecidability proof that we present, we first define the class $\mathcal{C}_r$ of transition systems which allows a single binary relation that is not restricted by a background theory. We show that $\texttt{INV}[\mathcal{C}_r, \mathcal{L}_{\forall^*}]$ is undecidable, and thus $\texttt{INV}[\mathcal{C}, \mathcal{L}_{\forall^*}]$ is undecidable for any extension $\mathcal{C}$ of $\mathcal{C}_r$ as well.

Definition 5.50 ($\mathcal{C}_r$)**.** We denote by $\mathcal{C}_r$ the class of EPR transition systems where Σ contains a single binary relation symbol r, any finite number of nullary relation symbols, and 4 constant symbols, and there is no background theory (i.e., $\Gamma = \emptyset$).

Therefore, the state space of $T \in \mathcal{C}_r$ is STRUCT[Σ] (no background theory). This subsection proves the following:

Theorem 5.51. *$\texttt{INV}[\mathcal{C}, \mathcal{L}_{\forall^*}]$ is undecidable for every class $\mathcal{C}$ that extends $\mathcal{C}_r$.*

To show undecidability of $\texttt{INV}[\mathcal{C}_r, \mathcal{L}_{\forall^*}]$, we present an encoding $\mathcal{E}_r$, which uses the single unrestricted binary relation r allowed by $\mathcal{C}_r$ to encode the counters. For the correctness of the reduction we must show *universal* witness formulas $\psi_{\mathcal{E}}$ (as per $\mathcal{L}_{\forall^*}$) that meet the conditions of Equation (5.47).

We use the same idea of Section 5.7.2, namely to encode the counters c_1, c_2, c_3 using 3 disjoint linked-lists whose heads are h_1, h_2, h_3, and to use the list lengths to encode counter values. However, this time we will use the relation r to directly capture next pointer edges. We first explain why the fact that $\mathcal{L}_{\forall^*}$ allows only universal invariants (as opposed to alternation-free) makes the reduction trickier, and then present the encoding that can be used for this reduction.

Encoding A zero-attempt is to make r be $\preceq$, and use the transition formulas and witness formulas to enforce the theory $\Gamma_{\preceq}$. This approach is bound to fail, since $\texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*}]$ is decidable, but it is useful to understand where exactly it fails. The second direction of the reduction (if M does not halt then there is no inductive invariant) will be correct exactly as it was in Section 5.7.2, but the first direction will fail: since we are targeting $\mathcal{L}_{\forall^*}$, we must present universal witness formulas and these do not exist. Intuitively, this is because with a universal formula we will only be able to approximate $c_i = \ell_i$ as $c_i \leq \ell_i$, which is not sufficient for Equation (5.47).

To make the reduction work, we must take advantage of the fact that r is not restricted by a background theory. We do this by using r to encode list edges directly. It is easy to use r to express linked-lists operations such as prepending to a linked-list, removing an element from the head of a list, and checking if the head has an outgoing edge. Thus, we can use r to write the formulas inc_i, dec_i, id_i, $zero_i$, and $init$ to express the same linked-lists operations as in Section 5.7.2, but this time using r and not $\preceq$. Figure 5.49(b1) provides an example of a structure that arises in this encoding for $c_1 = 3, c_2 = 0$ and $c_3 = 2$.

However, this also fails. The reason is that any universal formula is closed under substructure (unlike alternation-free formulas), and a structure that contains a linked-list

of length > 0, has a substructure where the head of the list does not have an r edge (if we remove the first node after the head from the domain), and this substructure will satisfy $zero_i$. This is demonstrated by Figure 5.49(b2) which is a substructure of Figure 5.49(b1), but $zero_3$ (wrongfully) holds in it. For this reason, any universal $\psi_{\mathcal{E}}$ cannot guarantee that $\psi_{\mathcal{E}}(\ell_1, \ell_2, \ell_3) \Rightarrow \neg zero_i$ for $\ell_i \neq 0$.

This can be fixed by adding another constant symbol t, to represent a common tail of the linked-lists, and changing the formulas as follows:

- inc_i will prepend a new node to list i (creating a new r edge)
- dec_i will remove a node from the start of list i, assuming $h_i \neq t$
- id_i will keep h_i unchanged
- $zero_i$ will test if $h_i = t$
- $init$ will say that r is empty and $head_i = t$ for $i = 1, 2, 3$

The key idea is that since the common tail is guarded by a constant, no "transition to substructure" can transform a list of length > 0 to a list of length 0. Note however that a transition to substructure can remove elements that are part of the r-path from h_i to t from the domain, thus making the list disconnected.

Figure 5.49(c1) provides an example of the resulting encoding $\mathcal{E}_r$ for $c_1 = 3, c_2 = 0$ and $c_3 = 2$. In this case, the substructure in Figure 5.49(c2) no longer satisfies $zero_3$.

Correctness The second correctness condition of the reduction (if M does not halt then there is no universal inductive invariant) is correct for this encoding by the same arguments provided for $\mathcal{E}_{\preceq}$ (it is not expressible in first-order logic to say that the length of an r-path is even). For the first correctness condition (if M halts there is a universal inductive invariant), we present universal witness formulas that meet the conditions of Equation (5.47).

Witness formulas For any $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, define the witness formula $\psi_{\mathcal{E}}(\ell_1, \ell_2, \ell_3)$ as:

$$\bigwedge_{i \in \{1,2,3\}} \forall x_0 \dots x_{\ell_i}. \bigwedge_{j=0}^{\ell_i} \left((h_i = x_0 \wedge \bigwedge_{k=0}^{j-1} r(x_k, x_{k+1})) \rightarrow (x_j = t \leftrightarrow j = \ell_i) \right)$$

These witness formulas are universal, and they satisfy the conditions for inductiveness of Equation (5.47). Intuitively, the witness formulas make sure that all r-paths of length at most ℓ_i starting from h_i are consistent with the fact that list i is of length ℓ_i: paths shorter

than ℓ_i must not end in t, and paths of length ℓ_i must end in t. We note that these witness formulas admit structures in which the three lists are not disjoint, but this is not problematic for the reduction.

5.7.4 Complexity of $\texttt{INV}[\mathcal{C}_{\mathbf{DP}}, \mathcal{L}_{\forall^*}]$ is Non-Elementary

As we saw in Section 5.5, $\texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*}]$ is decidable. In this section, we show that the complexity of $\texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*}]$ is non-elementary. Therefore this is also true for any $(\mathcal{C}, \mathcal{L})$ extending $(\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*})$, e.g., by the constructions presented in Section 5.6.

Theorem 5.52. *The complexity of $\texttt{INV}[\mathcal{C}_{DP}, \mathcal{L}_{\forall^*}]$ is non-elementary.*

Proof. We prove the theorem by a reduction from the reachability problem of lossy counter machines, which is known to have non-elementary complexity [211, 212], to the complement of $\texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*}]$. The reachability problem for lossy counter machines is: given a counter machine $M = (Q, c_1, \ldots, c_k)$ and $q_{\text{goal}} \in Q$, is there a trace of M under the lossy semantics that leads from the initial configuration $(q_1, 0, \ldots, 0)$, to a configuration with q_{goal}. Recall that the lossy semantics lets the value of any counter decrease non-deterministically in any transition.

The input of the reduction is (M, q_{goal}), and the output is $(T, P, L) \in (\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*})$ such that:

$$M \text{ has a lossy trace to } q_{\text{goal}} \;\Leftrightarrow\; (T, P, L) \notin \texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*}]$$

According to Section 5.5, $(\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*})$, satisfies the conditions of Theorem 5.6. Therefore, we can apply Corollary 5.12 and get:

$$(T, P, L) \notin \texttt{INV}[\mathcal{C}_{\text{DP}}, \mathcal{L}_{\forall^*}] \;\Leftrightarrow\; T^{\forall^*} \not\models P$$

Therefore, it suffices to show that

$$M \text{ has a lossy trace to } q_{\text{goal}} \;\Leftrightarrow\; T^{\forall^*} \not\models P \tag{5.53}$$

To construct $T = (\Sigma, \Gamma, \iota, \tau)$ and P that satisfy Equation (5.53), we use the same encoding $\mathcal{E}_{\preceq}$ presented in Section 5.7.2. The only difference is that for this reduction we do not need the special counter that always holds even values, and we just model the counters of the input machine M. We encode Q with nullary relations and the values of the k counters using $\preceq$ and k disjoint linked-lists whose heads are $h_1, \ldots, h_k$, where the lengths of the lists keep

the values of the counters. The formulas for the initial state and the transition relation are constructed as in Sections 5.7.1 and 5.7.2, and the safety property is given by $P = \neg q_{\text{goal}}$.

To see that the described T and P satisfy Equation (5.53), recall that $T^{\forall^*} \not\models P$ iff T has a $\forall^*$-relaxed trace to a state that violates P, where a $\forall^*$-relaxed trace can make transitions to substructures (since $\sqsubseteq_{\forall^*}$ is the substructure relation). The punch line is that for the encoding of Section 5.7.2, these transitions exactly correspond to decrements of the counters. For a linked-list represented by $\preceq$ and a constant for the head, any substructure is a linked-list with less elements. (For an example, see Figure 5.49(a1) and its substructure Figure 5.49(a2).) Therefore, by the construction of T we get that $\forall^*$-relaxed traces of T exactly correspond to lossy traces of the counter machine M, and thus Equation (5.53) holds, which completes the proof. □

5.8 Related Work for Chapter 5

Decidability for infinite-state systems via wqo's Following [5–7, 9], well-quasi-orders and well-structured (monotonic) transition systems (WSTSs) have become a standard technique for proving decidability of various problems over infinite-state systems (e.g., [76]). Often a WSTS is obtained by relaxing the semantics of a transition system to a *lossy* semantics. The classical examples of this are lossy channel systems [4], lossy counter machines (e.g., [168]) and vector addition systems or Petri nets (e.g., [150]). In [38, 86], this technique is applied to array-based transition systems, states are identified with finitely-generated models, and the quasi-order on states is the substructure relation.

The common approach in these works is to consider a system under lossy semantics or via an abstraction, and to analyze the decidability of the *safety* of the lossy/abstract system. Our work focuses on transition systems formalized using first-order logic, and takes the viewpoint of restricting potential inductive invariants to some language L, and then analyzing the decidability of *invariant inference* in L. The two views are connected: every particular language L can be regarded as an abstraction, where the abstract domain is formulas in L; or as a lossy semantics, with lossy transitions according to $\sqsubseteq_L$. Indeed, T^L can alternatively be formulated as a monotonic transition system, and this can be viewed as monotonic abstraction as in [9].

Our use of first-order logic together with a formalism that is parameterized by L, allows the systematic constructions presented in Section 5.6, which are able to construct restricted languages and wqo's that are applicable to complicated systems like the learning switch. From a practical perspective, using logic enables to use existing decision procedures in the invariant

inference process. The operation Avoid_L corresponds to concepts found both in IC3/PDR algorithms [34, 116] and in decision procedure based abstract interpreters, e.g., [220], that automate the process of inferring inductive invariants. Our work can be seen as studying the conditions required for termination of such procedures, and the conditions under which the underlying problem is undecidable and thus divergence occurs for infinitely many instances. nIndeed, in [116], our results have been used to obtain termination guarantees for $\text{PDR}^\forall$, a variant of IC3/PDR that finds universally quantified invariants, in cases where $\sqsubseteq_{\forall^*}$ is a wqo.

Linked-lists & counter machines In [32], it was observed that linked-list programs are counter machines. While the safety of counter machines is undecidable, the safety of lossy counter machines is decidable with non-elementary complexity [211]. In the case of linked-lists, our work can be seen as combining these two interesting results: "Linked-lists with *alternation-free* invariants are counter machines" (and therefore undecidable), and "Linked-lists with *universal* invariants are *lossy* counter machines" (and therefore decidable with non-elementary complexity).

In [10, 11], monotonic abstractions are defined for linked data structures. The wqo on heaps defined there is semantically similar to $\sqsubseteq_{\forall^*}$ with $\preceq$, but it is defined via tree operations and not via logic. This makes the result of Section 5.5 more powerful because it also provides unlimited unary predicates, as well as being amenable to the constructions of Section 5.6. The wqo of [10] results in a slightly less precise abstraction compared to that of Section 5.5, as it also allows edge deletion (which is not allowed in $\sqsubseteq_{\forall^*}$ with $\preceq$). Interestingly, the proof in [10] does not use Kruskal's Tree Theorem, which is the basis for our result.

Undecidability There are many undecidability results for safety of infinite state systems, e.g., [32, 39]. For the related problem `INV` of inferring inductive invariants we show two interesting undecidability results: for universal invariants over arbitrary domains; and for alternation-free formulas over the theory of linked-list reachability.

Completeness of abstract interpretation The theoretical study of the precision of abstract interpretation is an ongoing research area. Usually *completeness* for abstract interpretation means that the abstract domain is precise enough to prove all interesting safety properties, e.g., [87]. In our terms, this means that `INV` = `SAFE`, i.e., that all safe programs have an inductive invariant expressible in the abstract domain. Since we are interested in automatically analyzing Turing-complete programs (for which `SAFE` is undecidable), we consider the completeness problem at a later binding time. Essentially, `INV`

asks if the abstract domain is precise enough for a *given* property in a *given* program.

A very interesting twist on completeness of abstract interpreters is considered in [88] which algorithmically studies the completeness of certain abstract domains for certain questions.

Numeric domains For numeric domains (e.g., intervals), techniques such as widening and policy iteration (e.g. [83]) are used to prove the correctness of many infinite state systems. Surprising results in [82, 217] show that for some numeric domains and programs, precise information can be computed, thus solving `INV`. However, it is not known how to extend these techniques and results beyond numeric template domains, for example to the context of first-order logic considered here.

Chapter 6

Interactive Inference of Universal Invariants

This chapter is based on the results published in [186].

As we have seen in Chapters 3 and 4, first-order logic and specifically EPR can capture interesting and challenging verification problems. That is, one can encode both the system and its inductive invariant in such a way that verification conditions are in EPR, and then use automated solvers to reliably check the verification conditions, and if the invariant is not inductive, a *finite* counterexample to induction is provided. One problem that still remains is finding the inductive invariants themselves. In Chapter 5, we have explored this problem from a theoretical perspective and obtained some decidability and undecidability results.

In this chapter, we adopt a more practical mindset and attempt to answer the following question: how can we help the user find inductive invariants, beyond providing counterexamples in the classical guess and check process? We focus on universally quantified inductive invariants, and use the connection introduced in Chapter 5 between universal invariants, substructures, and diagrams. We also exploit the finite model property of EPR. Using these concepts, we develop a graphical interaction mode that allows the user to guide generalization from counterexamples to induction in an interactive process for finding universally quantified inductive invariants. The developed methodology is implemented as part of the Ivy deductive verification system, and we also present an empirical evaluation for several examples that have universal inductive invariants.

6.1 Overview & Illustration

This section provides an informal overview of the interactive verification methodology we develop, aimed at finding universally quantified inductive invariants. This methodology is implemented as part of the Ivy deductive verification system.

Design philosophy The methodology we develop is inspired by proof assistants such as Isabelle/HOL [181] and Coq [26] which engage the user in the verification process. We also aim to exploit the success of tools such as Z3 [61] which can be very useful for bug finding, verification, and static analysis, and on automated invariant inference techniques such as [116]. We aim to balance between the predictability and visibility of proof assistants, and the automation of decision procedures and automated invariant inference techniques.

Compared to fully automated techniques, we adopt a different philosophy which permits visible failures at the cost of more manual work from the users. Compared to proof assistants, we aim to provide the user with automated assistance, solving well-defined decidable problems. To obtain this, we restrict ourselves to systems that can be expressed as EPR transition systems, and focus on finding universally quantified inductive invariants. These restrictions guarantee that the tasks performed automatically are actually solving decidable problems.

Running example: leader election in a ring protocol We illustrate the interactive methodology on the example presented in Section 3.1.1, for a distributed leader election protocol in a ring topology. Figure 3.1 depicts the RML model of this protocol. The ring topology is modeled using the *btw* path relation, as explained in Section 3.3.3. Recall that we wish to verify the safety property that the protocol never elects more than one leader. As presented in Section 3.1.1, safety of this protocol can be proven with a universal inductive invariant, given by Equations (3.3) to (3.5). We now ask ourselves how we might discover such an inductive invariant without knowing it. We use this example to illustrate our methodology for finding inductive invariants by an interactive process between the user and well-defined, decidable, automated queries.

Graphical visualization of states A key ingredient in the interactive process is a graphical representation. Ivy displays states of the protocol as graphs, where the vertices represent elements, and edges represent relations and functions. As an example, consider the state of the leader election protocol depicted in Figure 6.1. This state has two nodes and two IDs, represented by vertices of different shapes. Unary relations are displayed using vertex labels. For example, in Figure 6.1, node 1 is labeled *leader*, and node 2 is labeled

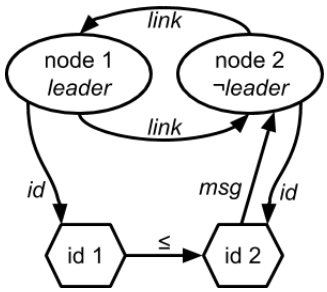


Figure 6.1: Graphical representation of a protocol state for the leader election in a ring protocol.

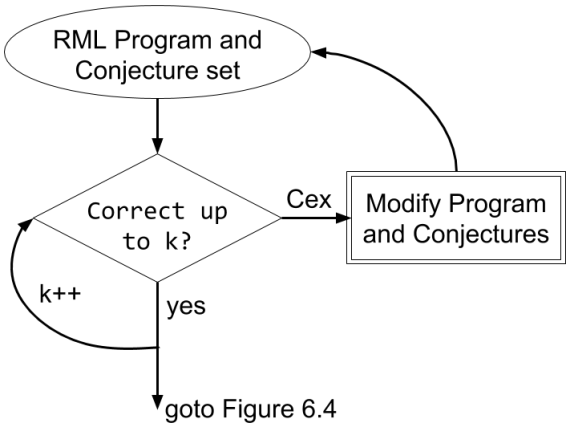


Figure 6.2: Flowchart of bounded verification.

$\neg leader$, denoting that the *leader* relation contains only node 1. Binary relations such as $\leq$ and *msg* are displayed using directed edges. Higher-arity relations are displayed by means of their projections or derived relations. For example, the ternary relation *btw* is displayed by the derived binary relation *link* which captures a single ring edge, and is defined by $link(x, y) \equiv \varphi_s(x, y)$ (as in Section 3.3.3, Figure 3.18). Functions such as *id* are displayed similarly to relations. The state depicted in Figure 6.1 contains two nodes, node 1 and node 2, such that the ID of node 1 is lower (by $\leq$) than the ID of node 2. A message with the ID of node 2 is pending at node 2, as indicated by the *msg* edge. This state is clearly not reachable in the protocol. However, as we shall see later, we encounter this state as part of a counterexample to induction, during the interactive search for the inductive invariant.

Bounded verification While our end goal is to verify the system for an unbounded number of execution steps (via a universally quantified inductive invariant), we also present a step of symbolic model debugging via bounded verification, i.e., unrolling the transition relation a bounded number of times. This step is extremely useful in practice, since one rarely writes a correct model on the first attempt, with errors ranging from minor typos to more serious issues. As we shall see, bounded unrolling will also be exploited for the

interactive search for an inductive invariant.

For example, suppose our initial modeling of the leader election protocol missed the axiom stating leaders have unique IDs (Figure 3.1 line 12). We can discover this by performing bounded verification with a bound of 4 transitions. This results in the error trace depicted in Figure 6.3. In this trace, node 1 identifies itself as a leader when it receives the message with the ID of node 2 since they have the same ID (id 1), and similarly for node 2, leading to violation of the assertion. After adding the missing axiom, we can run Ivy with an unrolling bound of 10 transitions to debug the model, and we do not get a counterexample trace.

While many tools offer debugging of formal models by bounded verification (e.g., Alloy [114], TLA+ [137]), we note that the bounded verification we perform here also exploits the restriction to EPR transition systems and EPR's finite model property. Unlike other verification systems, Ivy does not require an a priori bound on the size of the states considered (e.g., number of nodes). Instead, Ivy only requires a bound on the number of transitions to unroll. Of course, due to the properties of EPR, this also implies a bound on the size of the possible models, but this bound is implicit, and the user need not provide it.

In our experience, most protocols can be verified for about 10 transitions in a few minutes, which is extremely useful for eliminating typos and minor issues with the modeling of the protocol and/or formulation of the safety property. Once bounded verification does not find more bugs, the user can prove unbounded correctness by searching for an inductive invariant, as explained next.

Interactive search for universally quantified inductive invariants The second phase of the verification is to find a universally quantified inductive invariant that proves that the system is correct for any number of transitions. This phase requires more user effort but enables ultimate safety verification. Recall that an invariant I is *inductive* if: (i) all initial states satisfy I (**initiation**); (ii) every state satisfying I also satisfies the desired safety properties (**safety**); and (iii) I is closed under transitions of the system, i.e., a transition from any arbitrary state satisfying I results in a new state that also satisfies I (**consecution**).

If the user has a universally quantified inductive invariant in mind, Ivy can automatically check if it is indeed an inductive invariant. Due to the restrictions of EPR, this check is guaranteed to terminate with either a proof showing that the invariant is inductive or a finite counterexample to induction (CTI), which can be depicted graphically and presented to the user, as explained before. A CTI does not necessarily imply that the safety property is violated — only that I is not an inductive invariant. Coming up with inductive invariants for infinite-state systems is very difficult. Therefore, Ivy supports an interactive procedure

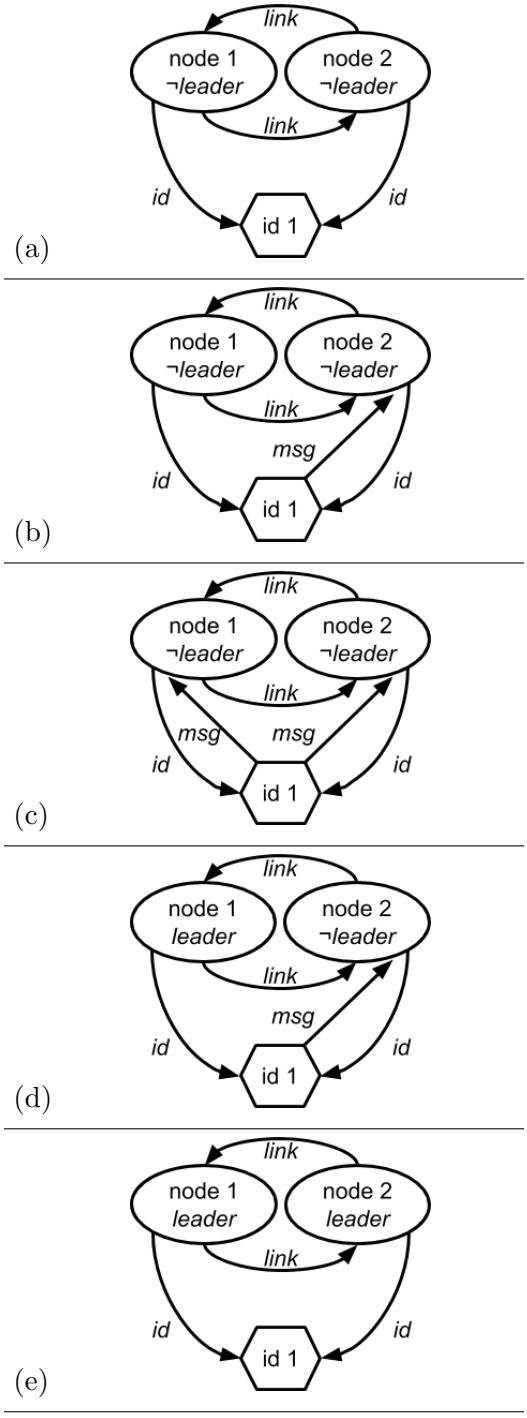


Figure 6.3: An error trace found by BMC for the leader protocol, when omitting the fact that node IDs are unique. (a) an initial state; (b) node 1 sent a message to node 2; (c) node 2 sent a message to node 1; (d) node 1 processed a pending message and became leader; (e) node 2 processed a pending message and became leader, there are now two leaders and the safety property is violated.

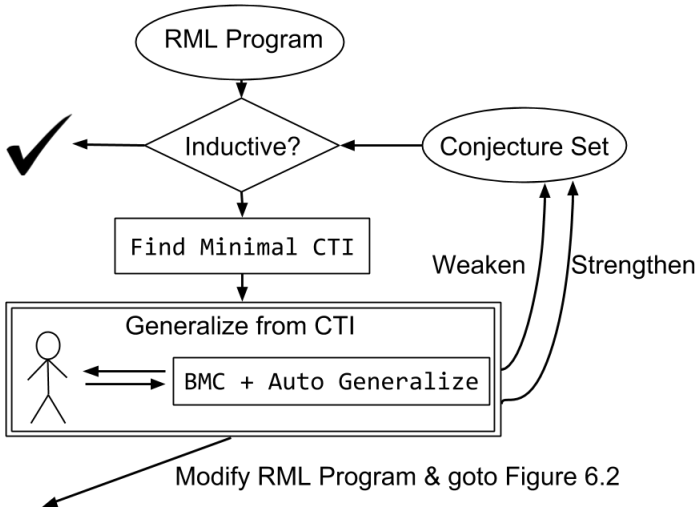


Figure 6.4: Flowchart of the interactive search for an inductive invariant.

for gradually obtaining an inductive invariant or deciding that the model or safety property need to be revised.

The search for an inductive invariant starts with a (possibly empty) set of universally quantified *conjectures*, whose conjunction is the current candidate inductive invariant. The search advances based on CTIs according to the procedure described in Figure 6.4. When a CTI is found it is displayed graphically, and the user has 3 options:

1. The user understands that there is a bug in the model or safety property, in which case they revise the RML program and start over in Figure 6.2. Note that in this case, the user may choose to retain some conjectures reflecting gained knowledge of the expected protocol behavior.

2. The user understands that one of the conjectures is wrong, in which case they remove it from the conjecture set, weakening the candidate invariant.

3. The user judges that the CTI is not reachable. This means that the invariant needs to be strengthened by adding a new conjecture. The new conjecture should eliminate the CTI, and should generalize from it. This is the most creative task, and our approach for it is explained below.

Graphical visualization of conjectures To allow the user to examine and modify possible conjectures, Ivy provides a graphical visualization of universally quantified conjectures. Such a conjecture asserts that some sub-configuration (i.e., logical substructure) of the system is not present in any reachable state. That is, any state of the system that contains this sub-configuration is not reachable. Ivy graphically depicts such conjectures by displaying

the forbidden sub-configuration. Sub-configurations are visualized similarly to the way states are displayed, but with a different semantics.

As an example consider the conjecture depicted in Figure 6.6 (b). The visualization shows two nodes and their distinct IDs; node 1 is shown to be a leader, while node 2 is not a leader. Furthermore, the ID of node 1 is lower (by $\leq$) than the ID of node 2. Note that no pending messages appear (no *msg* edges), and there is also no information about the topology (no *link* or *btw* edges). Viewed as a conjecture, this graph asserts that in any reachable state, there cannot be two nodes such that the node with the lower ID is a leader and the node with the higher ID is not a leader. Thus, this conjecture excludes infinitely many states with any number of nodes above 2 and any number of pending messages. It excludes all states that contain any two nodes such that the node with the lower ID is a leader and the node with the higher ID is not a leader.

Figure 6.6 (c) depicts an even stronger (more general) conjecture: unlike Figure 6.6 (b), node 2 is not labeled with $\neg$*leader* nor with *leader*. This means that the conjecture in Figure 6.6 (c) excludes all the states that contain two nodes such that the node with the lower ID is a leader, regardless of whether the other node is a leader or not.

Obtaining helpful CTIs Since we rely on the user to guide the generalization from CTIs, it is critical to display a CTI that is both easy to understand, and indicative of the proof failure. Therefore, Ivy searches for "minimal" CTIs. The procedure `Find Minimal CTI` used in Figure 6.4 (and explained in Section 6.2.3) automatically obtains a minimal CTI based on user provided minimization criteria. Examples include minimizing the number of elements, and minimizing certain relations (e.g., minimizing the *msg* relation to have few pending messages). This is performed automatically and reliably by reducing the minimization problem to a series of EPR queries.

Interactive generalization from CTIs When a CTI represents an unreachable state, we should strengthen the invariant by adding a new conjecture to eliminate the CTI. One possible universally quantified conjecture is the one which excludes all states that contain the concrete CTI as a sub-configuration (i.e., the diagram of the CTI, as presented in Chapter 5). However, this conjecture may be too specific, as the CTI contains many features that are not relevant to the failure. This is where generalization is required, or otherwise we may end up in a diverging refinement loop, always strengthening the invariant with more conjectures that are all too specific. (Even if $\sqsubseteq_{\forall^*}$ is a wqo, a naive iteration may take prohibitively long before its assured termination.)

C_0	$\forall n_1, n_2. \neg\big(leader(n_1) \wedge leader(n_2) \wedge n_1 \neq n_2\big)$
C_1	$\forall n_1, n_2. \neg\big(n_1 \neq n_2 \wedge leader(n_1) \wedge id(n_1) \leq id(n_2)\big)$
C_2	$\forall n_1, n_2. \neg\big(n_1 \neq n_2 \wedge msg(id(n_1), n_1) \wedge id(n_1) \leq id(n_2)\big)$
C_3	$\forall n_1, n_2, n_3. \neg\big(btw(n_1, n_2, n_3) \wedge msg(id(n_2), n_1) \wedge id(n_2) \leq id(n_3)\big)$

Figure 6.5: The conjectures found using Ivy for the leader election protocol. C_0 is the safety property, and the remaining conjectures (C_1 - C_3) were produced interactively. The conjunction $C_0 \wedge C_1 \wedge C_2 \wedge C_3$ is an inductive invariant for the protocol.

It is at this crucial point that we attempt to utilize the user's intuition, and not rely on automation. Ivy asks the user to guide generalization from CTIs. To achieve this, Ivy presents the user with a concrete CTI, and lets the user eliminate some of the features of the CTI that the user judges to be irrelevant (e.g., by marking some relations as irrelevant). This already defines a generalization of the CTI that excludes more states.

Next, the `BMC + Auto Generalize` procedure applies bounded verification (with a user specified bound) to check the user's suggestion, and also to generalize further. If the test fails, it means that the user's generalized conjecture is violated in a reachable state, and a concrete counterexample trace is displayed to let the user diagnose the problem. If the test succeeds (i.e., the bounded verification formula is unsatisfiable), Ivy automatically suggests a stronger generalization, based on a minimal unsatisfiable core. The user then decides whether to accept the suggested conjecture and add it to the invariant, or to change the parameters in order to obtain a different suggestion.

Next, we walk through this process for the leader election protocol, demonstrating the different stages, until we obtain an inductive invariant that proves the protocol's safety.

Illustration using the leader election protocol Figure 6.5 summarizes the 3 iterations Ivy required to find an inductive invariant for the leader election protocol. The initial set of conjectures contains only C_0, which is simply the specified safety property that there cannot be two distinct leaders elected.

In the first iteration, since C_0 alone is not inductive, Ivy applies `Find Minimal CTI`. This results in the CTI depicted in Figure 6.6 (a1). Figure 6.6 (a2) depicts a successor state of (a1) reached after node 2 receives the pending message with its ID. The state (a1) satisfies C_0, whereas (a2) violates it, making (a1) a CTI. After examining this CTI, the user judges that the state (a1) is unreachable, with the intuitive explanation that node 1 identifies itself as a leader despite the fact that node 2 has a higher ID. Thus, the user generalizes away the irrelevant information, which includes *msg* and the ring topology, resulting in the

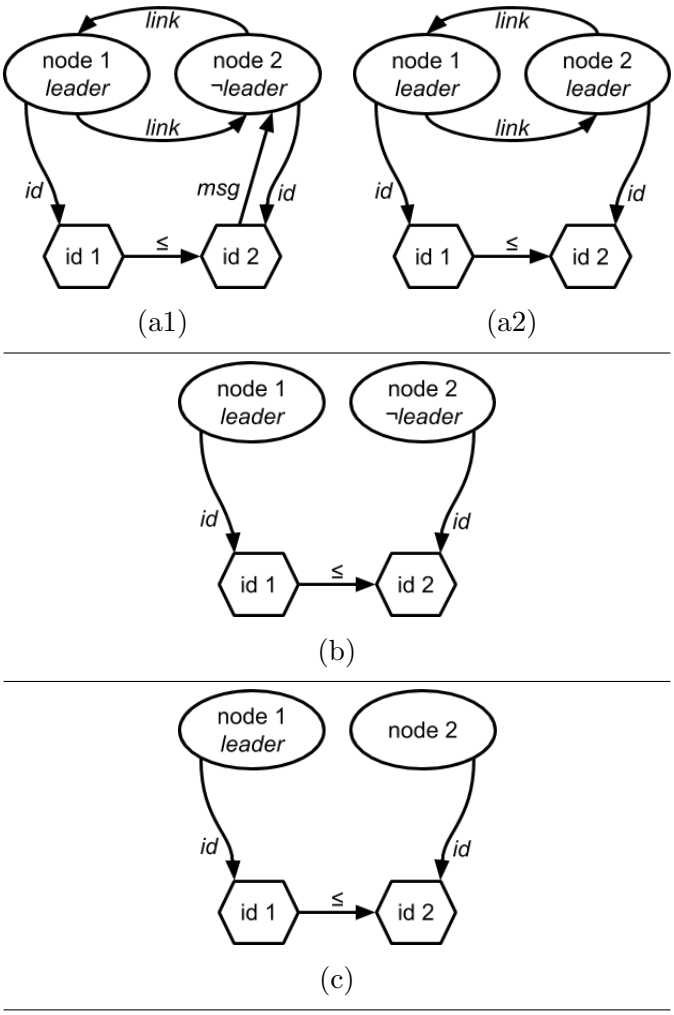


Figure 6.6: The 1st CTI generalization step for the leader protocol, leading to C_1. (a1) The CTI state that has one leader, but makes a transition to a state (a2) with two leaders. The root cause is that a node with non-maximal ID is a leader. (b) A generalization created by the user by removing the topology information and the *msg* relation (c) Further generalization obtained by `BMC + Auto Generalize`, which removed the fact that node 2 is not a leader.

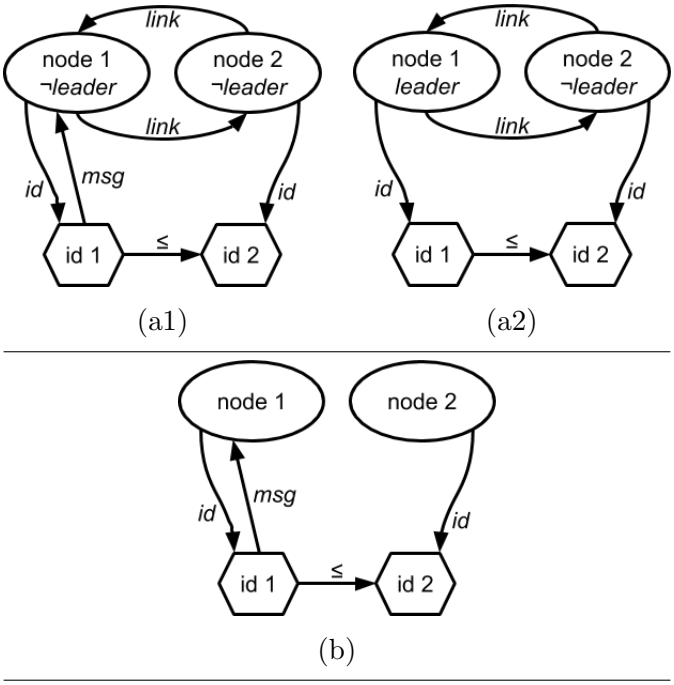


Figure 6.7: The 2nd CTI generalization step for the leader protocol, leading to C_2. (a1) The CTI state and its successor (a2) violating C_1. The root cause is that a node with non-maximal ID has a pending message with its own ID. (b) A generalization created by the user by removing the topology information and the leader relation. This generalization was validated but not further generalized by `BMC + Auto Generalize`.

generalization depicted in Figure 6.6 (b).

Next, the user applies `BMC + Auto Generalize` with bound 3 to this generalization. The test succeeds, and Ivy suggests the generalization in Figure 6.6 (c), where the information that node 2 is not a leader is also abstracted away. The user approves this generalization, which corresponds to conjecture C_1 shown in Figure 6.5, so C_1 is added to the set of conjectures.

If the user had used bound 2 instead of 3 when applying `BMC + Auto Generalize`, then Ivy would have suggested a stronger generalization that also abstracts the ID information, and states that if there are two distinct nodes, none of them can be a leader. This conjecture is bogus, but it is true for up to 2 transitions from the initial state (since with 2 nodes, a node can only become a leader after a send action followed by 2 receive actions). It is therefore the role of the user to select and adjust the bound for automatic generalization, and to identify bogus generalizations when they are encountered.

After adding the correct conjecture C_1, Ivy displays the CTI depicted in Figure 6.7 (a1) with its successor state (a2). Note that (a2) does not violate the safety property, but it violates C_1 that was added to the invariant, since a node with a non-maximal ID becomes a leader. The user examines (a1) and concludes that it is not reachable, since it has a pending message

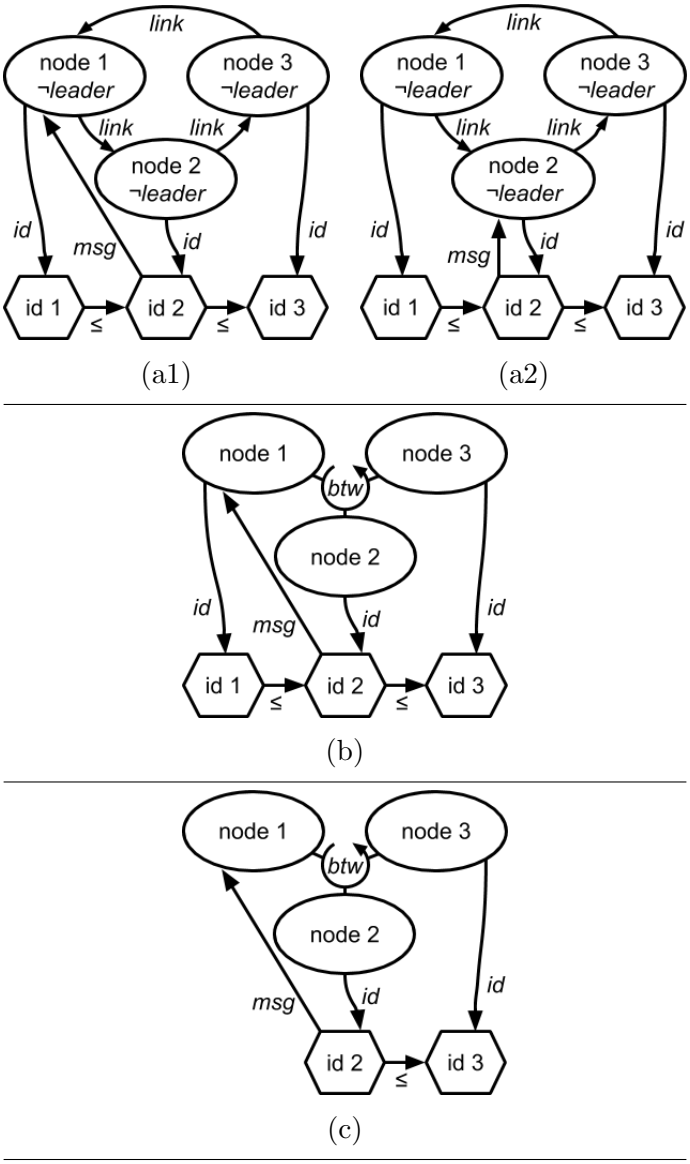


Figure 6.8: The 3rd CTI generalization step for the leader protocol, leading to C_3. (a1) The CTI state and its successor (a2) which violates C_2. The root cause is that node 1 has a pending message with the ID of node 2, even though node 3 is on the path from node 2 to node 1 and has an ID higher than node 2's ID (equivalently, node 2 is between node 1 and node 3 and has a lower ID than node 3's ID). (b) A generalization created by the user by removing the leader relation. The generalization does not contain *link*, only *btw*. (c) Further generalization obtained by `BMC + Auto Generalize`, which eliminated id1.

to node 1 with its own ID, despite the fact that node 2 has a higher ID. Here, the user again realizes that the ring topology is irrelevant and abstracts it away. The user also abstracts away the *leader* information. On the other hand, the user keeps the *msg* information, in accordance with the intuitive explanation of why the CTI is not reachable. The resulting user-defined generalization is depicted in Figure 6.7 (b). `BMC + Auto Generalize` with bound 3 validates this generalization for 3 transitions, but does not manage to generalize any further. Thus, the generalization is converted to C_2 in Figure 6.5 which is added to the invariant, and the process continues.

Finally, Figure 6.8 (a1) and (a2) depicts a CTI that leads to a violation of C_2. This CTI contains three nodes, with a pending message that appears to bypass a node with a higher ID. This time, the user does not abstract away the topology since it is critical to the reason the CTI is not reachable. The user only abstracts the *leader* information, which leads to the generalization depicted in Figure 6.8 (b). Note that in the generalized conjecture we no longer consider the *link* relation, but rather the *btw* relation. This expresses the fact that, as opposed to the concrete CTI, the conjecture generalizes from the specific topology of a ring with exactly 3 nodes, to a ring that contains it as a sub-configuration, i.e., a ring with at least 3 nodes that are not necessarily immediate neighbors of each other. We do require that the three nodes are ordered in such a way that node 2 is between node 1 and node 2 in the ring (equivalently, node 3 is between node 2 and node 1).

Applying `BMC + Auto Generalize` with a bound of 3 to Figure 6.8 (b) confirms this conjecture, and automatically abstracts away the ID of node 1, which results in the conjecture depicted in Figure 6.8 (c), which corresponds to C_3 in Figure 6.5. The user adds this conjecture to the invariant. After adding C_3, Ivy reports that $I = C_0 \wedge C_1 \wedge C_2 \wedge C_3$ is an inductive invariant for the leader election protocol.

6.2 Interactive Methodology for Safety Verification

In this section, we elaborate and formalize the ideas outlined in Section 6.1. Recall that we are interested in safety verification on an RML program. The RML program is compiled to a transition system $T = (\Sigma, \Gamma, \iota, \tau)$ and a safety property P (see Section 2.5), and we assume they form an EPR transition system. We allow stratified functions, but require that the conjunction of Γ, ι, τ, and $\neg P$, is in the many sorted EPR fragment (i.e., the functions and quantifier alternations are stratified).

The key idea is to combine automated analysis with user guidance in order to construct a universally quantified inductive invariant that proves safety. The next subsection describes

a preliminary stage for debugging the RML program, and the following subsections describe our interactive methodology of constructing universal inductive invariants.

6.2.1 Debugging via Symbolic Bounded Verification

Before starting the search for an inductive invariant, it makes sense to first search for bugs in the RML program. This can be done by unrolling the transition relation τ a bounded number of times. We define an assertion φ to be *k-invariant* if it holds in all states reachable after at most k transitions from an initial state. Thus, φ is k-invariant if and only if the following formula is unsatisfiable for any $j \leq k$:

$$\left(\bigwedge_{i=0}^{j} \wedge\Gamma(\Sigma_i)\right) \wedge \iota(\Sigma_0) \wedge \left(\bigwedge_{i=0}^{j-1} \tau(\Sigma_i, \Sigma_{i+1})\right) \wedge \neg\varphi(\Sigma_j) \tag{6.9}$$

where $\Sigma_i = \{a_i \mid a \in \Sigma\}$, and $\tau(\Sigma_i, \Sigma_{i+1})$ denotes the formula over vocabulary $\Sigma_i \uplus \Sigma_{i+1}$ obtained from τ when every symbol $a \in \Sigma$ is replaced by a_i and every symbol $a' \in \Sigma'$ is replaced by a_{i+1}. The formulas $\bigwedge\Gamma(\Sigma_i)$, $\iota(\Sigma_0)$, and $\varphi(\Sigma_j)$, are defined similarly.

Since we assume T to be an EPR transition system, checking k-invariance of any $\forall^*\exists^*$-formula φ is decidable. Furthermore, if such a φ is not k-invariant, we can obtain a finite model of Equation (6.9) for some $j \leq k$. This model represents an execution trace that executes j loop iterations and reaches a state that violates φ. This trace can be graphically displayed to the user as a concrete counterexample to the invariance of φ. Note that while checking k-invariance bounds the number of loop iterations, it does not bound the size of the states. Thus, if a property is found to be k-invariant, it holds in *all* states reachable by k iterations. This is in contrast to finite-state bounded model checking techniques that bound the state space.

The first step when using Ivy is to model the system at hand as an RML program, and then debug the program by checking k-invariance of the safety property P, as illustrated in Figure 6.2. If a counterexample is found, the user can examine the trace and modify the RML program to either fix a bug in the code, or to fix a bug in the specification (the assertions that determine P). Once no more counterexample traces exist up to a bound that satisfies the user, the user moves to the second step of constructing a universal inductive invariant that proves the safety of the system for unbounded number of iterations.

6.2.2 Interactive Search for Universal Inductive Invariants

Ivy assists the user in obtaining a universal inductive invariant. Recall that a formula I is an inductive invariant for a transition system $T = (\Sigma, \Gamma, \iota, \tau)$ if the verification conditions defined in Section 2.4.2 are unsatisfiable. Moreover, if I is a universally quantified formula, then the verification conditions are in EPR, so checking them is decidable, and if I is not an inductive invariant then we can obtain a finite state s that is a counterexample to one of the conditions. Such a state s is a *counterexample to induction (CTI)*.

Our methodology interactively constructs a universal inductive invariant represented as a conjunction of *conjectures*, i.e., $I = \bigwedge_{i=1}^{n} \varphi_i$. Each conjecture φ_i is a closed universal formula. In the sequel, we interchangeably refer to I as a set of conjectures and as the formula obtained from their conjunction.

Our methodology, presented in Figure 6.4, guides the user through an iterative search for a universal inductive invariant I by generalization from CTIs. We maintain the fact that all conjectures satisfy initiation $\Gamma, \iota \models \varphi_i$, so each CTI we obtain is always a state that satisfies all conjectures, but either violates the safety property P, or can lead to a state that violates one of the conjectures by executing a τ transition.

Initialization The search starts from a given set of conjectures as a candidate inductive invariant I. For example, I can be initialized to *true*. If P is universal, then I can initially include P (after checking that P is 0-invariant, i.e., satisfied by initial states). If the search starts after a modification of the RML program (e.g., to fix a bug or add a new feature to the model), then conjectures that were learned before can be reused. Additional initial conjectures can be computed by applying basic abstract interpretation techniques (e.g., using some predicate abstraction).

Iterations Each iteration starts by (automatically) checking whether the current candidate I is an inductive invariant. If I is not yet an inductive invariant, a CTI is presented to the user, which is a state s that either violates P (safety violation), or leads to a state that violates I via a τ transition (consecution violation). The user can choose to strengthen I by conjoining it with an additional conjecture that excludes the CTI from I. To this end, we provide automatic mechanisms to assist in *generalizing* from the CTI to obtain a conjecture that excludes it. In case of a consecution violation, the user may also choose to weaken I by eliminating one (or more) of the conjectures. The user can also choose to modify the RML program in case the CTI indicates a bug. This process continues until an inductive invariant is found.

For I to be an inductive invariant, all the conjectures φ_i need to be invariants (i.e., hold in all states reachable at the loop head). This might not hold in the intermediate steps of the search, but it guides strengthening and weakening: strengthening aims at only adding conjectures that are invariants, and weakening aims at identifying and removing conjectures that are not invariants. While strengthening and weakening are ultimately performed by the user, we provide several automatic mechanisms to assist the user in this process.

Obtaining a minimal CTI When I is not inductive, it is desirable to present the user with a CTI that is easy to understand, and not cluttered with many unnecessary features. This also tends to lead to better generalizations from the CTI. To this end, we automatically search for a "minimal" CTI, where the minimization parameters are defined by the user. Section 6.2.3 explains this in detail.

Interactive generalization To eliminate the CTI s, the user needs to either strengthen I to exclude s from I, or, in case of a consecution violation, to weaken I to include a state reachable from s via τ. Intuitively, if s is not reachable, then I should be strengthened to exclude it. If s is reachable, then I should be weakened. Clearly, checking whether s is reachable is infeasible. Instead, we provide the user with a generalization assistance for coming up with a new conjecture to strengthen I. The goal is to come up with a conjecture that is satisfied by all the reachable states. During the attempt to compute a generalization, the user might also realize that an existing conjecture is in fact not an invariant (i.e., it is not satisfied by all reachable states), and hence weakening is in order. In addition, the user might also find a modeling bug which means the RML program should be fixed.

Generalizations are explained and formalized in Section 6.2.4, and the interactive generalization process is explained in Section 6.2.5. Here again, the user defines various parameters for generalization, and Ivy automatically finds a candidate that meets the criteria. The user can further change the suggested generalization and can use additional automated checks to decide whether to keep it.

Note that if all the conjectures added by the user exclude only unreachable states (i.e., all are invariants), then weakening is rarely needed. As such, most of the automated assistance we provide focuses on helping the user obtaining "good" conjectures for strengthening, i.e., conjectures that do not exclude reachable states. Weakening will typically be used when some conjecture turns out to be "wrong" in the sense that it does exclude reachable states.

6.2.3 Obtaining Minimal CTIs

We refine the search for CTIs by trying to find a minimal CTI according to user adjustable measures. As a general rule, smaller CTIs are desirable since they are both easier to understand, which is important for interactive generalization, and more likely to result in more general (stronger) conjectures. The basic notion of a small CTI refers to the number of elements in its domain. However, other quantitative measures are of interest as well. For example, it is helpful to minimize the number of elements (or tuples) in a relation, e.g., if the relation appears as a guard for protocol actions (such as the *msg* relation in the leader election protocol of Figure 3.1). Thus, we define a set of useful minimization measures, and let the user select which ones to minimize, and in which order.

Minimization measures We consider the following measures for $s = (\mathcal{D}, \mathcal{I})$:

- Size of sort s: $|\mathcal{D}(\mathsf{s})|$.
- Number of positive tuples of relation r: $|\mathcal{I}(r)|$.
- Number of negative tuples of relation r: $|\{\overline{e} \mid \overline{e} \notin \mathcal{I}(r)\}|$.

Each measure m induces an order $\leq_m$ on structures, and each tuple $(m_1, \ldots, m_t)$ of measures induces a "smaller than" relation on structures which is defined by the lexicographic order constructed from the orders $\leq_{m_i}$.

Minimization procedure Given the tuple of measures to minimize, provided by the user, Algorithm 6.1 automatically finds a CTI that is minimal with respect to this lexicographic order. The idea is to conjoin ψ_{cti} (which encodes violation of inductiveness) with a formula ψ_{min} that is computed incrementally and enforces minimality. For a measure m, $\varphi_m(n)$ is an $\exists^*\forall^*$ clause stating that the value of measure m is no more than the number n. Such constraints are added to ψ_{min} for every m, by their order in the tuple, where n is chosen to be the minimal number for which the constraint is satisfiable (with the previous constraints). Finally, a CTI that obeys all the additional constraints is computed and returned.

For example, consider a k-ary relation r. We encode the property that the number of positive tuples of r is at most n as follows: $\exists \overline{x_1}, \ldots \overline{x_n}. \forall \overline{y}. \left(r(\overline{y}) \rightarrow \bigvee_{i=1}^{n} \overline{y} = \overline{x_i}\right)$, where $\overline{x_1}, \ldots, \overline{x_n}, \overline{y}$ denote k-tuples of logical variables.

Algorithm 6.1: Obtaining a Minimal CTI

```
1 if ψ_cti is unsatisfiable then return None;
2 ψ_min := true ;
3 for m in (m_1, ..., m_t) do
4     for n in 0, 1, 2, ... do
5         if ψ_cti ∧ ψ_min ∧ φ_m(n) is satisfiable then
6             ψ_min := ψ_min ∧ φ_m(n) ;
7             break
8 return s such that s ⊨ ψ_cti ∧ ψ_min
```

6.2.4 Formalizing Generalizations as Partial Structures

In this subsection we present the notion of a *generalization* and the notion of a *conjecture associated with a generalization*. These notions are key ingredients in the interactive generalization step described in the next subsection.

Recall that a CTI is a structure $s = (\mathcal{D}, \mathcal{I})$. Here it is useful for us to treat the interpretation of a relation symbol $r : \mathsf{s}_1, \ldots, \mathsf{s}_k$ as a function $\mathcal{I}(r) \colon \mathcal{D}(\mathsf{s}_1) \times \ldots \times \mathcal{D}(\mathsf{s}_k) \to \{0, 1\}$, and the interpretation of a function symbol $f : \mathsf{s}_1, \ldots, \mathsf{s}_k \to \mathsf{s}$ as a function $\mathcal{I}(f) : \mathcal{D}(\mathsf{s}_1) \times \ldots \times \mathcal{D}(\mathsf{s}_k) \times \mathcal{D}(\mathsf{s}) \to \{0, 1\}$ such that for any $x_1, \ldots, x_k \in \mathcal{D}$, $\mathcal{I}(f)(x_1, \ldots, x_k, y) = 1$ for exactly one element $y \in \mathcal{D}$. Essentially, we treat functions of arity k as "special" relations of arity $k + 1$ that relate each k-tuple to exactly one element. While this differs from the official definitions of Section 2.1, the difference is unessential, yet it makes the presentation of generalizations simpler and more uniform.

Generalizations of a CTI are given by *partial structures*, where relation symbols and function symbols are interpreted as *partial* functions. Formally:

Definition 6.10 (Partial Structures)**.** Given a vocabulary Σ and a domain $\mathcal{D}$, a *partial interpretation* $\mathcal{I}$ of Σ over $\mathcal{D}$ associates every k-ary relation symbol $r \in \Sigma$ with a *partial* function $\mathcal{I}(r) : \mathcal{D}^k \rightharpoonup \{0, 1\}$, and every k-ary function symbol $f \in \Sigma$ with a *partial* function $\mathcal{I}(f) : \mathcal{D}^{k+1} \rightharpoonup \{0, 1\}$, such that the sort restrictions are satisfied and also for any $x_1, \ldots, x_k \in \mathcal{D}$, $\mathcal{I}(f)(x_1, \ldots, x_k, y) = 1$ for *at most* one element $y \in \mathcal{D}$. A *partial structure* over Σ is a pair $(\mathcal{D}, \mathcal{I})$, where $\mathcal{I}$ is a partial interpretation of Σ over domain $\mathcal{D}$.

Note that a structure is a special case of a partial structure. Intuitively, generalization takes place when a CTI is believed to be unreachable, and a partial structure generalizes a CTI (structure) by turning some values ("facts") to be undefined or unspecified. For example, in a partial structure, $\mathcal{I}(r)(\bar{e})$ might remain undefined for some tuple $\bar{e}$. This is useful if the user believes that the structure is still unreachable, regardless of the value of $\mathcal{I}(r)(\bar{e})$. In Figures 6.6 to 6.8, (a1) and (a2) always represent total structures, while (b) and

(c) represent partial structures.

A natural *generalization partial order* can be defined over partial structures:

Definition 6.11 (Generalization Partial Order)**.** Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two partial interpretations of Σ over $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, such that $\mathcal{D}_2 \subseteq \mathcal{D}_1$. We say that $\mathcal{I}_2 \sqsubseteq \mathcal{I}_1$ if for every relation or function symbol $a \in \Sigma$, If $\overline{e} \in \mathrm{dom}(\mathcal{I}_2(a))$, then $\overline{e} \in \mathrm{dom}(\mathcal{I}_1(a))$ as well, and $\mathcal{I}_2(a)(\overline{e}) = \mathcal{I}_1(a)(\overline{e})$. For partial structures $s_1 = (\mathcal{D}_1, \mathcal{I}_1)$ and $s_2 = (\mathcal{D}_2, s_2)$ of Σ, we say that $s_2 \sqsubseteq s_1$ if $\mathcal{D}_2 \subseteq \mathcal{D}_1$ and $\mathcal{I}_2 \sqsubseteq \mathcal{I}_1$.

The generalization partial order extends the substructure relation of (total) structures. Intuitively, $s_2 \sqsubseteq s_1$ if the interpretation provided by s_1 is at least as "complete" (defined) as the interpretation provided by s_2, and the two agree on elements (or tuples) for which s_2 is defined. Thus, $s_2 \sqsubseteq s_1$ when s_2 represents more states than s_1, and we say that s_2 is a generalization of s_1.

From partial structures to conjectures Intuitively, every partial structure s represents an infinite set of structures that are more specific than s (they interpret more facts and contain more elements), and the conjecture that a partial structure induces excludes all these structures (states). Formally, a partial structure induces a universally quantified conjecture that is obtained as the negation of the *diagram of the partial structure*, where we extend classic definition of a diagram (given in Chapter 5, Definition 5.16) from structures to partial structures.

Definition 6.12 (Diagram for Partial Structures)**.** Let $s = (\mathcal{D}, \mathcal{I})$ be a *finite* partial structure of Σ and let $\mathcal{D}' = \{e_1, \ldots, e_{|\mathcal{D}'|}\} \subseteq \mathcal{D}$ denote the set of elements e_i for which there exists (at least one) relation or function symbol $a \in \Sigma$ such that e_i appears in some tuple of $\mathrm{dom}(\mathcal{I}(a))$. The *diagram* of s, denoted by $Diag(s)$, is the following formula over Σ:

$$\exists x_1 \ldots x_{|\mathcal{D}'|}.\, \texttt{distinct}(x_1, \ldots, x_{|\mathcal{D}'|}) \wedge \psi$$

where ψ is the conjunction of:

- $r(x_{i_1}, \ldots, x_{i_k})$ for every k-ary relation r in Σ and every $i_1, \ldots, i_k$ s.t. $\mathcal{I}(r)(e_{i_1}, \ldots, e_{i_k}) = 1$;
- $\neg r(x_{i_1}, \ldots, x_{i_k})$ for every k-ary relation r in Σ and every $i_1, \ldots, i_k$ s.t. $\mathcal{I}(r)(e_{i_1}, \ldots, e_{i_k}) = 0$;

- $f(x_{i_1}, \ldots, x_{i_k}) = x_j$ for every k-ary function f in Σ and every $i_1, \ldots, i_k$ and j s.t. $\mathcal{I}(f)(e_{i_1}, \ldots, e_{i_k}, e_j) = 1$; and
- $f(x_{i_1}, \ldots, x_{i_k}) \neq x_j$ for every k-ary function f in Σ and every $i_1, \ldots, i_k$ and j s.t. $\mathcal{I}(f)(e_{i_1}, \ldots, e_{i_k}, e_j) = 0$.

Intuitively, $Diag(s)$ is obtained by treating individuals in $\mathcal{D}$ as existentially quantified variables and explicitly encoding all the facts that are defined in s. Note that we treat constants as functions of 0 arity, and that for structures s that are not partial, $Diag(s)$ of Definition 6.12 coincides with that of Definition 5.16.

The negation of the diagram of a partial structure s constitutes a conjecture that is falsified by all structures that are more specific than s. This includes all structures that contain s as a substructure.

Definition 6.13 (Conjecture of a Partial Structure)**.** Let s be a finite partial structure. The *conjecture associated with* s, denoted $\varphi(s)$, is the universal formula equivalent to $\neg Diag(s)$.

Lemma 6.14. *Let s be a partial structure and let s' be a (total) structure such that $s \sqsubseteq s'$. Then $s' \not\models \varphi(s)$.*

Note that if $s_2 \sqsubseteq s_1$, then $\varphi(s_2) \models \varphi(s_1)$ i.e., a larger generalization results in a stronger conjecture.

This connection between partial structures and conjectures is at the root of our graphical interaction technique. We present the user with partial structures, and the user can control which facts to make undefined, thus changing the partial structure. The semantics of the partial structure is given by the conjecture associated with it. The conjectures C_1, C_2, and C_3 of Figure 6.5 are the conjectures associated with the partial structures depicted in Figure 6.6 (c), Figure 6.7 (b), and Figure 6.8 (c) respectively.

6.2.5 Interactive Generalization

In Ivy, generalization from a CTI, $s = (\mathcal{D}, \mathcal{I})$, is performed by an interactive process which consists of the following conceptual phases that are controlled by the user.

Coarse-grained manual generalization The user graphically selects an upper bound for generalization $s_u \sqsubseteq s$, with the intent to obtain a $\sqsubseteq$-smallest generalization s' of s_u. Intuitively, the upper bound s_u defines which elements of the domain may participate in the generalization and which tuples of which relations may stay interpreted in the generalization.

For example, if a user believes that the CTI remains unreachable even when some $\mathcal{I}(r)(\bar{e})$ is undefined, they can use this intuition to define the upper bound.

In Ivy the user defines s_u by graphically marking the elements of the domain that will remain in the domain of the partial structure. In addition, for every relation or function symbol a, the user can choose to turn all positive instances of $\mathcal{I}(a)$ (i.e., all tuples $\bar{e}$ such that $\mathcal{I}(a)(\bar{e}) = 1$) to undefined, or they can choose to turn all negative instances of $\mathcal{I}(a)$ to undefined. The user makes such choices by selecting appropriate checkboxes for every symbol.

In Figures 6.6 to 6.8, (b) depicts the upper bound s_u selected by the user according to the user's intuition.

Fine-grained automatic generalization via k-invariance Ivy searches for a $\sqsubseteq$-smallest generalization s' that generalizes s_u such that $\varphi(s')$ is k-invariant, where k is provided by the user.

This process begins by checking if $\varphi(s_u)$ is k-invariant using Equation (6.9). If verification fails, the user is presented with a trace that explains the violation. Based on this trace, the user can either redefine s_u to be less general ($\sqsubseteq$-greater), or they may decide to modify the RML program if a bug is revealed.

If $\varphi(s_u)$ is k-invariant, it means that the formula of Equation (6.9) for $\varphi = \varphi(s_u)$ is unsatisfiable. In this case, Ivy computes the minimal unsatisfiable core out of the literals of $\varphi(s_u)$ and uses it to define a most general ($\sqsubseteq$-smallest) s_m such that $\varphi(s_m)$ is still k-invariant. The partial structure s_m obtained by the minimal unsatisfiable core is displayed to the user as a candidate generalization (the user can also see the corresponding conjecture $\varphi(s_m)$).

The partial structures, s_m, obtained in this stage are depicted in Figures 6.6 and 6.8 (c). For the CTI of Figure 6.7, s_u, depicted in (b), is already minimal, and the unsatisfiable core minimization cannot remove any literals (so in this case $s_u = s_m$).

User investigates the suggested generalization After the automatic generalization found a stronger conjecture $\varphi(s_m)$ that is still k-invariant, the user must decide whether to add this conjecture to the candidate inductive invariant I. In order to make this decision, the user can check additional properties of the generalization (and the conjecture associated with it). For example, the user may check if it is k'-invariant for some $k' > k$. The user can also examine both the graphical visualization of s_m and a textual representation of $\varphi(s_m)$ and judge it according to their intuition about the system.

If the obtained conjecture does not seem correct, the user can choose to increase k and

try again. If a conjecture that appears bogus remains k-invariant even for large k, it may indicate a bug in the RML program that causes the behaviors of the program to be too restricted (e.g., an axiom or an assume that are too strong). The user may also choose to manually fine-tune the conjecture by re-introducing interpretations that became undefined. The user can also choose to change the generalization upper bound s_u or even ask Ivy for a different CTI, and start over. Eventually, the user must decide on a conjecture to add to I for the process to make progress.

6.3 Evaluation & Discussion

In this section we provide an empirical evaluation of the approach presented in this chapter, which is implemented as part of the Ivy deductive verification system [171, 186]. Ivy is implemented in Python and uses Z3 [61] for satisfiability testing. Ivy supports both the procedure for symbolic bounded verification described in Section 6.2.1 and the procedures for interactive construction of inductive invariants described in Sections 6.2.2 to 6.2.5.

6.3.1 Protocols

We consider several distributed protocols. Most importantly, we must evaluate the approach using protocols that can be proven with universally quantified inductive invariants. While some of the protocols we handle are challenging, they are in general simpler than the protocols handled in other parts of this thesis (specifically Chapter 4), whose proof requires inductive invariants that are not purely universally quantified. However, for protocols that do have universally quantified inductive invariants we gain a much more pleasant way of finding these invariants.

Lock server We consider a simple lock server example taken from Verdi [235, Figure 3]. The system contains an unbounded number of clients and a single server. Each client has a flag that denotes whether it thinks it holds the lock or not. The server maintains a list of clients that requested the lock, with the intent of always granting the lock to the client at the head of the list. A client can send a lock request to the server. When this request is received the server adds the client to the end of the list. If the list was previously empty, the server will also immediately send back a grant message to the client. A client that holds the lock can send an unlock message that, when received, will cause the server to remove the client from the waiting list, and send a grant message to the new head of the list. In this protocol, messages cannot be duplicated by the network, but they can be reordered.

Consequently, the same client can appear multiple time in the server's waiting list. The safety property to verify is that no two clients can simultaneously think they hold the lock.

Distributed lock protocol Next, we consider a simple distributed lock protocol taken from the IronFleet project [100, 110] that allows an unbounded set of nodes to transfer a lock between each other without a central server. Each node maintains an integer denoting its current epoch and a flag that denotes if it currently holds the lock. A node at epoch e that holds the lock, can transfer the lock to another node by sending a transfer message with epoch $e + 1$. A node at epoch e that receives a transfer message with epoch e' ignores it if $e' \leq e$, and otherwise it moves to epoch e', takes the lock, and sends a locked message with epoch e'. In this protocol, messages can be duplicated and reordered by the network. The safety property to verify is that all locked messages within the same epoch come from a single node.

Learning switch The learning switch network routing protocol was presented in detail in Chapter 5, and here we use the same modeling, presented in Figure 5.26.

Database chain consistency Transaction processing is a common task performed by database engines. These engines ensure that (a) all operations (reads or writes) within a transaction appear to have been executed at a single point in time (*atomicity*), (b) a total order can be established between the committed transactions for a database (*serializability*), and (c) no transaction can read partial results from another transaction (*isolation*). Recent work (e.g., [243]) has provided a chain based mechanism to provide these guarantees in multi-node databases. In this model the database is sharded, i.e., each row lives on a single node, and we wish to allow transactions to operate across rows in different nodes.

Chain based mechanisms work by splitting each transaction into a sequence of *subtransactions*, where each subtransaction only accesses rows on a single node. These subtransactions are executed sequentially, and traditional techniques are used to ensure that subtransaction execution does not violate safety conditions. Once a subtransaction has successfully executed, we say it has precommitted, i.e., the transaction cannot be aborted due to command in the subtransaction. Once all subtransactions in a transaction have precommitted, the transaction itself commits, if any subtransaction aborts the entire transaction is aborted. We used Ivy to show that one such chain transaction mechanism provides all of the safety guarantees provided by traditional databases.

The transaction protocol was modeled in RML using a the sorts transaction, node, key

(row), and subtransaction. Commit times are implicitly modeled by transactions (since each transaction has a unique commit time), and unary relations are used to indicate that a transaction has committed or aborted. We modeled the sequence of subtransactions in a transaction using the binary relation *opOrder* and tracked a transactions dependencies using the binary relation *writeTx* (indicating a transaction t wrote to a row) and a ternary relation *dependsTx* (indicating transaction t read a given row, and observed writes from transaction t'). To this model we added assertions ensuring that (a) a transaction reading row r reads the last committed value for that row, and (b) uncommitted values are not read. For our protocol this is sufficient to ensure atomicity.

Chord ring maintenance Chord is a peer-to-peer protocol implementing a distributed hash table [216]. In [240], Zave presented a model of the part of the protocol that implements a self-stabilizing ring. This was proved correct for the case of up to 8 participants, but the parameterized case was left open. We modeled Chord in Ivy and attempted to prove the primary safety property, which is that the ring remains connected under certain assumptions about failures. Our method was similar to Houdini [78] in that we described a class of formulas using a template, and used abstract interpretation to construct the strongest inductive invariant in this class. This was insufficient to prove safety, however. We took the abstract state at the point the safety failure occurred as our attempted inductive invariant, and used Ivy's interaction methods to diagnose the proof failure and correct the invariant. An interesting aspect of this proof is that, while Zave's proof uses the transitive closure operator in the invariant (and thus is outside any known decidable logic) we were able to interactively infer a suitable universally quantified inductive invariant that can be checked in EPR.

6.3.2 Results & Discussion

Next, we evaluate the effectiveness of the proposed methodology, and its implementation in Ivy. We begin, in Figure 6.15 by quantifying model size, size of the inductive invariant discovered and the number of CTIs generated when modeling the protocols described above. As can be seen, across a range of protocols, modeled with varying numbers of sorts and relation and function symbols, Ivy allowed us to discover inductive invariants in a modest number of interactive steps (as indicated by the number of CTIs generated in column **G**). However, Ivy is highly interactive and does not cleanly lend itself to performance evaluation (since the human factor is often the primary bottleneck). We therefore present here some observations from our experience using Ivy as an evaluation into its utility.

Protocol	S	RF	C	I	G
Leader election in ring	2	5	3	12	3
Lock server	5	11	3	21	8
Distributed lock protocol	2	5	3	26	12
Learning switch	2	5	11	18	3
Database chain replication	4	13	11	35	7
Chord ring maintenance	1	13	35	46	4

Figure 6.15: Evaluation of interactive search for universal invariants. Listed are protocols verified interactively with Ivy using the methodology presented in this chapter. **S** is the number of sorts in the model. **RF** is the number of relations and function symbols in the model. **C** is the size of the initial set of conjectures, measured by the total number of literals that appear in the formulas. **I** is the size of the final inductive invariant (also measured by total number of literals). **G** is the number of CTIs and generalizations that took place in the interactive search for the inductive invariant.

Modeling protocols in Ivy Models in Ivy are written in an extended version of RML. Since RML and Ivy are restricted to accepting code that can be translated into EPR formulas, they force some approximation on the model. For example, in the database commit protocol, expressing a constraint requiring that every subtransaction reads or writes at least one row is impossible in EPR, and we had to overapproximate to allow empty subtransactions.

Bounded verification Writing out models is notoriously error prone. We found Ivy's bounded verification stage to be invaluable while debugging models, even when we bounded ourselves to a relatively small number of steps (typically 3 – 9). Ivy displays counterexamples found through bounded verification using the same graphical interface as is used during inductive invariant search. We found this graphical representation of the counterexample made it easier to understand modeling bugs and greatly sped up the process of debugging a model.

Finding inductive invariants In our experience, using a graphical representation to select an upper bound for generalization was simple and the ability to visually see a concrete counterexample allowed us to choose a much smaller partial structure. In many cases we found that once a partial structure had been selected, automatic generalization quickly found the final conjecture accepted by the user.

Additionally, in some cases the conjectures suggested by Ivy were too strong, and indicated that our modeling excluded valid system behaviors. For example, when modeling the database example we initially used **assume**'s that were too strong. We detected this

when we saw Ivy reporting that a conjecture that seemed bogus is true even for a high number of transitions. After fixing the model, we could reuse work previously done, as many conjectures remained invariant after the fix.

Comparison to Coq and Dafny The lock server protocol is taken from [235], and thus allows some comparison of the safety verification process in Ivy with the methodology presented in this chapter, to the verification process in Coq and the Verdi framework. The size and complexity of the protocol description in Ivy is similar to Verdi, and both comprise of approximately 50 lines of code. When verifying safety with Verdi, the user is required to manually think of the inductive invariant, and then prove its inductiveness using Coq. For this protocol, [235] reports a proof that is approximately 500 lines long. With Ivy, the inductive invariant was found after 8 iterations of user guided generalizations, which took us less then an hour. Note that with Ivy, there is no need to manually prove that the invariant is inductive, as this stage is fully automated.

The distributed lock protocol is taken from [100], which allows a comparison with Dafny [148] and the IronFleet framework. This protocol took us a few hours to verify with Ivy. Verifying this protocol with Dafny took the IronFleet team a few days, when a major part of the effort was manually coming up with the inductive invariant [192]. Thus, for this protocol, the help Ivy provides in finding the inductive invariant significantly reduces the verification effort.

In both cases we are comparing the substantial part of the proof, which is finding and proving the inductive invariant. There are some differences in the encoding of this problem, however. For example, we use a totally ordered set where the Coq version of the lock server example uses a list. From the Coq version, executable code can be extracted, whereas we did not do this from the Ivy version.

Overall thoughts We believe that the interactive methodology presented here makes it easier for users to find inductive invariants, and provides a guided experience through this process. This is in contrast to the existing model for finding inductive invariants, where users must come up with the inductive invariant by manual reasoning. However, it is still limited to universally quantified inductive invariants, and generalizing it beyond this class can be a fruitful avenue for future research.

6.4 Related Work for Chapter 6

The idea of using decidable logics for program verification is quite old. For example, Klarlund *et al.* used monadic second order (MSO) logic for verification in the Mona system [102]. This approach has been generalized in the STRAND logic [158]. Similar logics could be used in the methodology we propose for interactively finding inductive invariants. However, EPR has the advantage that it does not restrict models to have a particular structure (for example a tree or list structure). Moreover as we have seen there are simple and effective heuristics for generalizing a counterexample model to an EPR formula. Finally, the complexity of EPR is relatively low (exponential compared to non-elementary) and it is implemented in efficient provers such as Z3.

Various techniques have been proposed for solving the parameterized model checking problem. Some achieve decidability by restricting the process model to specialized classes that have cutoff results [85] or can be reduced to well-structured transition systems (such as Petri nets) [7]. Such approaches have the advantages of being fully automated when they apply. However, they have high complexity and do not fail visibly, so the user cannot easily make progress. This and the restricted model classes make it difficult to apply these methods in practice.

There are also various approaches to parameterized model checking based on abstract interpretation. A good example is the Invisible Invariants approach [197]. This approach attempts to produce an inductive invariant by generalizing from an invariant of a finite-state system. However, like other abstract interpretation methods, it has the disadvantage of not failing visibly.

The kind of generalization heuristic we use here is also used in various model checking techniques, such IC3/PDR [34]. A generalization of this approach called PDR$^\forall$ can automatically synthesize universal invariants [116]. The method is fragile, however, and we were not successful in applying it to the examples verified here. Our goal in this work is to make this kind of technique interactive, so that user intuition can be applied to the problem.

There are also many approaches based on undecidable logics that provide varying levels of automation. Some examples are proof assistants such as Coq [26] or Isabelle/HOL [181] that are powerful enough to encode most of mathematics but provide little automation, and tools such as Dafny [148] that provide incomplete verification condition checking. The latter class of tools provide greater automation but do not fail visibly. Because of incompleteness they can produce incorrect counterexample models, and in the case of a true counterexample to induction they provide little feedback as to the root cause of the proof failure.

Part III

Liveness and Temporal Verification

Chapter 7

Reducing Liveness to Safety in First-Order Logic

This chapter is based on the results published in [189].

We now turn our attention to the problem of proving liveness and general temporal properties, i.e., other than safety. We are particularly motivated by the problem of verifying liveness properties of distributed protocols with unbounded resources and dynamic creations of objects. We are interested in handling both *unbounded-parallelism* (where the system is allowed to dynamically create processes), and *infinite-state per process*. In this chapter, we develop a transformation that reduces the temporal verification problem to the problem of verifying the safety of infinite-state systems expressed in pure first-order logic. This allows to leverage existing techniques for safety verification, and the other techniques developed in this thesis, to verify temporal properties of interesting distributed protocols, including some that have not been mechanically verified before.

We model infinite-state systems using first-order transition system specifications as presented in Section 2.4, and use first-order temporal logic (FO-LTL) (see e.g., [2, 162]) to specify temporal properties. This general formalism provides a powerful and natural way to model many infinite-state systems. It is particularly natural for distributed and multi-threaded systems, where quantifiers can be used to reason about the multiple nodes or threads, as well as messages, values, and other objects of the system, while supporting both unbounded-parallelism and infinite-state per process. The states of the system are modeled as first-order logic structures (rather than, say, fixed-length tuples of integer values), which allows a rich mechanism for formalizing various aspects of the state, such as dynamic sets of threads and processors, message channels, and unbounded local and global state. Such

models can also account for counters, i.e., variables over the natural numbers with operations such as increment, decrement, and comparison, using the techniques presented in Chapter 3. For example, a mutual exclusion protocol such as the ticket protocol uses two counters to implement a waiting queue in an unbounded array.

The fact that we go beyond fixed-length tuples of values to model a state and use first-order logic structures (with unboundedly large or even infinite domains) may seem like an extra burden on the task of proving liveness. We will show that, on the contrary, the formalism of first-order logic gives us a unique opportunity for proving liveness in a new manner.

Liveness properties As explained in Section 1.3, a liveness property specifies that *a good thing will happen*. For example, non-starvation for a mutual-exclusion protocol specifies that every request to access some critical resource will eventually be granted. It is typical of a liveness property that its validity depends on fairness assumptions (e.g., fair thread scheduling, eventual message delivery). For infinite-state systems, infinitely many fairness constraints may be needed (e.g., for unbounded-parallelism). A counterexample to a liveness property is an *infinite* fair execution, satisfying (possibly infinitely many) fairness constraints. Reasoning about liveness and fairness properties for a distributed protocol or any other concurrent program with a parameterized or dynamic number of threads and/or dynamic data structures is notoriously difficult.

To see why, consider a distributed protocol where the maximal number of steps until the *good thing* happens depends on the success of actions on a set of objects (threads, tasks in a list, nodes in a graph, messages in a queue, etc.). The set of objects may change dynamically and grow indefinitely through a sequence of interleaving actions (thread creation, etc.). In fact, in an infinite trace, this set itself can be infinite. As an example, consider a bug that causes newly created threads to become first-in-line for access to a critical resource. Then, a thread can become starved if new threads keep being created ahead of it. In the infinite counterexample trace, there is an infinite number of threads. Liveness proofs must therefore take into account both the control flow and infinitely many fairness assumptions to determine the possible sequences and prove that their interleaving actions cannot stall the progress towards the *good thing* forever.

Our approach: liveness-to-safety reduction In general, liveness-to-safety reductions are a recent theme in research on methods to prove program termination and liveness properties (see the discussion of related work in Section 7.7). The rationale is that we

have an ever growing arsenal of scalable techniques to automatically synthesize inductive invariants but we have only limited ways to synthesize ranking functions. Thus the goal is to shift as much burden as possible from the task of finding ranking functions to the task of finding inductive invariants. The problem is exacerbated by the fact that techniques to synthesize ranking functions apply mostly only to a limited range of data structures (essentially, linear arithmetic). Our contribution is a liveness-to-safety reduction that gets rid of the task of finding ranking functions altogether.

Note that the reduction we define is not a reduction in the complexity theoretic sense, and it involves abstraction. Indeed, this chapter provides some examples in which the reduction provably fails, i.e., it transforms a system that satisfies its specified liveness property into a system that does not satisfy its associated safety property. However, the reduction is sound, in the sense that proving the resulting safety property always ensures the validity of the original liveness property.

For finite-state systems, liveness can be proven through acyclicity (the absence of fair cycles in every execution). This is the classical liveness-to-safety reduction, a term coined in [27]. This also works for parameterized systems, where the state space is infinite, but actually finite (albeit unbounded) for every system instance [196]. It is well-known that, to prove a liveness property of an infinite-state system, an argument based on acyclicity would be unsound (an infinite-state system can be acyclic but non-terminating). We will show that, when we use a first-order logic to formalize *first-order fair transition systems*, there is a canonical way to derive an abstract semantics for which a suitable acyclicity test is sound.

While it is sound to test acyclicity on a finite-state system resulting from an abstraction, it is also void because in general, an abstraction that maps an infinite set of states to a finite set will introduce cycles in the resulting finite-state system (even if there were no cycles before). We avoid this by fine-tuning the abstraction individually for each execution, while abstracting only the cycle detection aspect (rather than the actual transitions of the system). Such fine-tuned abstraction is possible using the symbolic representation of the transition relation in first-order logic, as well as the first-order formulation of the fairness constraints.

The basic observation which we use here is that, once a finite domain of objects is fixed, there exist only finitely many first-order logic structures over the same signature, providing a natural finite abstraction by projection. To determine *how* to fix the finite domain of objects, we note that, an execution can be a counterexample only if it satisfies *all* fairness assumptions. However, to prove that a set of executions satisfies the given liveness property, in general we need only a finite number of fairness assumptions in any point in time. The

key idea here is that the finite set of needed fairness assumptions can be selected to fix a finite domain of objects, and vice versa.

Main results The contributions of this chapter can be summarized as follows:

- We define a parameterized fair cycle detection mechanism that is sound for proving fair termination of transition systems with both infinitely many states and infinitely many fairness constraints.
- We instantiate the parameterized mechanism in a uniform way for transition systems expressed in first-order logic, exploiting the *footprint* of transitions. For such systems, we obtain an *algorithmic* reduction from verification of arbitrary temporal properties to verification of safety properties.
- We extend the applicability of the reduction by allowing a user to specify a nesting structure which breaks the termination argument into levels.
- We demonstrate the utility of our approach by applying it to verify liveness properties of several interesting protocols: ticket protocol (with unbounded-parallelism), alternating bit protocol, TLB shootdown protocol, and three variants of Paxos, including Stoppable Paxos. Our evaluation indicates that in many cases the safety problem obtained from the reduction can be verified using verification conditions in first-order logic, and specifically in the decidable EPR fragment.
- To the best of the author's knowledge, we provide the first mechanized liveness proof for both TLB Shootdown and Stoppable Paxos. Interestingly, Stoppable Paxos is tricky enough that [140] prove its liveness using an informal proof of about 3 pages of temporal-logic reasoning.

7.1 Overview

In this section we present the main ideas of our approach for proving temporal properties of infinite-state systems using first-order logic, enabled by our novel liveness-to-safety reduction.

Section 7.1.1 introduces a running example that we use to illustrate our approach. Section 7.1.2 shows how to specify infinite-state systems and their temporal properties using first-order temporal logic (FO-LTL), a well established combination of pure (uninterpreted) first-order logic and linear temporal logic (see e.g., [164]). Our approach for temporal verification of such systems consists of two steps, summarized in Figure 7.1. In the first

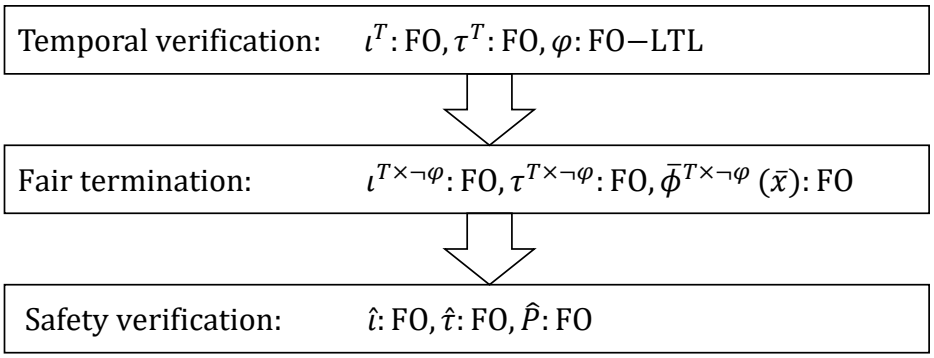


Figure 7.1: Flow of temporal verification via liveness-to-safety reduction in first-order logic. The input model is given by a first-order specification of the initial states and transition relation, and by a temporal specification in FO-LTL. It is then transformed to a fair transition system (fully specified in first-order logic, without FO-LTL), whose fair traces are exactly those traces of the original model that violate its temporal specification (this step is explained in further detail in Chapter 8). The main contribution described in this chapter is the reduction from this fair transition system to a safety problem, by constructing a new transition system (specified in first-order logic) such that if it does not reach its error state, then the input model satisfies its temporal specification.

step, we reduce verification of temporal properties to verification of fair termination, where the transition system and the fairness constraints are specified in first-order logic (this is in resemblance to the finite state case, and explained in more detail in the next chapter, Section 8.2). In the second step, we reduce the problem of fair termination of a first-order transition system, to a safety verification problem for a first-order transition system. The latter reduction is the main contribution of this chapter, sketched in Section 7.1.3.

What We Gain Once the temporal verification is reduced to safety verification of a first-order transition system, the safety property can be semi-automatically proven by supplying an inductive invariant (in first-order logic), and then proving the resulting verification conditions using first-order theorem provers. Furthermore, as our examples show, in many cases the resulting verification conditions are in the decidable EPR fragment.

Reducing temporal verification to verification conditions in first-order logic (and EPR in particular) has both theoretical and practical benefits. Theoretically, in contrast to more powerful logics, first-order logic has a complete proof system. Practically, great progress has been made in automated first-order theorem proving (e.g., SPASS [234], Vampire [207], iProver [123]), including support for EPR. Our approach allows to leverage this vast progress for temporal verification.

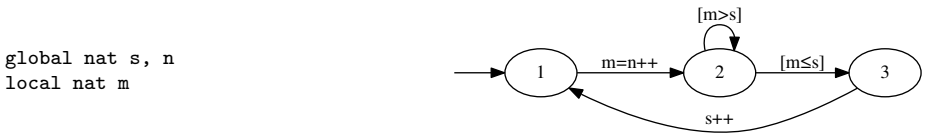


Figure 7.2: The ticket protocol for mutual exclusion. Threads in state 1 are idle; a thread in state 2 is waiting to enter the critical section; and state 3 is the critical section.

7.1.1 A Running Example

We illustrate our approach using the *ticket protocol* for ensuring mutual exclusion with non-starvation among multiple threads, depicted in Figure 7.2. The ticket protocol is an idealized version of spinlocks used in the Linux kernel [54]. The protocol uses natural numbers as ticket values. The global state contains a variable, n, that records the next available ticket, and a variable, s, that records the ticket that is currently being served. Each thread contains a local ticket variable m. Each thread that wishes to enter the critical section runs the code depicted in Figure 7.2, where it first acquires a ticket by setting a local variable m to n and atomically incrementing the next available ticket n. It then waits until its ticket m is equal to s, the ticket that is served. When this happens, it enters the critical section. When it exits the critical section, it increases s, allowing the next thread to be served.

We note that the ticket protocol may be run by any number of threads. In fact, the ticket protocol supports the *unbounded-parallelism* model, in which new threads may be created during the run of the protocol. Thus, when considering infinite traces of the ticket protocol, we cannot assume a finite set of threads, not even an unbounded one. Similarly, the set of ticket numbers is infinite. The latter is true even if we assume that the number of threads is finite, since threads may attempt to enter the critical section infinitely often.

7.1.2 First-Order Temporal Specification

We formalize temporal verification problems using first-order transition system specifications as presented in Section 2.4 to capture the system of interest, and formulas in FO-LTL to capture temporal specifications. Notice that the FO-LTL specification goes beyond pure first-order logic, since the semantics of linear temporal logic (where time ranges over the natural numbers) is not first-order expressible (see e.g., [2]). We now illustrate this formalism using the ticket example.

In our running example, the formalization of the protocol as a first-order transition system $T^t = (\Sigma^t, \Gamma^t, \iota^t, \tau^t)$ is straightforward, using total orders to abstract the counters as explained in Chapter 3. The formula for τ^t appears in full detail later in this chapter in Figure 7.4. More interesting is the temporal specification of the ticket protocol, i.e.,

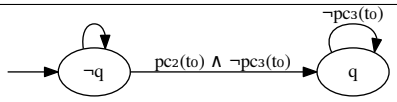
Liveness Property	$\forall x : \mathsf{thread}.\ \Box\,(pc_2(x) \rightarrow \Diamond\, pc_3(x))$
Fairness Assumption	$\forall x : \mathsf{thread}.\ \Box \Diamond\ scheduled(x)$
Temporal Spec. (φ)	$(\forall x : \mathsf{thread}.\ \Box \Diamond\ scheduled(x)) \rightarrow \forall x : \mathsf{thread}.\ \Box\,(pc_2(x) \rightarrow \Diamond\, pc_3(x))$
$\neg\varphi$	$(\forall x : \mathsf{thread}.\ \Box \Diamond\ scheduled(x)) \wedge (\exists x : \mathsf{thread}.\ \Diamond\,(pc_2(x) \wedge \Box\, \neg pc_3(x)))$
$Skolem(\neg\varphi)$	$(\forall x : \mathsf{thread}.\ \Box \Diamond\ scheduled(x)) \wedge \Diamond\,(pc_2(t_0) \wedge \Box\, \neg pc_3(t_0))$
$T_{Skolem(\neg\varphi)}$	 $\phi_1(x) = scheduled(x)$ $\phi_2 = q$

Figure 7.3: Expressing the temporal specification of the ticket protocol. The liveness property and fairness assumptions are expressed in FO-LTL. The FO-LTL specification of the protocol states that the fairness assumptions imply the liveness properties. To obtain a fair transition system that captures infinite traces that violate the specification, we first negate the specification and Skolemize the result (t_0 is a Skolem constant from the subformula $\exists x : \mathsf{thread}.\ \Diamond(pc_2(x) \wedge \Box\, \neg pc_3(x))$). We then convert it to the depicted fair transition system $T_{Skolem(\neg\varphi)}$. $T_{Skolem(\neg\varphi)}$ has two states, labeled q and $\neg q$ (q is a new nullary relation), and fairness constraints $\phi_1(x)$, ϕ_2.

non-starvation. Figure 7.3 depicts the FO-LTL specification for the ticket protocol. The specification we wish to verify is that every thread that requests to enter the critical section (i.e., reaches location 2), eventually enters (i.e., reaches location 3). This should hold under the fairness assumption that all the threads are scheduled infinitely often. Thus, the FO-LTL specification is that the fairness assumption implies liveness. Figure 7.3 also depicts the negation of the specification, and a "monitor" that tracks it, which we explain next.

Reduction from FO-LTL specification to fair termination As a first step in our approach for verifying that a transition system given by $T = (\Sigma, \Gamma, \iota, \tau)$ in first-order logic satisfies a specification given by an FO-LTL formula φ, we follow an approach which generalizes the standard automata theoretic approach from the propositional case to the first-order case. Specifically, we reduce the problem of verifying a temporal specification for a transition system, to the problem of checking that a *fair transition system* has no infinite fair traces.

A *fair transition system* is specified by $(\Sigma, \Gamma, \iota, \tau, \{\phi_1(\bar{x}), \ldots, \phi_n(\bar{x})\})$, where $\Sigma, \Gamma, \iota, \tau$ specify the state space, initial states and transitions as usual (see Section 2.4), and each $\phi_i(\bar{x})$ is a first-order formula with free variables, used to specify Büchi acceptance conditions, which we call fairness constraints. The semantics of the fairness constraints requires that a fair trace must satisfy each fairness constraint infinitely often, for *every* assignment to the free variables. They thus capture *infinitely many* fairness constraints, for example, that every thread is scheduled infinitely often.

Given a first-order transition system specification $T = (\Sigma, \Gamma, \iota, \tau)$ and an FO-LTL specification φ, we would like to check if $T \models \varphi$. To reduce this problem to fair termination, we first negate φ and Skolemize it. We then convert the negation of the specification to a fair transition system, whose fair traces are precisely those that violate φ. This can be done using standard techniques, one of which is presented in Section 8.2. This process is illustrated for the ticket example in Figure 7.3 (the fair transition system is simplified for purposes of presentation, but still encodes the correct specification). We then take the product of the original transition system T, and the transition system that encodes the negation of φ. The result is a fair transition system, whose fair traces are in one-to-one correspondence to runs of the T that violate φ. From this point, all that remains is to check if the resulting fair transition system is empty or not (i.e., does it have a fair trace). This reduction to fair termination is both sound and complete.

For the ticket example, the result is a fair transition system with an additional Skolem constant t_0 and an additional nullary relation q that represents the state of an automaton that is depicted in Figure 7.3, which encodes the negation of the liveness property. The resulting transition system has two fairness constraints: $\phi_1 = \mathit{scheduled}(x)$, $\phi_2 = q$. Note that the first constraint has a free thread variable, in order to enforce that every thread is scheduled infinitely often.

7.1.3 Reducing Fair Termination to Safety

The core result presented in this chapter is a technique for reducing the fair termination problem of a fair transition system $(\Sigma, \Gamma, \iota, \tau, \{\phi_1(\bar{x}), \ldots, \phi_n(\bar{x})\})$ to a safety problem given by $(\widehat{\Sigma}, \widehat{\Gamma}, \widehat{\iota}, \widehat{\tau})$ and $\widehat{P}$, when both systems are specified in first-order logic. This allows standard methods and other methods developed in this thesis for proving safety to be used.

7.1.3.1 Intuition from the Ticket Protocol Example

Before presenting our reduction, let us gain some intuition from the example of the ticket protocol. Consider the following intuitive argument that explains why the ticket protocol satisfies its specification. We need to show that there is no infinite trace in which every thread is scheduled infinitely often, but at some point it time, say k_0, thread t_0 takes a ticket (enters pc_2) and from k_0 on, t_0 never enters the critical section (pc_3). We observe that to show no such trace exists, it suffices to consider the finite set A of threads and ticket numbers that are active at k_0, that is, all threads that were scheduled prior to k_0, and all ticket numbers allocated prior to k_0. The reason is that for any interval $[k_1, k_2]$ later than k_0

in which all the threads in A are scheduled, *the state of at least one of them changed*, when restricted to the allocated tickets in A. This is because one of these threads is the one being serviced. In other words, when projecting the states of the protocol to the finite set A, there is no abstract cycle that visits all the fairness constraints induced by A. This resembles the liveness-to-safety reduction of [27] in the case of finite-state systems. Next, we present our reduction and discuss this relation in more detail.

7.1.3.2 Reducing Fair Termination to Safety

The essence of our reduction from fair termination to safety is to identify a family of finite execution traces, such that every fair trace must contain one of them as a prefix. After such a family is identified, we are left with the safety problem of showing that no such finite trace is reachable. In the finite-state case, the classical reduction of [27] uses for this purpose the family of *fair cycles* — lasso shaped finite executions that visit the same state twice, while visiting every fairness constraint at least once in the loop.

A naive use of the family of fair cycles fails for infinite-state systems, and for two reasons. First, with infinitely many states, a trace can be infinite without repeating the same state twice. Second, with infinitely many fairness constraints, the condition that a finite path visits all fairness constraints is inadequate: an infinite trace may visit every fairness constraint infinitely often, but without having any finite segment that visits all of them even once.

When considering fair transition systems specified in first-order logic, both problems can be eliminated if we consider a *finite* subset D of elements. We can then project the states of the system into D, and obtain a *finite abstraction* of the state space, such that any infinite (concrete) trace will revisit some *abstract state* infinitely often. Furthermore, we can use D to define a finite set of fairness constraints F_D, which we obtain by instantiating the fairness constraints $\phi(x)$ only with elements from D. For the ticket example, this means we would only require that every thread is scheduled for a finite set of threads. Now, in every infinite fair trace, there must be a D-abstract fair cycle, i.e., an infinite fair trace must visit two states s_1 and s_2 such that $s_1|_D = s_2|_D$, and every fairness constraint in F_D is visited by the path from s_1 to s_2. This characterization, combined with a mechanism for computing D described next, provides the desired family of finite execution traces, such that every fair trace must contain one of them as a prefix.

Consider the ticket example. We see that the above argument captures the essence of the intuitive proof we presented, provided that the set D contains all threads active at k_0, i.e., the time where t_0 obtains its ticket number, and all ticket numbers allocated at that time.

Indeed, for the ticket example, as well as for other interesting examples, one cannot use any *a priori* fixed finite set D. A key enabling insight we present is to determine a different D for different traces, in a way which ensures that for any fair trace, we would assign a finite set D. In the ticket example, we can define D to be the set of threads scheduled and ticket numbers allocated prior to k_0. A key question is how to determine D in the general case.

Dynamic abstraction and fairness selection using footprint For determining D, we present a natural solution that is sufficient to fully automate the reduction (i.e., we do not ask for the user's help), while being precise enough to capture challenging examples. Our solution is to maintain a finite set of elements that is the *footprint* of a finite execution prefix. Transitions are usually *local*, in the sense that each transition affects (and is affected by) a finite set of elements. We show that we can syntactically extract this finite set from the transition relation τ. For the ticket example, the footprint of a prefix would precisely be all the threads scheduled, and all the ticket numbers allocated.

This gives rise to a natural way to define D. Given a trace, wait for a time k_0 when a certain condition is met, and then let D be the footprint of the execution prefix up to k_0. In the ticket example, a suitable condition is that thread t_0 obtains its ticket number. In the general case, we must choose a condition such that every infinite fair trace eventually meets it. Our canonical solution for this is to wait for a finite subset of the fairness constraints, which is determined by the initial state. In particular, it will include all nullary fairness constraints, i.e., fairness constraints given by formulas without free variables. For the ticket example, $\phi_2 = q$ is a nullary fairness constraint, ensuring that k_0 is after t_0 obtains its ticket number. This works well for other interesting examples as well.

Realization of the reduction With these definitions of D and k_0, we define an algorithmic reduction of fair termination to safety in first-order logic. Given a fair first-order transition system, the reduction augments it by a monitor that identifies finite execution traces that (1) reach the point k_0 in which D is set to the footprint of the execution so far, and (2) afterwards visit two states s_1 and s_2 such that $s_1|_D = s_2|_D$ and every fairness constraint in F_D is visited by the path from s_1 to s_2. Upon identifying such a prefix, the monitor enters an error state. The output of the reduction is the augmented transition system and the safety property that the error state is not reachable.

Note that, while the soundness of our reduction relies on a notion of a dynamic finite abstraction, we do not construct an abstract transition system. In fact, the reduction does not incur any abstraction of the transitions. The finite abstraction is only used by the

monitor that augments the transition system, for abstract fair cycle detection.

Parameterized systems Parameterized systems are a special case of infinite-state systems, where each value of the parameter defines a finite-state instance of the system, but the parameter itself is unbounded. In first-order logic, this means we are only interested in traces over finite domains. In this case, it is sound to prove fair termination by proving the absence of fair cycles. This can be viewed as a special case of our approach, where the footprint includes all elements of the domain. Note that for this setting, the reduction to safety is complete, i.e., if the original transition system has no infinite fair traces, then the resulting system will be safe. While our main interest here is in systems that are truly infinite-state, it is nice to note that our general formalism maintains completeness in the special case of parameterized systems.

7.1.4 A Nested Termination Argument

For finite-state systems and for parameterized transition systems the reduction from fair termination to fair cycle detection is complete. For infinite-state systems the reduction we present is sound but not complete. This is expected, since fair termination is theoretically harder than safety[1]. Incompleteness of the reduction means that for some instances where the input system has no fair traces, the output system will not be safe. While incompleteness is unavoidable, we would like a reduction that works for examples of interest. However, the reduction presented in the previous section fails for a particular class of natural examples.

Intuitively, this happens when progress is not obtained in a bounded number of steps after we fix the finite set of elements used for abstract fair cycle detection. To handle such examples, we develop a more powerful reduction that relies on a *nesting structure* provided by the user. A nesting structure divides the transitions of a system into several nested levels. Given such a structure, one can use abstract fair cycle detection for each level separately, while assuming subsequent levels terminate.

7.2 Preliminaries

In this section we extend the formalism presented in Chapter 2 to temporal specifications using FO-LTL. We also present the fair transition systems in first-order logic, the analog of (generalized) Büchi automata for first-order transition systems.

[1] For transition systems in first-order logic, it is easy to show that fair termination is Π^1_1 hard (see [2]), while safety is in the arithmetical hierarchy.

7.2.1 First-Order Linear Temporal Logic (FO-LTL)

To specify temporal properties of first-order transition systems we use First-Order Linear Temporal Logic (FO-LTL), which combines LTL with first-order logic (see e.g., [2]). For simplicity of presentation, we consider only the "globally" ($\Box$) and the "eventually" ($\Diamond$) temporal operators. Our approach also extends to other standard temporal operators (i.e., "next" and "until").

Syntax Given a first-order vocabulary Σ, FO-LTL formulas are defined by:

$$f ::= r(t_1, \ldots, t_n) \mid t_1 = t_2 \mid \neg f \mid f_1 \vee f_2 \mid \exists x.f \mid \Box f$$

where r is an n-ary relation symbol in Σ, x is a variable, each t_i is a term over Σ (defined as in first-order logic) and $\Box$ denotes the "globally" temporal operator. We also use the standard shorthand for the "eventually" temporal operator: $\Diamond f = \neg\Box\neg f$, and the usual shorthands for logical operators (e.g., $\forall x.f = \neg\exists x.\neg f$).

Semantics FO-LTL formulas over Σ are interpreted over infinite sequences of states (first-order structures) over Σ, with the same domain $\mathcal{D}$. Atomic formulas are interpreted over states (which are first-order structures), the temporal operators are interpreted as in traditional LTL, and first-order quantifiers are interpreted over the shared domain $\mathcal{D}$.

Formally, the semantics is defined w.r.t. an infinite sequence of states $\pi = s_0, s_1, \ldots$ and an assignment σ that maps variables to $\mathcal{D}$ — the shared domain of all states in π. We sometimes omit σ when the formula has no free variables. We define $\pi^i = s_i, s_{i+1}, \ldots$ to be the suffix of π starting at index i. The semantics is defined as follows.

$$\begin{aligned}
\pi, \sigma \models r(t_1, \ldots, t_n) &\Leftrightarrow s_0, \sigma \models r(t_1, \ldots, t_n) \\
\pi, \sigma \models t_1 = t_2 &\Leftrightarrow s_0, \sigma \models t_1 = t_2 \\
\pi, \sigma \models \neg\psi &\Leftrightarrow \pi, \sigma \not\models \psi \\
\pi, \sigma \models \psi_1 \vee \psi_2 &\Leftrightarrow \pi, \sigma \models \psi_1 \text{ or } \pi, \sigma \models \psi_2 \\
\pi, \sigma \models \exists x.\psi &\Leftrightarrow \text{exists } d \in \mathcal{D} \text{ s.t. } \pi, \sigma[x \mapsto d] \models \psi \\
\pi, \sigma \models \Box\psi &\Leftrightarrow \text{forall } i \geq 0,\ \pi^i, \sigma \models \psi
\end{aligned}$$

We say that a first-order transition system $(\Sigma, \Gamma, \iota, \tau)$ satisfies a closed FO-LTL formula φ over Σ if all of its traces satisfy φ (i.e., for every $\mathcal{D}$).

$$\Sigma^t = \{n : \mathsf{ticket}, s : \mathsf{ticket}, m : \mathsf{thread} \to \mathsf{ticket}, \leq (\mathsf{ticket}, \mathsf{ticket}), pc_1(\mathsf{thread}), pc_2(\mathsf{thread}), pc_3(\mathsf{thread}), scheduled(\mathsf{thread})\}$$
$$\Gamma^t = \{\psi_{\text{total order}}\}$$
$$\iota^t = \psi_{\min}(n) \wedge \psi_{\min}(s) \wedge \forall x : \mathsf{thread}.\ pc_1(x) \wedge \neg pc_2(x) \wedge \neg pc_3(x) \wedge \psi_{\min}(m(x)) \wedge \neg scheduled(x)$$
$$\tau^t = \exists x : \mathsf{thread}.\ \tilde{\tau}(x)$$
$$\tilde{\tau}(x) = (\tau_{12}(x) \vee \tau_{22}(x) \vee \tau_{23}(x) \vee \tau_{31}(x)) \wedge \forall y : \mathsf{thread}. scheduled'(y) \leftrightarrow x = y$$
$$\tau_{12}(x) = \psi_{12}(x) \wedge m'(x) = n \wedge (\forall y : \mathsf{thread}.\ x \neq y \to m'(y) = m(y)) \wedge \psi_{\text{succ}}(n, n') \wedge s' = s$$
$$\tau_{22}(x) = \psi_{22}(x) \wedge m(x) > s \wedge (\forall y : \mathsf{thread}.\ m'(y) = m(y)) \wedge n' = n \wedge s' = s$$
$$\tau_{23}(x) = \psi_{23}(x) \wedge m(x) \leq s \wedge (\forall y : \mathsf{thread}.\ m'(y) = m(y)) \wedge n' = n \wedge s' = s$$
$$\tau_{31}(x) = \psi_{31}(x) \wedge (\forall y : \mathsf{thread}.\ m'(y) = m(y)) \wedge n' = n \wedge \psi_{\text{succ}}(s, s')$$
$$\psi_{ij}(x) = pc_i(x) \wedge pc'_j(x) \wedge \left(\bigwedge_{k \neq j} \neg pc'_k(x)\right) \wedge \bigwedge_k \forall y : \mathsf{thread}.\ x \neq y \to (pc'_k(y) \leftrightarrow pc_k(y))$$
$$\psi_{\min}(x) = \forall y : \mathsf{ticket}.\ x \leq y$$
$$\psi_{\text{succ}}(x, y) = x < y \wedge \forall z : \mathsf{ticket}.\ x < z \to y \leq z$$
$$\psi_{\text{total order}} = (\forall x : \mathsf{ticket}.\ x \leq x) \wedge (\forall x, y, z : \mathsf{ticket}.\ x \leq y \wedge y \leq z \to x \leq z) \wedge$$
$$(\forall x, y : \mathsf{ticket}.\ x \leq y \wedge y \leq x \to x = y) \wedge (\forall x, y : \mathsf{ticket}.\ x \leq y \vee y \leq x)$$

Figure 7.4: First-order logic specification of the ticket protocol. The vocabulary includes uninterpreted relations and functions, and a total order on ticket values $\leq$, axiomatized by $\psi_{\text{total order}}$.

7.2.2 Fair Transition Systems

A (Büchi) *fair transition system* is a tuple $(S, S_0, R, \mathcal{F})$ where (S, S_0, R) are a usual transition system (as in Section 2.3), and $\mathcal{F} \subseteq \mathcal{P}(S)$ is a (possibly infinite) set of *fairness constraints*. An infinite trace $\pi = s_0, s_1, \ldots$ is *fair* if it visits every fairness constraint $F \in \mathcal{F}$ infinitely often, i.e., for every $F \in \mathcal{F}$ the set $\{i \mid s_i \in F\}$ is infinite.

Fair termination The fair transition system $(S, S_0, R, \mathcal{F})$ *terminates* if it has no fair (infinite) traces.

Specification in first-order logic As usual, we specify transition systems in first-order logic, by providing $(\Sigma, \Gamma, \iota, \tau)$. As an example, Figure 7.4 depicts the first-order logic specification of the ticket protocol. In this chapter, we simplify the formal presentation by handling only relations and constant symbols from the vocabulary, as function symbols can be treated as special relations with increased arity.

A first-order logic specification of a *fair* transition system is $(\Sigma, \Gamma, \iota, \tau, \Phi)$, where $\Sigma, \Gamma, \iota, \tau$ are as usual, and $\Phi = \{\phi_1, \ldots, \phi_n\}$, where each ϕ_i is a formula over Σ that may contain free variables. For each domain $\mathcal{D}$, the semantics of $(\Sigma, \Gamma, \iota, \tau, \Phi)$ is a fair transition system $(S, S_0, R, \mathcal{F})$ where S, S_0, R are as usual, and $\mathcal{F}$ is a possibly infinite set of fairness constraints

defined by Φ:

$$\mathcal{F} = \{F_\phi(\bar{e}) \mid \phi(x_1, \ldots, x_n) \in \Phi,\ \bar{e} \in \mathcal{D}^n\}$$
$$\text{where } F_\phi(\bar{e}) = \{s \in S \mid s, [x_1 \mapsto e_1, \ldots, x_n \mapsto e_n] \models \phi(x_1, \ldots, x_n)\}$$

Note that each fairness formula $\phi \in \Phi$ with free variables induces a set of fairness constraints of the form $F_\phi(\bar{e})$, for each $\bar{e} \in \mathcal{D}^n$. Observe that since $\mathcal{D}$ may be infinite, ϕ may stand for infinitely many such fairness constraints.

We say that $(\Sigma, \Gamma, \iota, \tau, \Phi)$ *terminates* if it has no fair (infinite) traces, i.e., if for any $\mathcal{D}$, the fair transition system defined for $\mathcal{D}$ terminates.

7.2.3 Reducing FO-LTL Verification to Fair Termination

Given any closed formula f in FO-LTL over vocabulary Σ, we can construct a fair transition system (monitor) over an extended vocabulary $\Sigma^f \supseteq \Sigma$ such that the pointwise-projection of its set of fair traces on Σ is exactly the set of all traces that satisfy f. This is a straightforward extension of the classical construction of a Büchi automaton for an LTL formula used in the automata theoretic approach to verification [233, 236], except that instead of a finite-state automaton we obtain a fair first-order transition system. We defer explaining this in detail to Section 8.2 in the next chapter.

To check whether a given transition system $(\Sigma, \Gamma, \iota, \tau)$ satisfies f we can then take the product of the "monitor" of $\neg f$ and the original transition system. Proving that the original transition system satisfies the FO-LTL formula f reduces to proving that the product transition system has no fair traces. This reduction is sound and complete, meaning that $(\Sigma, \Gamma, \iota, \tau)$ satisfies f if and only if the fair transition system terminates. As such, from now on we focus on proving fair termination. This allows us to verify arbitrary FO-LTL properties.

7.2.3.1 Illustration on the Ticket Protocol

Consider the first-order logic specification of the ticket protocol, $(\Sigma^t, \Gamma^t, \iota^t, \tau^t)$, presented in Figure 7.4. Its desired FO-LTL specification, and the steps of constructing a monitor for its negation are presented in Figure 7.3. The constructed monitor is a fair transition system, defined over vocabulary $\{pc_2(\mathsf{thread}),\ pc_3(\mathsf{thread}),\ \mathit{scheduled}(\mathsf{thread}),\ t_0 : \mathsf{thread},\ q\}$. We compose it with the transition system of the ticket protocol. The result is a fair transition system $(\Sigma, \Gamma, \iota, \tau, \Phi)$ defined below, whose fair (infinite) traces represent traces of the protocol

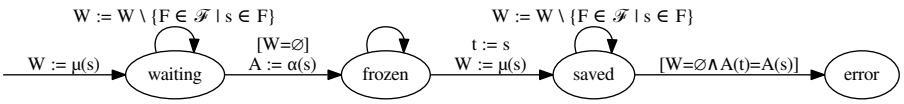


Figure 7.5: Monitor that checks the (α, μ)-acyclicity condition. The monitor uses auxiliary state: $W \in \mathcal{P}_{fin}(\mathcal{F})$, $A : S \to X$, and $t \in S$. The variable s denotes the current state of the monitored transition system. W is a finite set of fairness constraints for which the monitor is waiting. A is used to store $\alpha(s_{k_f})$, and t is used to store s_{k_1} (see Definition 7.8). The monitor starts with an initial condition that W is $\mu(s_0)$, and then goes into waiting state, in which it updates W by removing fairness constraints that have been satisfied. Once $W = \emptyset$, a fair prefix is obtained, hence the monitor non-deterministically decides whether k_f is reached, in which case it freezes the abstraction by setting A to $\alpha(s_{k_f})$. It then non-deterministically chooses when to fix k_1. When this happens, the monitor saves s_{k_1} in t, and also resets W to $\mu(s_{k_1})$. It then keeps updating W, and checks if we reach a point in which $W = \emptyset$ and $A(t) = A(s)$, i.e., a point k_2 s.t. $[k_1, k_2]$ is μ-fair, and $\alpha(s_{k_f})(s_{k_1}) = \alpha(s_{k_f})(s_{k_2})$. If this happens, the monitor goes to the error state. The error state is thus reachable in the product of the monitor with the transition system if and only if the (α, μ)-acyclicity condition is violated.

that violate the specification, hence its fair termination implies correctness of the protocol:

$$\Sigma = \Sigma^t \cup \{t_0 : \mathsf{thread},\ q\}$$

$$\Gamma = \Gamma^t$$

$$\iota = \iota^t \wedge \neg q$$

$$\tau = \exists x : \mathsf{thread}.\ \tilde{\tau}(x) \wedge t_0' = t_0 \wedge \tau_q$$

$$\text{where } \tau_q = (\neg q \wedge \neg q') \vee (\neg q \wedge q' \wedge pc_2'(t_0) \wedge \neg pc_3'(t_0)) \vee (q \wedge q' \wedge \neg pc_3'(t_0))$$

$$\Phi = \{\phi_1(x), \phi_2\} \text{ where } \phi_1(x) = \mathit{scheduled}(x),\ \phi_2 = q$$

7.3 Reducing Fair Termination to Safety in First-Order Logic

In this section we introduce our approach for proving fair termination using first-order logic. The approach is based on a reduction from fair termination to safety that is applied to transition systems in first-order logic. We start by presenting the reduction parameterized by a dynamic abstraction function and a fairness selection function, and establishing its soundness. Next, we show how to apply the reduction using first-order logic, by defining a canonical dynamic abstraction and fairness selection functions.

7.3.1 Parametric Reduction via Dynamic Abstraction

For clarity of the presentation, we first present the reduction of fair termination to safety in a semantic way, ignoring the first-order logic aspects. This semantic reduction is parameterized by a dynamic finite abstraction function, and a dynamic fairness selection function (defined below). In Section 7.3.2, we show how to algorithmically derive these functions for transition systems that are specified in first-order logic.

Fix a fair transition system $(S, S_0, R, \mathcal{F})$. Roughly speaking, the termination argument for proving that $(S, S_0, R, \mathcal{F})$ has no fair traces is based on showing that no fair abstract cycle exists. To ensure soundness of this argument, we use a dynamic finite abstraction function that must abstract states into a finite set, and we use a dynamic fairness selection function that must select a finite set of fairness constraints, to ensure that eventually they are all satisfied in a finite trace (facilitating the existence of a fair abstract cycle). Formally:

Definition 7.6 (Dynamic Finite Abstraction)**.** A *dynamic finite abstraction function* is a function $\alpha : S \to S \to X$, where X is any set, and for any $s \in S$ the range of $\alpha(s)$ is finite.

Definition 7.7 (Dynamic Fairness Selection)**.** A *dynamic fairness selection function* is a function $\mu : S \to \mathcal{P}_{fin}(\mathcal{F})$ (where $\mathcal{P}_{fin}(\mathcal{F})$ denotes the set of finite subsets of $\mathcal{F}$). Given a trace (either infinite or finite) $\pi = s_0, s_1, \ldots$ and $0 \leq k < k' < |\pi|$, we say that the $[k, k']$ segment of π is μ-fair if $\forall F \in \mu(s_k).\ \{s_{k+1}, \ldots, s_{k'-1}\} \cap F \neq \emptyset$.

Recall that we consider transition systems with infinitely many fairness constrains. (In first-order logic transition systems, the fairness formulas induce infinitely many fairness constraints, by instantiating the free variables.) In this setting, an infinite trace can be fair without containing any finite segment that visits all fairness constraints. The dynamic fairness selection function is therefore used to select a finite subset of the fairness constraints that is "relevant" for a particular state (note that each fairness constraint that is selected for $\mu(s)$ is taken as a whole from $\mathcal{F}$; that is, μ does not change the individual fairness constraints, and only selects finitely many fairness constraints for each state). Note that in an infinite fair trace, for every k there exists a $k' > k$ such that the $[k, k']$ segment is μ-fair.

To prove that $(S, S_0, R, \mathcal{F})$ has no fair traces, we require a dynamic finite abstraction function α and a dynamic fairness selection function μ such that there are no (α, μ)-abstract fair cycles, as defined below.

Definition 7.8 ((α, μ)-Abstract Fair Cycle)**.** A finite trace $s_0, s_1, \ldots, s_n$ contains an (α, μ)-*abstract fair cycle* if there are $k_f < k_1 < k_2 \leq n$ such that the segments $[0, k_f]$ and $[k_1, k_2]$

are μ-fair, and we have $\alpha(s_{k_f})(s_{k_1}) = \alpha(s_{k_f})(s_{k_2})$. The (α, μ)-*acyclicity condition* requires that no trace contains an (α, μ)-*abstract fair cycle*.

Intuitively, the acyclicity condition requires that any fair segment $[k_1, k_2]$ that follows a fair prefix $[0, k_f]$ cannot form a cycle when states are abstracted by the finite abstraction associated with s_{k_f}. The index k_f can therefore be viewed as the "freeze" point of the finite abstraction, after which no fair abstract cycle is allowed. Note that the condition requires that no matter which freeze point is selected (as long as the prefix $[0, k_f]$ is μ-fair), the finite abstraction $\alpha(s_{k_f})$ is precise enough to exclude a fair abstract cycle.

Lemma 7.9. *Let $(S, S_0, R, \mathcal{F})$ and α and μ be s.t. the (α, μ)-acyclicity condition is satisfied, then $(S, S_0, R, \mathcal{F})$ has no fair traces.*

Proof. Assume to the contrary that $(s_k)_{k=0}^{\infty}$ is a fair trace of $(S, S_0, R, \mathcal{F})$. Let k_f be an index such that the $[0, k_f]$ segment is μ-fair. Consider the sequence $\left(\alpha(s_{k_f})(s_k)\right)_{k=0}^{\infty}$. Since the range of $\alpha(s_{k_f})$ is finite, there must be an infinite subsequence that consists of a constant $x \in X$. Let $k_1 \geq k_f$ be the index of the first occurrence of x after k_f. Due to fairness, there must exist $k_2 > k_1$ such that the $[k_1, k_2]$ segment is μ-fair. The reason is that a fair trace must visit every element of $\mathcal{F}$ infinitely often. This ensures that whenever we fix k_1 and select the finite subset of $\mathcal{F}$ given by $\mu(s_{k_1})$, we will encounter all elements of $\mu(s_{k_1})$ after some finite number of transitions. In particular, this segment can be extended such that $\alpha(s_{k_f})(s_{k_2}) = x$ (since x repeats infinitely often), i.e., $\alpha(s_{k_f})(s_{k_1}) = \alpha(s_{k_f})(s_{k_2})$. This contradicts the acyclicity condition. □

Monitoring the (α, μ)-acyclicity condition Since an (α, μ)-abstract fair cycle is manifested in a finite prefix of the trace, the (α, μ)-acyclicity condition (Definition 7.8) is a safety property of $(S, S_0, R, \mathcal{F})$. Hence, given α and μ, Lemma 7.9 lets us reduce fair termination to safety checking.

The safety property can be further simplified by creating a monitor that checks the presence of (α, μ)-abstract fair cycles. Figure 7.5 presents such a monitor that is parameterized by α and μ. The monitor runs in parallel to the transition system and tracks violations of the (α, μ)-acyclicity condition, i.e., presence of (α, μ)-abstract fair cycles. That is, the monitor checks for the presence of a trace $s_0, s_1, \ldots$ of $(S, S_0, R, \mathcal{F})$ where for some k_f the segments $[0, k_f]$ and $[k_1, k_2]$ are μ-fair but $\alpha(s_{k_f})(s_{k_1}) = \alpha(s_{k_f})(s_{k_2})$. To identify the μ-fair prefix $[0, k_f]$, the monitor starts in state "waiting" and uses a set $W \in \mathcal{P}_{fin}(\mathcal{F})$ of fairness constraints that is initialized to $\mu(s_0)$; while the monitor is in state "waiting", W is updated whenever the current state s of the monitored transition system visits one of the fairness

constraints. When $W = \emptyset$, the prefix is μ-fair, hence the monitor non-deterministically selects the "freeze" point k_f and stores the corresponding finite abstraction $\alpha(s_{k_f})$ in $A : S \to X$. It then moves to state "frozen" from which it non-deterministically selects k_1, where it saves s_{k_1} in $t \in S$ and initializes W to $\mu(s_{k_1})$ in order to identify a μ-fair segment. It then continues to state "saved" where it updates W in each step, and when $W = \emptyset$, i.e., $[k_1, k_2]$ is a μ-fair segment, if a state s is encountered where $A(s) = A(t)$, then the monitor goes to state "error" and declares violation.

Clearly, the error state is reachable in the product of the monitor with the transition system if and only if the (α, μ)-acyclicity condition is violated. Therefore, the reduction from fair termination to safety constructs the product of the monitor with the transition system. Fair termination then reduces to checking that the product transition system, denoted $(\widehat{S}, \widehat{S_0}, \widehat{R})$, satisfies the safety property $\widehat{P}_{\neg \text{error}}$ defined by the set of states where the monitor is not in its error state. Note that the product transition system has no fairness constraints, since we are only interested in its safety (i.e., in its finite traces). We also note that $(\widehat{S}, \widehat{S_0}, \widehat{R})$ is not an abstract transition system. The use of the dynamic finite abstraction α does not incur any abstraction of the transitions, and it is only used for checking the (α, μ)-acyclicity condition.

Lemma 7.10. *Let $(S, S_0, R, \mathcal{F})$ be a transition system, α a dynamic finite abstraction function and μ a dynamic fairness selection function, and let $(\widehat{S}, \widehat{S_0}, \widehat{R})$ be the transition system obtained from the composition of (S, S_0, R) with the (α, μ)-acyclicity monitor specified in Figure 7.15. If $(\widehat{S}, \widehat{S_0}, \widehat{R})$ satisfies the safety property $\widehat{P}_{\neg error}$, then $(S, S_0, R, \mathcal{F})$ has no fair traces.*

7.3.2 Uniform Reduction in First-Order Logic

We now present the realization of the reduction for a first-order transition system $T = (\Sigma, \Gamma, \iota, \tau, \Phi)$. The main ingredients are the algorithmic extraction of a dynamic finite abstraction function α and a fairness selection function μ based on the *footprint* of a trace, and the realization of the acyclicity monitor based on these functions as a transition system in first-order logic.

Tracking the footprint of a trace The dynamic finite abstraction and fairness selection functions formalized in the previous section rely on finiteness arguments. The first key idea that allows us to realize these functions in first-order logic is to augment the first-order transition system $(\Sigma, \Gamma, \iota, \tau, \Phi)$ with a new unary relation d that will always contain a finite

number of elements, and will intuitively accumulate all the elements that the trace has seen or affected so far.

Assume, without loss of generality, that ι and τ are given in the following form:

$$\iota = \exists x_1, \ldots, x_n.\ \widetilde{\iota}(x_1, \ldots, x_n) \qquad \tau = \exists x_1, \ldots, x_m.\ \tilde{\tau}(x_1, \ldots, x_m)$$

Then, define an augmented transition system $(\Sigma^d, \Gamma^d, \iota^d, \tau^d, \Phi^d)$ where

$$\begin{aligned}
\Sigma^d &= \Sigma \cup \{d^1\} \\
\Gamma^d &= \Gamma \\
\iota^d &= \exists x_1, \ldots, x_n.\ \widetilde{\iota} \wedge \forall x.\ d(x) \leftrightarrow \left(\textstyle\bigvee_{i=1}^{n} x = x_i \vee \bigvee_{c \in \Sigma} x = c\right) \\
\tau^d &= \exists x_1, \ldots, x_m.\ \tilde{\tau} \wedge \forall x.\ d'(x) \leftrightarrow \left(d(x) \vee \textstyle\bigvee_{i=1}^{m} x = x_i \vee \bigvee_{c \in \Sigma} x = c'\right) \\
\Phi^d &= \Phi
\end{aligned}$$

Intuitively, the d relation contains the elements that affect or are affected by the transition, captured by the existentially quantified variables and constants. For both sequential programs and distributed algorithms, each transition usually interacts with a finite set of elements (e.g., threads, memory locations, values, etc.). This set is sometimes called the *footprint* of a transition. The idea of d is that it is updated so that it includes the footprint of all the transitions in the trace so far.

The augmentation of the transition system with d has two benefits, captured by the following lemmas. First, it does not affect termination:

Lemma 7.11. *$(\Sigma^d, \Gamma^d, \iota^d, \tau^d, \Phi^d)$ terminates if and only if $(\Sigma, \Gamma, \iota, \tau, \Phi)$ terminates.*

Second, it provides a way to select a finite set of elements in each reachable state:

Lemma 7.12. *Let $s = (\mathcal{D}, \mathcal{I})$ be a reachable state of $(\Sigma^d, \Gamma^d, \iota^d, \tau^d, \Phi^d)$. Then the interpretation of d, $\mathcal{I}(d) \subseteq \mathcal{D}$, is a finite set.*

The reason is that d initially contains at most $n + |C|$ elements, and with each transition it grows by at most $m + |C|$ elements, where C denotes the set of constant symbols in Σ.

Dynamic finite abstraction and fairness selection functions based on footprints Now, we apply the reduction from fair termination to safety on $(\Sigma^d, \Gamma^d, \iota^d, \tau^d, \Phi^d)$. To do so, we define the finite abstraction function for $(\Sigma^d, \Gamma^d, \iota^d, \tau^d, \Phi^d)$ by projecting all relations to the finite set given by d (at k_f), and we define the fairness selection function by taking all

the fairness constraints over all elements of d. Given an interpretation $\mathcal{I}$ and a set A, define the projection of $\mathcal{I}$ to A as $\mathcal{I}|_A = \lambda r.\, \mathcal{I}(r) \cap A^{arity(r)}$. Observe that if A is finite, then the range of $\lambda\mathcal{I}.\mathcal{I}|_A$ is finite. For $s = (\mathcal{D}, \mathcal{I})$, define $s|_A = (A, \mathcal{I}|_A)$.

Definition 7.13 (First-order α and μ)**.** For any state $s = (\mathcal{D}, \mathcal{I})$ of $(\Sigma^d, \Gamma^d, \iota^d, \tau^d, \Phi^d)$, let

$$\alpha(s) = \lambda s'.\, s'|_{\mathcal{I}(d)}$$

$$\mu(s) = \{F_\phi(\bar{e}) \mid \phi(x_1, \ldots, x_n) \in \Phi,\ \bar{e} \in (\mathcal{I}(d))^n\} \qquad \text{see Section 7.2 for the definition of } F_\phi(\bar{e})$$

Since $\mathcal{I}(d)$ is finite, the range of $\alpha(s)$ is finite, and so is $\mu(s)$[2].

Implementing the acyclicity monitor in first-order logic Having defined the dynamic finite abstraction function α and the fairness selection function μ, in order to complete the reduction of fair termination to safety it remains to implement the monitor presented in Figure 7.5 for α and μ in first-order logic, and compose it with $(\Sigma^d, \Gamma^d, \iota^d, \tau^d, \Phi^d)$.

Figure 7.15 presents the realization of the monitor in first-order logic, using a vocabulary $\Sigma^m \supseteq \Sigma^d$ which we describe next. The control states of the monitor are tracked using nullary relations *waiting*, *frozen*, *saved*, *error* $\in \Sigma^m$. The key ideas are to "freeze" the abstraction at k_f by copying the relation $d \in \Sigma^d$ at the freeze point into an auxiliary unary relation $a \in \Sigma^m$, and to "select" the relevant subset of the fairness constraints (both initially and when moving to the "saved" state) by introducing an auxiliary relation $w_i \in \Sigma^m$ for each quantified fairness constraint ϕ_i and initializing it to all tuples in d. To implement the check whether $A(t) = A(s)$ (indicating that an abstract state is repeated), the monitor remembers t by introducing a copy $\Sigma_s = \{a_s \mid a \in \Sigma\}$ of Σ which is set when moving to the "saved" state. Then, the equality of the abstract states is checked by comparing the relation and constant symbols in Σ_s (representing the old state t) and their counterparts in Σ (representing the current state s) on the elements in a. This mimics applying the projection which defines the abstraction at k_f. We denote the first-order specification of the monitor by $(\Sigma^m, \iota^m, \tau^m)$.

The product of $(\Sigma^d, \Gamma^d, \iota^d, \tau^d, \Phi^d)$ with the monitor is then the first-order transition

[2] Strictly speaking, this is only true for the reachable states of $(\Sigma^d, \Gamma^d, \iota^d, \tau^d)$, since there are unreachable states in which d contains infinitely many elements. However, the soundness result is unaffected by this, since the proof of Lemma 7.9 only applies α and μ to reachable states.

cmd	first-order realization
$W := \mu(s)$	$\bigwedge_i \forall \bar{x}.\ w_i'(\bar{x}) \leftrightarrow \bigwedge_j d(x_j)$
$W := W \setminus \{F \in \mathcal{F} \mid s \in F\}$	$\bigwedge_i \forall \bar{x}.\ w_i'(\bar{x}) \leftrightarrow (w_i(\bar{x}) \wedge \neg\phi_i(\bar{x}))$
$[W = \emptyset]$	$\bigwedge_i \forall \bar{x}.\ \neg w_i(\bar{x})$
$A := \alpha(s)$	$\forall x.\ a'(x) \leftrightarrow d(x)$
$t := s$	$\bigwedge_{r\in\Sigma} \forall \bar{x}.\ r_s'(\bar{x}) \leftrightarrow r(\bar{x}) \ \wedge\ \bigwedge_{c\in\Sigma} c_s' = c$
$[A(t) = A(s)]$	$\bigwedge_{r\in\Sigma} \forall \bar{x}.\ \left(\bigwedge_j a(x_j)\right) \rightarrow (r_s(\bar{x}) \leftrightarrow r(\bar{x})) \wedge \bigwedge_{c\in\Sigma}(a(c) \vee a(c_s)) \rightarrow c = c_s$

Figure 7.15: Realization in first-order logic of commands from Figure 7.5. The first-order logic realization uses the following relations: w_i for each $\phi_i \in \Phi$ (w_i has the same arity as ϕ_i), to implement W; a unary relation a, capturing A by recording the interpretation of d at k_f; a relation r_s for every $r \in \Sigma$ (with the same arity), to implement t and capture a copy of the interpretation of all state relations at k_1.

system (without fairness constraints) $\widehat{T} = (\widehat{\Sigma}, \widehat{\Gamma}, \widehat{\iota}, \widehat{\tau})$, where

$$\widehat{\Sigma} = \Sigma^m = \Sigma^d \cup \Sigma_s \cup \{\mathit{waiting}, \mathit{frozen}, \mathit{saved}, \mathit{error}, a\} \cup \{w_i \mid \phi_i \in \Phi\}$$

$$\widehat{\Gamma} = \Gamma^d = \Gamma$$

$$\widehat{\iota} = \iota^d \wedge \iota^m$$

$$\widehat{\tau} = \tau^d \wedge \tau^m$$

The following theorem summarizes the reduction from fair termination to safety in first-order logic, as well as its correctness:

Theorem 7.14. *Let $T = (\Sigma, \Gamma, \iota, \tau, \Phi)$ be a fair transition system in first-order logic, and let $\widehat{T} = (\widehat{\Sigma}, \widehat{\Gamma}, \widehat{\iota}, \widehat{\tau})$ be the transition system defined above. If $\widehat{T}$ satisfies the safety property $\widehat{P} = \neg\mathit{error}$, then T has no fair traces.*

The reduction from fair termination to safety is linear (both in time and space) in the size of (the description of) T. The size of (the description of) the resulting first-order transition system $\widehat{T}$ is approximately twice as big as that of T (due to the copied relations).

Handling finite domains In some settings, we have additional knowledge that some sets of elements are always finite. For example, we could be interested in a distributed or multithreaded protocol without unbounded-parallelism. That is, we consider the protocol only when it is running on a fixed, finite (albeit unbounded) set of nodes or threads. Such a protocol can still use some infinite sorts representing integer values, messages, etc. In such a setting, we may include all threads, nodes, or any other set that is known to be finite into the initial condition for d. Another example is where we might include all elements smaller than some constant representing a natural number initially in d. The soundness of

the reduction is maintained, as long as d is guaranteed to be finite in all reachable states.

In particular, in the case of parameterized systems, where all the sorts are finite, we can include *all* elements of the domain in the initial condition of d, while preserving soundness of the reduction. In fact, in this case, fair cycle detection is sound without any abstraction nor fairness selection. Thus, the transition system of the monitor can be simplified by eliminating both d and a, and starting the monitor in the "frozen" state, while initializing W to contain all fairness constraints, and including all elements in the cycle detection (instead of just the ones in a).

Using derived relations and ghost code to increase precision Our dynamic abstraction of a state is computed completely automatically by projecting the first-order structure to the set of elements determined by the footprint $\mathcal{I}(d)$ at the freeze point. The precision of the abstraction can thus be increased by using additional derived relations, which can allow the projection to distinguish between more states. For example, consider a vocabulary with a unary relation r. Suppose that two states, s_1 and s_2, differ in their interpretation of r, where $\mathcal{I}_1(r) = \emptyset$ and $\mathcal{I}_2(r) = \{e\}$, but e is not included in the finite set by which the abstraction is computed. In this case, $\alpha(s_f)(s_1) = s_1|_{\mathcal{I}_f(d)} = s_2|_{\mathcal{I}_f(d)} = \alpha(s_f)(s_2)$. However, if we add to the vocabulary a nullary derived relation r' that tracks the formula $\exists x.\, r(x)$, then we will now have that $\alpha(s_f)(s_1) \neq \alpha(s_f)(s_2)$, due to the different interpretation of r'. We used such derived relations for the liveness verification of Paxos protocols (see Section 7.6.1). Another possible way to increase the precision of the abstraction is by adding ghost code that increases the footprint of transitions. This will also cause the finite set of elements to which we project to be larger, making the abstraction more precise.

7.3.3 Detailed Illustration for Ticket Protocol

We now illustrate our reduction of fair termination to safety on the ticket protocol. Let $(\Sigma, \Gamma, \iota, \tau, \Phi)$ be the fair transition system defined in Section 7.2.3.1 for the ticket protocol. To prove the fair termination of $(\Sigma, \Gamma, \iota, \tau, \Phi)$ we apply our reduction to safety. The first step is to augment the transition system by tracking the footprint. This results in the following transition system, T^d (note that we add a unary relation for each sort):

$$\Sigma^d = \Sigma \cup \{d_{\mathsf{thread}}(\mathsf{thread}),\ d_{\mathsf{ticket}}(\mathsf{ticket})\}$$

$$\Gamma^d = \Gamma^t$$

$$\iota^d = \iota \wedge (\forall x : \mathsf{thread}.\ d_{\mathsf{thread}}(x) \leftrightarrow (x = t_0)) \wedge (\forall x : \mathsf{ticket}.\ d_{\mathsf{ticket}}(x) \leftrightarrow (x = n \vee x = s))$$

$$\tau^d = \exists x : \mathsf{thread}.\ \tilde{\tau}(x) \wedge t_0' = t_0 \wedge \tau_q \wedge \big(\forall y : \mathsf{thread}.\ d'_{\mathsf{thread}}(y) \leftrightarrow (d_{\mathsf{thread}}(y) \vee x = y)\big) \wedge \big(\forall y : \mathsf{ticket}.\ d'_{\mathsf{ticket}}(y) \leftrightarrow (d_{\mathsf{ticket}}(y) \vee y = n' \vee y = s')\big)$$

Next, we construct the first-order realization of the monitor of Figure 7.5 and compose it with the augmented transition system. This completes the reduction from fair termination to safety reduction, and results in the following transition system, $\hat{T}$:

$$\widehat{\Sigma} = \Sigma^d \cup \{r_s \mid r \in \Sigma\} \cup \{\mathit{waiting},\ \mathit{frozen},\ \mathit{saved},\ \mathit{error}\} \cup \{a_{\mathsf{thread}}(\mathsf{thread}),\ a_{\mathsf{ticket}}(\mathsf{ticket}),\ w_1(\mathsf{thread}),\ w_2\}$$

$$\widehat{\Gamma} = \Gamma^d = \Gamma^t$$

$$\widehat{\iota} = \iota^d \wedge \mathit{waiting} \wedge \neg \mathit{frozen} \wedge \neg \mathit{saved} \wedge \neg \mathit{error} \wedge (\forall x : \mathsf{thread}.\ w_1(x) \leftrightarrow d_{\mathsf{thread}}(x)) \wedge w_2$$

$$\widehat{\tau} = \tau^d \wedge (\tau_{\mathrm{wait}} \vee \tau_{\mathrm{freeze}} \vee \tau_{\mathrm{save}} \vee \tau_{\mathrm{error}})$$

$$\tau_{\mathrm{wait}} = (\mathit{waiting}' \leftrightarrow \mathit{waiting}) \wedge (\mathit{frozen}' \leftrightarrow \mathit{frozen}) \wedge (\mathit{saved}' \leftrightarrow \mathit{saved}) \wedge \big(\forall x : \mathsf{thread}.\ w_1'(x) \leftrightarrow (w_1(x) \wedge \neg \mathit{scheduled}(x))\big) \wedge (w_2' \leftrightarrow (w_2 \wedge \neg q))$$

$$\tau_{\mathrm{freeze}} = \mathit{waiting} \wedge \neg \mathit{waiting}' \wedge \mathit{frozen}' \wedge \neg \mathit{saved}' \wedge (\forall x : \mathsf{thread}.\ \neg w_1(x)) \wedge \neg w_2 \wedge \big(\forall x : \mathsf{thread}.\ a'_{\mathsf{thread}}(x) \leftrightarrow d_{\mathsf{thread}}(x)\big) \wedge \big(\forall x : \mathsf{ticket}.\ a'_{\mathsf{ticket}}(x) \leftrightarrow d_{\mathsf{ticket}}(x)\big)$$

$$\tau_{\mathrm{save}} = \mathit{frozen} \wedge \neg \mathit{waiting}' \wedge \neg \mathit{frozen}' \wedge \mathit{saved}' \wedge \big(\forall x : \mathsf{thread}.\ w_1'(x) \leftrightarrow d_{\mathsf{thread}}(x)\big) \wedge w_2' \wedge n_s' = n \wedge s_s' = s \wedge q_s' \leftrightarrow q \wedge \forall x : \mathsf{thread}.\ m_s'(x) = m(x) \wedge \bigwedge\nolimits_k pc_{ks}'(x) \leftrightarrow pc_k(x)$$

$$\tau_{\mathrm{error}} = \mathit{saved} \wedge \mathit{error}' \wedge (\forall x : \mathsf{thread}.\ \neg w_1(x)) \wedge \neg w_2 \wedge q_s \leftrightarrow q \wedge \Big(\forall x : \mathsf{thread}.\ a_{\mathsf{thread}}(x) \rightarrow \bigwedge\nolimits_k pc_{ks}(x) \leftrightarrow pc_k(x)\Big) \wedge (\forall x : \mathsf{ticket}.\ a_{\mathsf{ticket}}(x) \rightarrow (n_s = x \leftrightarrow n = x) \wedge (s_s = x \leftrightarrow s = x)) \wedge (\forall x : \mathsf{thread}, y : \mathsf{ticket}.\ a_{\mathsf{thread}}(x) \wedge a_{\mathsf{ticket}}(y) \rightarrow (m_s(x) = y \leftrightarrow m(x) = y))$$

Inductive invariant To provide a better understanding of the entire verification process, we discuss the inductive invariant that establishes the safety of the system $\hat{T}$ that results from the reduction for the example of the ticket protocol. The inductive invariant includes

some rather straightforward properties of reachable states of the ticket protocol. The most interesting part of the inductive invariant is the one that establishes the connection between the protocol state and the footprint, as well as the absence of fair abstract cycles. This part is:

$$\forall x : \mathsf{thread}.\ (pc_2(x) \vee pc_3(x)) \rightarrow d_{\mathsf{thread}}(x)$$

$$\forall x : \mathsf{ticket}.\ x \leq n \rightarrow d_{\mathsf{ticket}}(x)$$

$$(\mathit{frozen} \vee \mathit{saved}) \rightarrow \forall x : \mathsf{thread}.\ (pc_2(x) \vee pc_3(x)) \wedge m(x) \leq m(t_0) \rightarrow a_{\mathsf{thread}}(x)$$

$$(\mathit{frozen} \vee \mathit{saved}) \rightarrow \forall x : \mathsf{ticket}.\ x \leq m(t_0) \rightarrow a_{\mathsf{ticket}}(x)$$

$$\forall x : \mathsf{ticket}.\ s \leq x < n \rightarrow \exists y : \mathsf{thread}.\ m(y) = x \wedge (pc_2(y) \vee pc_3(y))$$

$$\mathit{saved} \rightarrow \forall x : \mathsf{thread}.\ (m_s(x) = s_s \wedge pc_{2_s}(x) \wedge \neg w_1(x)) \rightarrow$$
$$((pc_1(x) \wedge m(x) = s_s) \vee (pc_2(x) \wedge m(x) > m(t_0)) \vee (pc_3(x) \wedge m(x) = s_s))$$

$$\mathit{saved} \rightarrow \forall x : \mathsf{thread}.\ (m_s(x) = s_s \wedge pc_{3_s}(x) \wedge \neg w_1(x)) \rightarrow$$
$$((pc_1(x) \wedge m(x) = s_s) \vee (pc_2(x) \wedge m(x) > m(t_0)))$$

Note that this invariant establishes the fact that all threads in pc_2 or pc_3 are in d_{thread}, and that all allocated ticket numbers are in d_{ticket}. It then establishes that after the freeze point, a_{thread} and a_{ticket} include all the threads ahead of t_0, and all the ticket numbers lower than the ticket of t_0. The rest of the invariant establishes the existence of the thread whose state must change in every fair segment. Notice in particular the use of $\neg w_1(x)$, which expresses the fact that thread x has been scheduled since the save point (for threads that were active at the save point). This structure is typical, and represents the structure of the inductive invariants we obtained in all of our examples.

We note that for the ticket example, the resulting verification conditions fall into the decidable EPR fragment of first-order logic (all quantifier alternations are stratified). This means that for the ticket protocol, our reduction actually allows decidable deductive verification of liveness.

7.4 Capturing Nested Termination Arguments

While sound, the reduction presented in the previous section loses completeness even for systems that are of practical interest. In particular, it does not capture a form of termination arguments we call *nested*, which are needed for interesting protocols. In this section we

```
initially: sender_i = receiver_i = 0, sender_bit = receiver_bit = 0

process sender

action send_data:
    data_channel.send(
        sender_array[sender_i], sender_bit)

action receive_ack:
    b := ack_channel.receive()
    if b = sender_bit:
        sender_i := sender_i + 1
        sender_bit := !sender_bit
```

```
process receiver

action receive_data:
  d, b := data_channel.receive()
  if b = receiver_bit:
      receiver_array[receiver_i] := d
      receiver_i := receiver_i + 1
      receiver_bit := !receiver_bit

action send_ack:
    if receiver_i > 0:
        ack_channel.send(!receiver_bit)
```

Figure 7.16: The alternating bit protocol for transition of messages using lossy first-in-first-out (FIFO) channels. The protocol comprises of two processes, `sender` and `receiver`, and two lossy FIFO channels, `data` and `ack`.

present a more powerful reduction that relies on a user-provided *nesting structure* that captures such arguments. Chapter 8 presents a more advanced solution to this issue, based on *temporal prophecy*, that subsumes the technique developed in this section. However, the technique developed here is simpler and provides useful insight.

We start by presenting the example of the alternating bit protocol for motivation and intuition. We show that for this protocol, the reduction presented in Section 7.3 is incomplete, i.e., it results in an unsafe transition system. We then present the more powerful reduction based on a nesting structure.

7.4.1 Alternating Bit Protocol

The alternating bit protocol (ABP) is a classic communication algorithm for transition of messages using lossy first-in-first-out (FIFO) channels. The protocol involves a sender and a receiver, which operate according to the code in Figure 7.16. The protocol uses two channels, the `data` channel from the sender to the receiver, and the `ack` channel, from the receiver to the sender. The sender and the receiver each have a state bit, initialized to 0, which acts as a "sequence number". We assume that the sender initially has an (infinite) array `sender_array` filled with values that we wish to transmit to the receiver. The sender keeps sending the first value with a "sequence number bit" of 0. When the receiver receives this value, it will store it in `receiver_array`, flip its bit, and start sending an acknowledgment message for sequence bit 0. When the sender receives this, it will also flip its bit and start sending the second value from the array with sequence bit 1. The sender and receiver realize when they should move on to the next value by comparing their bit to the sequence bit of the incoming messages. Assuming the array is infinite, the protocol continues to send values and the receiver keeps receiving them.

For this protocol, we wish to prove the following progress property, asserting that every element of the sender's data array is eventually transferred to the receiver:

$$\forall i.\ \Diamond\ receiver\_array[i] = sender\_array[i]$$

The protocol satisfies the above property under suitable fairness assumptions. For simplicity of the presentation, in this section we take the fairness assumptions to be that the 4 actions `send_data`, `send_ack`, `receive_ack`, and `receive_data` are called infinitely often. The more realistic fairness assumptions are explained in Section 7.6.

As explained in Section 7.2.3, verifying the progress property under the fairness assumptions is equivalent to proving termination of the fair transition system obtained by the product of the system with a "monitor" for the (Skolemized) negation of the specification:

$$\left(\bigwedge_{A \in \mathcal{A}} \Box \Diamond\ A \text{ is called}\right) \wedge \Box\ receiver\_array[i_0] \neq sender\_array[i_0]$$

where $\mathcal{A} = \{\texttt{send\_data}, \texttt{send\_ack}, \texttt{receive\_ack}, \texttt{receive\_data}\}$.

7.4.2 Inadequacy of the Fair Termination to Safety Reduction for ABP

The ABP protocol satisfies its specification, which ensures that the above fair transition system has no fair traces. Nevertheless, applying the liveness-to-safety reduction of Section 7.3 results in an unsafe transition system. This is already the case for the parametric reduction described in Section 7.3.1 with any α and μ, due to the following lemma:

Lemma 7.17. *Let $(S^{ABP}, S_0^{ABP}, R^{ABP}, \mathcal{F}^{ABP})$ be the fair transition system whose fair termination characterizes the correctness of ABP. The (α, μ)-acyclicity condition does not hold for any dynamic finite abstraction function α and fairness selection function μ.*

Proof. First, we observe that the (α, μ)-acyclicity condition is monotone in μ. Namely, the more fairness constraints μ selects, the "easier" it is for the acyclicity condition to hold. Since in this case $\mathcal{F}^{\text{ABP}}$ is finite (it contains 4 constraints, corresponding to the four actions in $\mathcal{A}$), we restrict ourselves to the case of $\mu = \lambda s.\ \mathcal{F}^{\text{ABP}}$, and show that no suitable α exists. (As explained above, if the condition holds for some α and μ, it will also hold with this μ.)

We prove the lemma by providing a family of execution prefixes of the form $w_f \cdot w_n$ for $n \in \mathbb{N}$, such that w_f is fair (i.e., visits all fairness constraints), and w_n contains n fair segments (i.e., n disjoint segments, each of them visits all the fairness constraints). Now, consider any dynamic finite abstraction function α and assume that it satisfies the acyclicity

condition. Let s_f be the state at the end of w_f. The range of $\alpha(s_f)$ must be finite, and let N denote its cardinality. Now, consider the execution prefix $w_f \cdot w_{N+2} = s_0, \ldots, s_m$. Let $k_f < k_1 < \ldots < k_{N+1}$ be such that k_f is the index of the last state in w_f (so $[0, k_f]$ is fair, and $s_{k_f} = s_f$), and such that for every $1 \leq i \leq N$, the segment $[k_i, k_{i+1}]$ is fair. Now, by the acyclicity condition $\alpha(s_f)(s_{k_i}) \neq \alpha(s_f)(s_{k_j})$ for every $1 \leq i < j \leq N+1$. However, the cardinality of the range of $\alpha(s_f)$ is N, so by the pigeonhole principle we reached a contradiction.

To see that the alternating bit protocol contains such a family of execution prefixes, take:

$$
\begin{aligned}
w_f &= (\texttt{send\_data}) \cdot (\texttt{receive\_data}) \cdot (\texttt{send\_ack}) \cdot (\texttt{receive\_ack}) \\
w_n &= y_n \cdot z_n \text{ where:} \\
y_n &= (\texttt{send\_data})^n \cdot (\texttt{send\_ack})^n \cdot (\texttt{receive\_data}) \cdot (\texttt{receive\_ack}) \\
z_n &= ((\texttt{send\_data}) \cdot (\texttt{receive\_data}) \cdot (\texttt{send\_ack}) \cdot (\texttt{receive\_ack}))^{n-1}
\end{aligned}
$$

First, note that w_f is fair, y_n is fair, and z_n contains $n-1$ fair segments, so w_n indeed contains n fair segments. Let us consider the values transferred in such an execution prefix. After w_f, the first value of the array is transferred to the receiver. After y_n, the second value is transferred, but the data channel contains $n-1$ copies of the second value, and the acknowledgment channel contains $n-1$ acknowledgments for the first value. After z_n, these $n-1$ copies are all received, but the third value of the sender array has not yet been transferred to the receiver. Therefore, if we take $i_0 = 2$ (the index of the third value), then these are indeed execution prefixes of $(S^{\text{ABP}}, S_0^{\text{ABP}}, R^{\text{ABP}}, \mathcal{F}^{\text{ABP}})$. □

The key idea of the above proof is that after the freeze point k_f, any given α actually bounds the number of fair segments that the system can take. However, as we saw, for the alternating bit protocol there can be no such bound: there is no bound on the number of (fair) steps that are needed from the time a message is sent for index i to the time its acknowledgment is received.

However, once a data message for some value is sent, we *can* give a bound on how many fair segments can happen before it is received — the bound is the amount of messages ahead of it in the channel (note that even if it is dropped, it will be re-transmitted). A similar argument holds for acknowledgment messages — once an acknowledgment message is sent, we can bound the number of fair segments before it (or a copy of it) is received. Intuitively, this gives the alternating bit protocol a flavor of a nested loop. The outer loop iterates through the array, each iteration of it corresponds to a new value being transmitted. The

inner loop keeps retransmitting data and acknowledgment messages until they are received.

This intuition leads to a more general liveness-to-safety reduction, which can prove the liveness of the alternating bit protocol. The motivating idea is to split the transitions of the transition system into several nested levels, and to prove the fair termination of every level separately, while assuming the inner levels terminate. For the alternating bit protocol, the inner level will consist of transitions in which the sender bit and the receiver bit do not change, and the outer level will consist of transitions which change those bits.

Next, we formalize this nested reduction. We later return to the alternating bit protocol and explain its proof.

7.4.3 Reduction with Nesting Structure

Fix a fair transition system $(S, S_0, R, \mathcal{F})$. In order to support termination proofs of systems such as the alternating bit protocol, we introduce a nesting structure.

Definition 7.18 (Nesting Structure)**.** For a fair transition system $(S, S_0, R, \mathcal{F})$, a *nesting structure* with n levels is $\bar{\eta} = \langle \eta^0, \ldots, \eta^{n-1} \rangle$ such that for every $0 \leq i < n$, $\eta^i \subseteq S \times S$, $\eta^0 = R$, and for every $0 \leq i < n-1$, $\eta^{i+1} \subseteq \eta^i$. Given a nesting structure, and a trace $\pi = s_0, s_1, \ldots$, we define the *level* of position k by $\mathit{level}_\pi(k) = \max\{i \mid (s_k, s_{k+1}) \in \eta^i\}$.

To prove that $(S, S_0, R, \mathcal{F})$ has no fair traces, we require a nesting structure $\bar{\eta}$, a dynamic finite abstraction function α and a dynamic fairness selection function μ such that the following condition holds:

Definition 7.19 (Nested Acyclicity Condition)**.** The $(\bar{\eta}, \alpha, \mu)$-*nested acyclicity condition* requires that for any trace $\pi = s_0, s_1, \ldots$, for any nesting level $0 \leq i < n$ and for any $k_e \leq k_f < k_1 < k_2$ such that: (i) $\forall k_e \leq k < k_2$. $\mathit{level}_\pi(k) \geq i$, and (ii) $\mathit{level}_\pi(k_1) = \mathit{level}_\pi(k_2) = i$, and (iii) the segments $[k_e, k_f]$ and $[k_1, k_2]$ are μ-fair, we have $\alpha(s_{k_f})(s_{k_1}) \neq \alpha(s_{k_f})(s_{k_2})$.

By augmenting a transition system with a nesting structure, we decompose the fair termination proof to a proof for each level, assuming the subsequent levels terminate. This is analogous to proving that a program with nested loops terminates by proving that each loop terminates under the assumption that the inner ones terminate.

Lemma 7.20. *Let $(S, S_0, R, \mathcal{F})$ and $\bar{\eta}$ and α and μ be s.t. the above conditions are satisfied (Definitions 7.6, 7.7, 7.18 and 7.19), then $(S, S_0, R, \mathcal{F})$ has no fair traces.*

Proof. Assume to the contrary that $\pi = (s_k)_{k=0}^{\infty}$ is a fair trace of $(S, S_0, R, \mathcal{F})$. Consider the sequence $(\mathit{level}_\pi(k))_{k=0}^{\infty}$, and let i_m be the minimal level that appears infinitely often

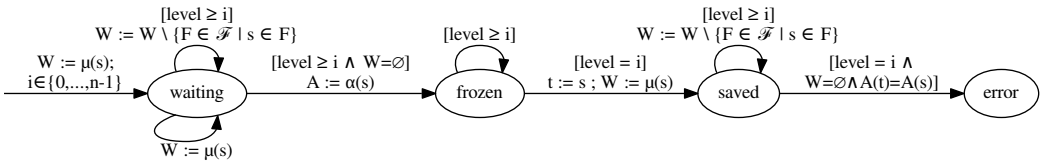


Figure 7.21: Monitor that checks the $(\bar{\eta}, \alpha, \mu)$-acyclicity condition. The monitor extends the monitor depicted in Figure 7.5 to also track the nesting level. It uses an auxiliary state $0 \leq i < n$, which is initialized non-deterministically. The variable *level* denotes the level of the current position of the monitored transition system, computed based on $\bar{\eta}$. The monitor non-deterministically selects when to fix k_e, the entry position to level i, and from that point on behaves similarly to the monitor in Figure 7.5, except that it also makes sure that the level always remains at least i, and that the level of the points k_1 and k_2 where $\alpha(s_{k_f})(s_{k_1}) = \alpha(s_{k_f})(s_{k_2})$ is exactly i. The error state is reachable in the product of the monitor and the monitored transition system if and only if the $(\bar{\eta}, \alpha, \mu)$-acyclicity condition is violated.

in this sequence. We can now choose a subsequence $k_0 < k_1 < \ldots$ s.t. $level_{k_i} = i_m$ and $\forall k_0 \leq k.\ i_m \leq level_k$. Let j_f be the first index such that the $[k_0, k_{j_f}]$ segment is μ-fair. Consider the sequence $\left(\alpha(s_{k_{j_f}})(s_{k_j})\right)_{j=0}^{\infty}$. Since the range of $\alpha(s_{k_{j_f}})$ is finite, there must be an infinite subsequence which is a constant $x \in X$. Let $j_1 \geq j_f$ be the index of the first occurrence of x after j_f. Due to fairness, there must exist $j_2 > j_1$ such that the $[k_{j_1}, k_{j_2}]$ segment is μ-fair. In particular, this segment can be extended such that $\alpha(s_{k_f})(s_{k_{j_2}}) = x$ (since x repeats infinitely often), i.e., $\alpha(s_{k_f})(s_{k_{j_1}}) = \alpha(s_{k_f})(s_{k_{j_2}})$. Now, for $k_e = k_0$, $k_f = k_{j_f}$, $k_1 = k_{j_1}$ and $k_2 = k_{j_2}$, this contradicts the nested acyclicity condition. □

A monitor for the nested acyclicity condition Similarly to the non-nested acyclicity condition, we depict in Figure 7.21 a monitor, parameterized by α, μ and $\bar{\eta}$, that tracks violations of the $(\bar{\eta}, \alpha, \mu)$-acyclicity condition. The key difference compared to the monitor of the non-nested condition is the tracking of the nesting level. To identify violations of the condition in every nesting level, the monitor non-deterministically selects a level $0 \leq i < n$. It then uses the self edge in state "waiting" to non-deterministically select the "entry" point k_e to level i after which an abstract fair cycle in level i will be detected. From this point on, it makes sure that the level remains at least i, and that the states that close an abstract cycle are both encountered at level exactly i. If this happens, a violation is detected and the monitor moves to state "error".

When using the monitor from Figure 7.21 instead of the one given in Figure 7.15, Lemma 7.10 extends to formalize the soundness of the reduction of fair termination to safety verification for every α, μ, and nesting structure $\bar{\eta}$.

Realization in first-order logic To realize the reduction using a nesting structure in first-order logic, we let the user specify the nesting structure $\bar{\eta} = \langle \eta^0, \ldots, \eta^{n-1} \rangle$ by providing $n-1$ two vocabulary formulas $\varphi_\eta^1, \ldots, \varphi_\eta^{n-1}$ (i.e., formulas over $\Sigma \cup \Sigma'$), and then define:

$$\eta^i = \left\{ (s, s') \in S \times S \mid (s, s') \models \left(\tau \wedge \bigwedge\nolimits_{1 \leq j \leq i} \varphi_\eta^j \right) \right\}$$

Note that this satisfies $\eta^0 = R$, and for every $0 \leq i < n-1$, $\eta^{i+1} \subseteq \eta^i$. With this definition, it is straightforward to construct a first-order transition system that implements the monitor of Figure 7.21. We also note that for the degenerate case of $n = 1$, i.e., a single level, this construction is identical to that of Section 7.3.

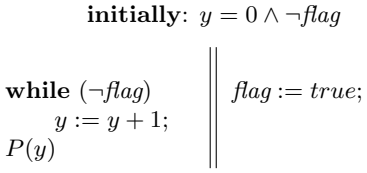
Nesting structure for ABP Finally, we demonstrate the use of a nesting structure on the alternating bit protocol. For the alternating bit protocol, we need 2 levels, which means we specify $\bar{\eta}$ by providing the following formula: $\varphi_\eta^1 = (\mathit{sender\_bit}' \leftrightarrow \mathit{sender\_bit}) \wedge (\mathit{receiver\_bit}' \leftrightarrow \mathit{receiver\_bit})$. This lets the "inner loop" consist of the transitions that do not change the sender or the receiver's bit, and effectively lets the proof of the "outer loop" assume that in any fair trace, the bits change infinitely often. Using this nesting structure, the resulting transition system is safe, and we are able to verify its safety using an inductive invariant in first-order logic (actually, in EPR). More interesting details about this example appear in Section 7.6.1.

7.5 Limitations of Our Reduction in First-Order Logic

Incompleteness of first-order inductive invariant for proving safety One kind of incompleteness that affects the verification process occurs after our liveness-to-safety reduction. It could be the case that the transition system produced by our reduction is safe, but its safety cannot be proven by any inductive invariant expressible in first-order logic. In such cases, one can employ the known methods of adding ghost code and auxiliary predicates (to the resulting system) to express the inductive invariant. However, even with these methods, there is no complete proof system for safety of first-order transition systems, as verifying their safety is in general undecidable.

For the rest of this section, we discuss incompleteness of the reduction itself, i.e., cases where the resulting transition system is unsafe.

Incompleteness of the reduction in first-order logic While sound, our reduction is incomplete. As demonstrated in Section 7.4.2 for the non-nested reduction, some incomplete-



$P_1(y) :=$ **while** $(y > 0)$
$\quad y := y - 1;$
$P_2(y) := y := y + 2;$
$\quad$ **while** $(y > 0)$
$\quad\quad y := y - 1;$
$P_3(y) :=$ compute $Ackermann(y, y)$

Figure 7.22: Program that non-deterministically chooses a natural number y and performs a computation $P(y)$ depending on y. The different programs P_1, P_2, P_3 are used to demonstrate the limits of our verification technique.

ness is incurred already for the parametric reduction, i.e., for any choice of α and μ. That is, there are some fair transition systems with no infinite fair traces, such that for any α and μ, the reduction results in an unsafe transition system. However, we consider the more interesting incompleteness incurred by our uniform α and μ in first-order logic, that is, the use of the footprint and structure projection. In this section, we shed some light on this incompleteness using examples in which it manifests. However, we start by a simple example for which the reduction is successful, to illustrate the small differences that can cause it to fail.

The AnyY program Consider the program in Figure 7.22 where $P(y)$ is instantiated with $P_1(y)$. The program contains two threads, and we assume they are scheduled fairly. Consider the transition system that results from applying the reduction of Section 7.3.2 to this program. In this transition system, the freeze point will be after the flag is set to true (since we wait for both threads to be scheduled). At that point, the footprint will contain all elements seen so far, which are all elements smaller than or equal to y. By projecting to this finite set, the abstraction can distinguish between all states of the remaining execution. Therefore, no abstract fair cycle is present, and the transition system is safe.

The AnyY+2 program Now, consider the same program in Figure 7.22 where $P(y)$ is instantiated with $P_2(y)$. If we use the footprint to define α as before, the resulting abstraction will not be able to distinguish between the values $y_f + 1$ and $y_f + 2$, where y_f is the value of y at the freeze point, since both these values will not be in the relation d. As a result, the obtained transition system will not be safe. This happens despite the fact that with the parametric version of the reduction of Section 7.3, we can actually find a suitable α such that the resulting transition system is safe (for μ we simply take both fairness constraints, as there are only two). Take $\alpha = \lambda s.\, \lambda s'.\, \min(s(y) + 2, s'(y))$, where $s(y)$ denotes the value of y at state s. Thus, $\alpha(s)$ abstracts states into the finite set $\{0, \ldots, s(y) + 2\}$. This abstraction

distinguishes between all states of the remaining execution, so the reduction results in a safe transition system.

While the first-order reduction of Section 7.3.2 fails for this program, the first-order reduction of Section 7.4 can succeed, if we use a 2-level nesting structure that puts the while loop in $P_2(y)$ into a separate level.

The AnyAck program As seen in the proof of Lemma 7.17, for a program with a finite set of fairness constraints, the (α, μ)-acyclicity condition requires that from any reachable freeze point s, the system cannot take more than N fair segments after the freeze point, where N is the cardinality of $\alpha(s)$. In first order logic, $N = O(exp(poly(k)))$, where k is the length of the path from the initial state to the freeze point. This is because the size of the footprint d is linear in k, and the number of different projections of structures from a fixed vocabulary to a set of n elements is $O(exp(poly(n)))$.

Applying this argument to programs of the form of Figure 7.22, we obtain that the reduction of Section 7.3.2 must fail if $P(y)$ takes more than exponential time in y. By using a nesting structure and the reduction of Section 7.4, this limitation is lifted. However, one can still show that the number of steps that $P(y)$ may do, can be computed by a primitive recursive function over y. This means that if we let $P(y)$ be a program that computes the Ackermann function of y, no nesting structure can make the reduction in first-order logic to work. This is in sharp contrast to the parametric reduction, which can prove programs of this form for any terminating $P(y)$ without a nesting structure at all, simply by letting $\alpha(s)$ contain as many elements as the number of steps $P(s(y))$ takes to terminate.

Summary of limitations In Section 7.4.2 we already saw an example where the parametric reduction without nesting function fails, but adding a nesting structure lets even the first-order reduction succeed. In this section, we saw examples for which the parametric reduction succeeds without a nesting structure (for a suitable α), but the first order reduction either requires a nesting structure (AnyY+2), or even fails for any nesting structure (AnyAck). This shows that the theoretical characterization of the proof theoretic power of our formalism is intriguing, and we consider it an attractive direction for future investigation. However, from the practical perspective, when considering distributed protocols (that do not ordinarily lead to non-primitive recursion), we expect our reduction in first-order using a nesting structure to provide a powerful enough proof system. This is already demonstrated by the challenging examples we considered in this work.

7.6 Evaluation

To evaluate the applicability of the presented approach, we used it to verify liveness properties of several challenging protocols. For safety verification, we used the Ivy deductive verification system [171, 186], which uses the Z3 theorem prover [61] to discharge verification conditions. For each example, we manually applied our reduction to obtain a safety verification problem[3]. We then used Ivy to interactively find an inductive invariant that proves safety of the resulting system, and to automatically check the resulting verification conditions.

For all examples except the TLB shootdown, the resulting verification conditions are in the EPR decidable fragment. However, even for the TLB shootdown, Z3 was able to verify the inductive invariant in a few minutes. Figure 7.23 lists the examples, along with the run times for checking verification conditions. The artifact of [189] contains the Ivy files for all described examples. The artifact is available at https://www.cs.tau.ac.il/~odedp/reducing-liveness-to-safety-in-first-order-logic/, or alternatively at https://dl.acm.org/citation.cfm?id=3158114&picked=formats.

Next we discuss the interesting features of each example (Section 7.6.1), and the user experience and effort required in the verification process (Section 7.6.2), which is an important aspect in evaluating the practicality of our approach.

7.6.1 Examples

Ticket Protocol The example of the ticket protocol has been discussed in detail throughout this chapter. We note that in order to have the verification conditions in EPR, we modeled the local variable m using a relation, rather than a function. This allows us to use a $\forall \mathsf{ticket}.\ \exists \mathsf{thread}$ quantifier alternation which is needed in the inductive invariant (see Section 7.3.3), without breaking stratification. We also emphasize that our proof shows the non-starvation even for the case of unbounded-parallelism, i.e., unbounded dynamic thread creation.

Alternating Bit Protocol The alternating bit protocol was described in Section 7.4.1. Our model and proof are naturally performed in EPR. We model the FIFO channels using dynamic totally ordered sets. Section 7.4.1 presented simplified fairness constraints. For the evaluation, we used the standard fairness constraints for each channel: if messages are infinitely often sent, then messages are infinitely often received. The proof is done using a

[3] In Chapter 8 we develop a deeper integration of the liveness verification technique into Ivy, where there is no need to manually apply the reduction, and the notion of temporal prophecy introduced there provides a clean syntax that allows the user to specify invariants directly for the system that results from the reduction.

Protocol	$< \infty$	∞	$\lvert\bar{\eta}\rvert$	$\mathbf{C}_s$	$\mathbf{C}_e$	$\mathbf{C}_m$	VC	t [sec]
Ticket Protocol	–	threads, tickets	1	15	3	19	EPR	3.7
Alternating Bit Protocol	$\{i : \mathsf{index} \mid i < i_0\}$	indices, values, messages	2	13	2	20	EPR	6.9
TLB Shootdown	–	processors, pagemaps, entries	3	22	9	60	FO	219
Paxos	nodes	ballots, values	1	9*	11	22	EPR	8.4
Multi-Paxos	nodes	ballots, values, instances	1	10*	13	32	EPR	9.0
Stoppable Paxos	nodes	ballots, values, instances	1	14*	14	34	EPR	9.9

Figure 7.23: Protocols for which we verified liveness. For each protocol, $< \infty$ reports what is assumed to be finite (but unbounded), and ∞ reports what is allowed to be truly infinite. $|\bar{\eta}|$ reports the number of nesting levels used in the proof. $\mathbf{C}_s$, $\mathbf{C}_e$, and $\mathbf{C}_m$ list the number of conjectures used in the inductive invariant (which provides some measure of user effort, see Section 7.6.2), split into: conjectures used to prove safety of the original protocol ($\mathbf{C}_s$), conjectures added for the liveness proof that express additional properties of the original protocol ($\mathbf{C}_e$), and conjectures that prove the safety of the transition system resulting from the liveness-to-safety reduction by relating the state of the protocol and the state of the monitor ($\mathbf{C}_m$). VC mentions if the resulting verification conditions are in EPR or FO (general first-order logic). Finally, $\mathbf{t}$ reports the run time (in seconds) for checking the verification conditions using Ivy and Z3. The experiments were performed on a laptop running 64-bit Linux, with a Core-i7 1.8 GHz CPU. Z3 version 4.5.0 was used, along with the latest version of Ivy at the time (commit `7ce6738`). Z3 uses heuristics which employ randomness. Therefore, each experiment was repeated 10 times using random seeds, and we report the mean.

* For the Paxos examples, not all conjectures in $\mathbf{C}_s$ were used in the liveness proof, in order to avoid quantifier alternations that would result in verification conditions outside of EPR. Nevertheless, $\mathbf{C}_s$ counts all conjectures needed to prove safety, in order to compare the difficulty of proving safety and the difficulty of proving liveness.

nesting structure with 2 levels, as described in Section 7.4.

Another interesting feature of this example is that we must assume there are only finitely many indices smaller than i_0 (the Skolem constant from the negation of $\forall i. \Diamond\, receiver\_array[i] = sender\_array[i]$). Indeed, for a domain in which the indices are a total order which includes unreachable elements, the system does not actually satisfy the progress property for every i_0. The assumption that i_0 is reachable is cleanly expressed in our formalism by letting the initial condition for d include all indices smaller than i_0. A possible alternative to this would be to explicitly model a concurrent action that fills the sender array with values, and rephrase the progress property such that for every array entry, once it is filled at the sender it must eventually be copied to the receiver. This will actually have the same effect, as it will make sure that by the freeze point, all elements smaller than i_0 are included in d.

We note that the alternating bit protocol was also considered by [8]. The liveness property they verify is that the sender and the receiver change their bits infinitely often. Thus, comparing with our proof, they only verified that the "inner-loop" terminates, while we prove the more natural specification that every array entry is eventually transmitted.

TLB Shootdown The TLB shootdown algorithm [29] is used in the Mach operating system. Modern processors use page tables to translate from virtual to physical memory. These page tables are cached in the processors in the Translation Look-aside Buffer (TLB). When some processor changes the page table, it interrupts all other processors currently using the page table and waits for them to receive the interrupt before changing the entries. In [106], an abstracted version of this algorithm was formally verified, after adding one critical atomic region to the code that prevents an error path. They only showed safety of the algorithm, and to the best of the author's knowledge, the work presented here is the first to mechanically prove its liveness property.

The algorithm itself runs in four phases: (1) the initiator interrupts all processors using a page table, (2) the interrupted processors set a flag that they are deactivated, (3) when every processor set the flag, the initiator changes the page table and finishes, (4) the responders can then continue and flush their TLB. The algorithm is further complicated by the fact that a processor can non-deterministically choose to act as initiator in which case it doesn't respond to interrupts.

We show that each processor will either infinitely often run through the main loop or infinitely often trough the responder loop, which shows that both the initiator and the loop body of the responder terminate. Similarly to the ticket protocol, we implemented the lock

operation as spin-locks to expose dead-locks as non-terminating runs. We also added strong fairness assumptions for the page table lock; otherwise, a process can be blocked indefinitely when waiting for the lock.

We took the code from [106] and translated it into Ivy. The resulting transition system has 24 program locations. To apply our reduction, we use a nesting structure with 3 levels. Interestingly, our proof is sound even in the case where processors are added dynamically and the number of processors is unbounded.

Paxos, Multi-Paxos, and Stoppable Paxos As explained in Chapter 4, the Paxos family of protocols is widely used in practice to build fault-tolerant distributed systems, and verifying the safety and liveness of protocols in the family constitutes a current verification challenge [70, 100, 235]. Recall that protocols in the family are variations on the Paxos consensus algorithm [135, 136]. Under certain fairness assumptions, Paxos allows a set of crash-prone nodes in an asynchronous network to solve the consensus problem, i.e., to agree on a common decision taken among values that the nodes propose. Fairness assumptions are unavoidable, as purely asynchronous fault-tolerant distributed consensus is impossible [77]. We prove that Paxos eventually reaches a decision under the standard fairness assumptions as formalized, e.g., in [140]. We assume that there is a node l and a majority of nodes Q such that (i) no action of l or of any node in Q can become forever enabled and never executed, (ii) every message sent between l and the nodes in Q is eventually delivered, and (iii) eventually, no node different from l tries to propose values. For comparison, [138] contains an informal proof of about a page of the same property under similar assumptions.

Practical systems use more complex protocols in the Paxos family such as Multi-Paxos [136], which allows a set of nodes to agree on a growing log (i.e., a sequence of values). Under the Paxos fairness assumptions, we prove that every position in the log is eventually agreed upon.

An important aspect of distributed systems is their dynamic nature: nodes crash and are replaced, and participants may arrive or leave the system dynamically. In such a dynamic environment, it is necessary to be able to reconfigure the set of participants of a protocol. Stoppable Paxos [140, 142], also explained in Section 4.6.4, extends Multi-Paxos with the ability to stop all participating nodes by deciding on a special *stop* command, such that all nodes terminate with the same final log. The set of participants can then be reconfigured, after which log replication can resume using a new incarnation of Stoppable Paxos. Compared to alternative approaches to reconfiguration, Stoppable Paxos has both practical advantages (e.g., it does not limit parallelism in the absence of reconfiguration) and

aesthetic advantages [142]. However, it is one of the most intricate protocol in the Paxos family, exhibiting a dependency between rounds (ballots) and instances not present in other Paxos variants. As admitted by [140], "getting the details right was not easy". Specifically, the liveness of Stoppable Paxos requires a careful argument, since one must show that nodes cannot be stuck in a situation where no command can be decided due to a stop command that is perceived to be choosable, while at the same time the stop command never gets decided either.

We model a single incarnation of Stoppable Paxos (thus the set of participants is fixed) and we prove that, under the Paxos fairness assumptions, for every position i in the log, eventually either a value is agreed upon or Stoppable Paxos stops with a final log of length strictly smaller than i. Our proof is the first mechanically-checked liveness proof of Stoppable Paxos. For comparison, [140] gives an informal but detailed proof of the liveness property of Stoppable Paxos (under the same fairness assumptions we use here) in about 3 pages of temporal-logic reasoning. Compared to this informal proof, proving liveness of Stoppable Paxos with our approach is more succinct and seems less tedious, and of course has the benefit that the proof is mechanically checked.

Our proofs for Paxos protocols are based on the Ivy models presented in Chapter 4, with the addition of the temporal specification in FO-LTL and invariants for proving the safety of the system after the reduction. In all three cases, when applying the liveness-to-safety reduction, we exploit the fact that the set of nodes is finite (albeit unbounded) by initially adding all nodes to the relation d. We also use derived relations to increase the precision of the abstraction, e.g., by projecting away the value component of the relation modeling proposals in order to track whether some value has been proposed despite that value not being in the footprint at the freeze point. The proof requires only a single level.

7.6.2 Discussion of User Experience

Verifying the safety of the transition system resulting from the reduction An important aspect of our approach is the effort required from the user to prove the safety of the reduced system. Our experience has been positive: in all of the examples we considered, we managed to find inductive invariants for the reduced systems with reasonable effort, comparable to the effort required for proving safety of the original system.

Recall that the vocabulary of the reduced system consists of the vocabulary of the original system and the vocabulary of the temporal property and the monitor. Existing invariants (from the safety proof) of the original system can be reused unchanged, and then additional

invariants are needed to prove acyclicity. Thinking of these invariants requires the user to understand the meaning of the new relations. This can be viewed as a different way of thinking about termination, and in our experience, when the user gets used to this way of thinking, finding invariants that prove acyclicity is comparable to finding usual safety invariants (which is indeed a creative task that requires a sophisticated user). During this process, Ivy's feedback in the form of graphically displayed counterexamples to induction proved helpful, in the same way that it helps to find a usual safety invariant.

The invariant for the reduced system of the ticket protocol (Section 7.3.3) gives a good sense of the style (and complexity) of the invariants used to prove acyclicity. The form of this invariant is representative of the other examples we considered: the invariant is composed of parts that assert that a and d are large enough (the abstraction is precise enough), and parts that assert the existence of a difference between the current state and the saved state.

Figure 7.23 lists for each example the number of conjectures needed to prove the safety of the original system ($\mathbf{C}_s$), the number of additional conjectures that only refer to the vocabulary of the original system ($\mathbf{C}_e$), and the number of conjectures that relate symbols of the original system to symbols of the monitor ($\mathbf{C}_m$). Each conjecture is a conjunct in the inductive invariant, and they are the "mental building blocks" of the invariant (most conjectures are expressed in one line). We thus use conjectures as a measure of the difficulty of finding inductive invariants. As the figure shows, the number of conjectures needed to prove liveness is $2-4$ times higher than the number required to prove safety. However, many of these conjectures are quite repetitive, and we found the effort required to prove liveness comparable to the effort required to prove safety.

Understanding that a nesting structure is needed Another interesting point that requires sophistication from the user is the realization that a nesting structure is needed. This is somewhat analogous to the realization that a lexicographic ranking function is needed rather than a simple one in traditional termination proofs.

Our experience and intuition is that a nesting structure is needed when the transition system has an implicit "nested loop" structure. An indication of this is usually the fact that an important object is "missing" from the abstraction, as seen in a counterexample to induction. This intuition is demonstrated in Section 7.4.2 for the alternating bit protocol. While the alternating bit protocol does not syntactically contain a nested loop structure, there is an implicit nested loop present in the traces of the system: the inner loop transmits a single value (by retransmitting data and ack messages), and the outer loop iterates through the array of values. The verification of the TLB protocol contains more examples of the

same flavor, where in the "outer loop" a thread is waiting for another thread to free a lock and in the "inner loop" the other thread checks that all other threads have set a flag.

We note that when attempting to prove such a system without a suitable nesting structure, the user may try to find an inductive invariant for the reduced system (which is actually not safe), and during this process the counterexamples to induction provided by Ivy are helpful in assisting the user to realize that a nesting structure is needed (in a way which is similar to how counterexample to induction can help detect a bug).

7.7 Related Work for Chapter 7

There is an enormous body of literature on automatic or deduction-style temporal reasoning; see, e.g., [67] or [93] and pointers therein. These works share common goals with our work (and some also use first-order logic), but are mostly orthogonal to the topic of our work, which is the reduction of liveness to safety.

[72] prove liveness properties of parameterized systems by a liveness-to-safety reduction, and by applying the method of invisible invariants [197] to prove the resulting safety properties. They handle "bounded response" properties of the form $\Box(q \rightarrow \Diamond r)$, where a global bound exists on the number of rounds between q and r. In the perspective of response properties, our approach can handle cases in which there is no global bound on the number of rounds between q and r. Dynamic abstraction allows the bound to be dynamic, i.e., to be determined at the point where q happens, while a nesting structure can handle cases where no finite (dynamic) bound exists.

Liveness-to-safety reductions based on acyclicity in the style of [27] have been extended in [213] to the settings of regular model checking, push-down systems, and timed automata. A contribution of our work is that we show how to extend acyclicity to the setting of first-order logic transition systems, which is beyond the above mentioned models. A key difference is that for regular model checking, push-down systems and timed automata, a dynamic abstraction (and the notion of a freeze point) is not needed. Instead, these restricted formalisms admit a fixed abstraction which makes cycle detection sound (e.g., since the set of states reachable from any initial state is finite).

In [60], liveness of infinite-state systems is automatically proven by implicit predicate abstraction combined with well-founded relations. While their approach can automatically handle some infinite-state systems, it ultimately relies on a finite-state abstraction for the liveness-to-safety reduction, whereas our approach can handle systems in which no finite abstraction can be used to prove liveness.

Another interesting formalism for temporal verification of infinite-state systems is rewriting logic [172, 174], and the linear temporal logic of rewriting (LTLR) [173]. This formalism, based on term rewriting, is extremely expressive, and can also express unbounded-parallelism. Indeed, our use of free variables in expressing infinitely many fairness constraints is analogous to rewriting logic's notion of localized and parameterized fairness constraints. Techniques for model checking temporal properties of rewrite systems have been developed in [18–20]. Other than the differences in underlying formalism (term rewriting vs. first-order transition systems), these techniques differ from our work in that they use abstraction and simulation (via rewriting logic's narrowing) to obtain a symbolic state space that is finite (or with finitely many states reachable from any initial state). This is in contrast to our notion of dynamic abstraction and freezing. An interesting direction for future work could be to adapt our dynamic abstraction and acyclicity condition to the setting of rewriting logic.

We next discuss the reduction of liveness to safety for general programs which relies on the concept of *transition invariants* from [200, 202], a concept which originates from size-change analysis [24, 145]. Here, the safety property is the validity of a transition invariant. The burden of finding a ranking function for the input program is thus shifted to the burden of finding an inductive invariant for the input program. The shift of burden has proven useful in many approaches to proving program termination and liveness for many classes of programs; see, e.g., [48, 49, 128, 132, 146, 176, 198]. Since the transition invariant is translated from a set of ranking functions, one still has to find ranking functions. For arithmetic programs, this is in general considered a minor task, mostly because the ranking functions can be derived from automatic termination proofs for comparatively small programs of a specific form [199]. However, these methods do not apply directly to systems like distributed protocols with states of a rich structure as considered by our work.

In the approach of *trace abstraction* for proving program termination and general LTL properties in [64, 101], a set of ranking functions (which is constructed for a set of small programs of a specific form, as in the approach discussed above) is here translated to a set of finite-state Büchi automata (and not to a transition invariant). We then need to check the inclusion between Büchi automata (instead of checking the validity of a transition invariant). The inclusion check gets reduced to checking emptiness of a Büchi automaton, and thus to checking repeated reachability in a finite-state graph.

The advertised goal of the work in [73] is a liveness-to-safety reduction for the verification of concurrent programs with an unbounded number of threads (the ticket protocol is given as a motivating example). Following the approach of *trace abstraction* as in [64, 101], a set

of ranking functions is now translated to a class of (infinite-state) Büchi automata (over an infinite alphabet) for which non-emptiness cannot be reduced to repeated reachability. Again, since the approach relies on ranking functions, it is not clear how the approach can be applied directly to systems like distributed protocols handled by our approach.

The works of [8, 55, 229] are related to our work in that it also avoids the constraint-based synthesis of ranking functions, replacing it essentially by new forms of widening in a CTL-style backwards fixpoint iteration. The approaches have not yet been extended to deal with general liveness and fairness and to deal with the data structures typically found in distributed protocols.

In the termination analysis for heap-manipulating programs presented in [160], the use of ranking functions is avoided by a novel form of reasoning over sets of heap elements, sets which are *a priori* known to be finite. However, as we have seen, for liveness of distributed systems the key is to identify when to fix a finite set, a problem which does not exists when considering sequential heap-manipulating programs. A similar problem does arise in verification of concurrent data structures (e.g., [93]), but the dominant approach for proving liveness and termination in this context is to use ranking functions.

The work in [30, 120–122] addresses the problem of verifying liveness by using a restricted language, the formalism of Threshold Automata, for specifying distributed algorithms operating in a partially synchronous communication mode. This restriction allows them to derive decision procedures based on cutoff theorems, both for safety and liveness. The derivation of the bounds on the lengths of counterexamples seems specific to the formalism, and not related to the implicit bounds in our setting. It is not clear whether the formalism can capture the complexity of the distributed protocols considered in our work.

In the context of regular model checking [31], the technique developed in [221, 222] contains an ingredient that bears similarity to dynamic abstraction. In these works, system states are represented by finite words, and the developed technique assumes that the reachability relation is itself regular, and as a result derives decidability of several model checking problems. The analogue of dynamic abstraction in this setting is realized by fixing some length n at a "freeze point" (so n depends on the trace), and from this point forward projecting words to their prefix of length n, obtaining a finite abstraction. Our representation of states as first-order structures (combined with the use of quantifiers in inductive invariants), rather than finite words, provides greater flexibility; and protocols as complex as the ones considered in our evaluation are beyond the scope of the above mentioned technique. More recently, regular model checking techniques have also been

applied to prove liveness properties of probabilistic and random systems [149, 156], and it would be interesting to try to generalize the technique we present here to such systems as well.

To summarize, none of the existing approaches proposes a liveness-to-safety reduction that addresses the problem of verifying liveness for distributed protocols and other systems modeled with first-order logic. In particular, our notion of dynamic finite abstraction and our acyclicity condition differ from existing works that either use ranking functions, or a fixed abstraction (with finitely many states reachable from any initial state). Furthermore, none of the existing approaches has demonstrated its practical potential on verification problems at the scale of the examples considered in our evaluation.

Chapter 8

Temporal Prophecy

This chapter is based on the results published in [190].

In this chapter, we introduce *temporal prophecy* as a proof technique that improves the precision of the liveness-to-safety reduction developed in Chapter 7. Temporal prophecy provides a more powerful alternative to the nesting structure, and also allows to integrate the proof technique naturally into the Ivy deductive verification system. As we shall see, temporal prophecy also leads to robustness of the proof method, which is manifested by a theorem of closure under *first-order reasoning*, with cut elimination as a special case.

There are various techniques in the literature that transform the problem of verifying liveness of a system to the problem of verifying safety of a different system. These transformations (a.k.a. reductions) compose the system with a device that has the known property that some safety condition P implies liveness. The classical example of this is proving termination of a while loop with a ranking function. In this case, the device evaluates a chosen function r on loop entry, where the range of r is a well-founded set. The safety property P is that r decreases at every iteration, which implies that the loop must terminate.

A related reducion, due to Armin Biere [27], applies to finite-state (possibly parameterized) systems. The safety property P is, in effect, that no state occurs twice, from which we can infer termination. In the infinite-state case, this can be generalized using a function f that projects the program state onto a finite set. We can think of this as a ranking that tracks the set of unseen values of f and is ordered by set inclusion. However, the property that no value of f occurs twice is simpler to verify, since the composed device can non-deterministically guess the recurring value. In general, the effectiveness of a liveness-to-safety transformation depends strongly on the difficulty of the resulting safety proof problem.

Other methods can be seen as instances of this general approach. For example, the Terminator tool [51] might be seen as combining the ranking and the finite projection approaches. Another approach by Fang *et al.* applies a collection of ad-hoc devices with known safety-to-liveness properties to prove liveness of parameterized protocols [72]. Our focus here is the reduction presented in Chapter 7, which uses a dynamically chosen finite projection that depends on a finite prefix of the system's execution.

In the case of infinite-state systems, these transformations from liveness verification to safety verification are not precise reductions. That is, while safety implies liveness, a counterexample to the safety property P does not in general imply a counterexample to liveness. For example, in the projection method, a terminating infinite-state system may have runs whose length exceeds the finite range of any chosen projection f, forcing some value to repeat.

In this chapter, we show that the precision of a liveness-to-safety transformation can be usefully increased by the addition of *prophecy variables*. These variables are expressed as first-order LTL formulas. For example, suppose we augment the state of the system with a variable $r_{\Box p}$ that tracks the truth value of the proposition $\Box p$, which is true when p holds in all future states. We can soundly add two constraints to the transition system. To the transition relation, we add $r_{\Box p} \leftrightarrow (p \wedge r_{\Box p}{}')$, where $r_{\Box p}{}'$ denotes the value of the prophecy variable in the post-state. We also add the fairness constraint that $r_{\Box p} \vee \neg p$ holds infinitely often. These constraints are typical of tableau constructions that convert a temporal formula to a symbolic automaton. As we show in this chapter, the additional information they provide refines the trace set of the transformed system, potentially eliminating false counterexamples.

In particular, we will show how to integrate tableau-based prophecy with the liveness-to-safety transformation of Chapter 7. We show that the precision of this transformation is consequently increased. The result is that we can prove properties that otherwise would not be directly provable using the technique, unless a nesting structure is used. Furthermore, temporal prophecy leads to robustness of the proof method, which is manifested by a cut elimination theorem. Temporal prophecy also provides a natural way to integrate the liveness-to-safety transformation into the Ivy deductive verification system.

This chapter makes the following contributions:

- Introduce the notion of temporal prophecy, including prophecy formulas and prophecy witnesses, *via* a first-order LTL tableau construction.
- Show that temporal prophecy increases the proof power (i.e., precision) of the safety-to-liveness reduction of Chapter 7, and further show that the properties provable with

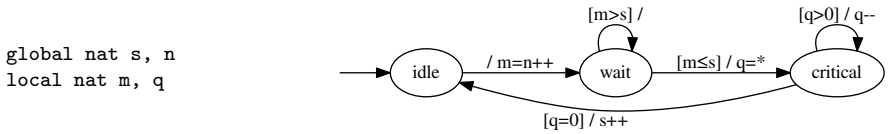


Figure 8.1: The ticket mutual exclusion protocol with task queues. Edges are labeled by condition / action.

temporal prophecy are closed under *first-order reasoning*, with cut elimination as a special case.

- Integrate the liveness-to-safety reduction based on dynamic abstraction and temporal prophecy into the Ivy deductive verification system, deriving the prophecy formulas from an inductive invariant provided by the user (for proving the safety property).

- Demonstrate the effectiveness of the approach on some challenging examples that cannot be handled by the reduction without temporal prophecy (or a nesting structure).

- Demonstrate that prophecy witnesses can eliminate quantifier alternations in the verification conditions generated for the safety problem obtained after the reduction, facilitating decidable reasoning.

8.1 Illustrative Example: Ticket with Task Queues

We motivate and illustrate temporal prophecy using the ticket protocol with task queues, depicted in Figure 8.1. This protocol is a modified version of the ticket protocol used as the running example of Chapter 7 (Figure 7.2). Like the ticket protocol of Chapter 7, this protocol obtains mutual exclusion with non starvation among multiple threads. It may be run by any number of threads, and also allows dynamic spawning of threads. Each thread can be in one of three states: idle, waiting to enter the critical section, or in the critical section. The right to enter the critical section is determined by a ticket number. A global variable n, records the next available ticket, and a global variable s, records the ticket currently being served. Each thread has a local variable m that records the ticket it holds. A thread only enters the critical section when $m \leq s$. Unlike the ticket protocol of Chapter 7, here once a thread enters the critical section, it handles tasks that accumulated in its task queue, and stays in the critical section until its queue is empty (tasks are only added to the queue when the thread is outside the critical section). In Figure 8.1, this is modeled by the task counter q, a thread-local variable which is non-deterministically set when a thread enters the critical section (to account for the unbounded, but finite, number of tasks), and is then decremented

in each step. When $q = 0$ the thread leaves the critical section, and increments s to allow other threads to be served.

The protocol is designed to satisfy the following first-order temporal property:

$$(\forall x.\Box\Diamond \mathit{scheduled}(x)) \rightarrow \forall y.\Box\,(\mathit{wait}(y) \rightarrow \Diamond \mathit{critical}(y))$$

That is, if every process is scheduled infinitely often, then every waiting process eventually enters its critical section. (Note that we encode fairness assumptions as part of the temporal property.)

Insufficiency of the liveness-to-safety reduction While the temporal property is clearly satisfied by the ticket protocol, proving it is challenging for liveness-to-safety transformations. First, due to the unbounded values obtained by the ticket number and the task counter, and also due to dynamic spawning of threads, this example does not belong to the class of parameterized systems [196], where a simple lasso argument is sound (and complete) for proving liveness. Second, while using a finite abstraction can recover soundness, no fixed finite abstraction is precise enough to show the absence of a lasso-shaped counterexample in this example. The reason is that a thread can go to the waiting state (*wait*) with any number of threads waiting "ahead of it in line".

For cases where no finite abstraction is sufficiently precise to prove liveness, we may instead apply the liveness-to-safety transformation of Chapter 7. This transformation relaxes the requirement of proving absence of lassos over a fixed finite abstraction, and instead requires one to prove absence of lassos over a *dynamic* finite abstraction that is only determined after some prefix of the trace (allowing for better precision). Soundness is maintained since the abstraction is still finite. Technically, the technique requires to prove that no *abstract cycle* exists. An abstract cycle (Definition 7.8), which we also call an *abstract lasso*, is a finite execution prefix that (i) visits a *freeze point*, at which a finite projection (abstraction) of the state space is fixed; (ii) the freeze point is followed by two states that are equal in the projection, which we call the *repeating states*; and (iii) all fairness constraints are visited both before the freeze point and between the repeating states.

Unlike fixed finite abstractions, dynamic abstractions allow us to prove that an eventuality holds if there is a finite upper bound on the number of steps required *at the time the eventuality is asserted* (the freeze point). The bound need not be fixed *a priori*. Unfortunately, due to the non-determinism introduced by the task counter q, each of the k threads ahead of t in line could require an unbounded number of steps to leave the critical section, and this

number is not yet determined when t makes its request. As a result, there is an abstract lasso which freezes the abstraction when t makes its request, after which some other thread t_0 enters the critical section and loops, decrementing its task counter q. Since the value of the task counter of t_0 is not captured in the abstraction, the loop does not change the abstract state. This spurious abstract lasso prevents this liveness-to-safety transformation from proving the property.

Temporal prophecy to the rescue The key to fixing this problem by temporal prophecy is to predict the future to the extent that a bound on the steps required for progress is determined at the freeze point. Surprisingly, this is accomplished by the use of one temporal prophecy variable corresponding to the truth value of the following formula:

$$\exists x. \Diamond \Box \mathit{critical}(x).$$

If this formula is initially true, there is some thread t_0 that eventually enters the critical section and stays there. At this point, we can prove it eventually exits (a contradiction) because the number of steps needed for this is bounded by the current task counter of t_0. Operationally, the freeze point is delayed until $\Box \mathit{critical}(x)$ holds at which point t_0's task counter is captured in the finite projection, ruling out an abstract lasso. On the other hand if the prophecy variable is initially false, then all threads are infinitely often out of the critical section. With this fairness constraint, thread t requires only a finite number of steps to be served, determined by the number of threads with lesser tickets. Operationally, the extra fairness constraint extends the lasso loop until the abstract state must change, ruling out an abstract lasso.

Though the liveness-to-safety transformation via dynamic abstraction and abstract lasso detection cannot handle the problem as given, introducing suitable temporal prophecy eliminates the spurious abstract lassos. Some spurious lassos are eliminated by postponing the freeze point, thus refining the finite abstraction, and others are eliminated by additional fairness constraints on the lasso loop. This example is explained in greater detail in Section 8.3.3.

8.2 Tableau for FO-LTL

In order to define temporal prophecy, we present a standard tableau construction for FO-LTL formulas that results in a *fair* first-order transition system (as defined in Section 7.2). This

is a straightforward extension of the classical construction of a Büchi automaton for an LTL formula used in the automata theoretic approach to verification [233, 236]. However, unlike the classical construction, we define the tableau for a set of formulas, not necessarily a single temporal formula.

For an FO-LTL formula φ, we denote by $sub(\varphi)$ the set of subformulas of φ, defined in the usual way. In the sequel, we consider a finite set A of FO-LTL formulas that is closed under subformulas, i.e. for every $\varphi \in A$, $sub(\varphi) \subseteq A$. Note that A may contain formulas with free variables.

Definition 8.2 (Tableau vocabulary)**.** Given a finite set A as above over a first-order vocabulary Σ, the *tableau vocabulary* for A, denoted Σ_A, is obtained from Σ by adding a fresh relation symbol $r_{\Box\varphi}$ of arity k for every formula $\Box\varphi \in A$ with k free variables.

Recall that $\Box$ is the only primitive temporal operator we consider (a similar construction can be done for other operators). The symbols added in Σ_A will be used to "label" states by temporal subformulas that are satisfied by all outgoing fair traces. To translate temporal formulas over Σ to first-order formulas over Σ_A we use the following definition.

Definition 8.3. For an FO-LTL formula $\varphi \in A$ (over Σ), its first-order representation, denoted $\mathrm{FO}[\varphi]$, is a first-order formula over Σ_A, defined inductively, as follows.

$$\begin{aligned}
\mathrm{FO}[\varphi] &= \varphi \quad \text{if } \varphi = r(t_1, \ldots, t_n) \text{ or } \varphi = t_1 = t_2 \\
\mathrm{FO}[\Box\psi(\overline{x})] &= r_{\Box\psi(\overline{x})}(\overline{x}) \\
\mathrm{FO}[\neg\psi] &= \neg\mathrm{FO}[\psi] \\
\mathrm{FO}[\psi_1 \vee \psi_2] &= \mathrm{FO}[\psi_1] \vee \mathrm{FO}[\psi_2] \\
\mathrm{FO}[\exists x.\psi] &= \exists x.\mathrm{FO}[\psi]
\end{aligned}$$

Note that $\mathrm{FO}[\varphi]$ has the same free variables as φ. We can now define the tableau for A as a fair transition system specification in first-order logic.

Definition 8.4 (Tableau transition system)**.** The *tableau transition system* for A is the first-order fair transition system $T_A = (\Sigma_A, \Gamma_A, \iota_A, \tau_A, \Phi_A)$ where Σ_A is defined as above,

and:

$$\Gamma_A = \emptyset$$

$$\iota_A = \mathit{true}$$

$$\tau_A = \bigwedge_{\Box\varphi \in A} \forall \overline{x}.\ (r_{\Box\varphi}(\overline{x}) \leftrightarrow (\mathrm{FO}\,[\varphi(\overline{x})] \wedge r_{\Box\varphi}{}'(\overline{x})))$$

$$\Phi_A = \{\mathrm{FO}\,[\Box\varphi(\overline{x}) \vee \neg\varphi(\overline{x})] \mid \Box\varphi(\overline{x}) \in A\}$$

Note that the original symbols in Σ (and Σ') are not constrained by τ_A, and may change arbitrarily with each transition. However, the $r_{\Box\varphi}$ relations are updated in accordance with the property that $\pi, \sigma \models \Box\varphi$ iff $s_0, \sigma \models \varphi$ and $\pi^1, \sigma \models \Box\varphi$ (where $\pi = s_0, s_1, \ldots$ is a trace and φ is a first-order formula over Σ). The definition of the fairness constraints Φ_A ensures that in a fair trace, an eventuality cannot be postponed forever (note that $\Box\varphi(\overline{x}) \vee \neg\varphi(\overline{x})$, used above, is equivalent to $\Diamond\neg\varphi(\overline{x}) \rightarrow \neg\varphi(\overline{x})$).

The next claims summarize the properties of the tableau. Lemma 8.5 states that the FO-LTL formulas over Σ that hold in the outgoing fair traces of a tableau state correspond to the first-order formulas over Σ_A that hold in the state. Lemma 8.6 states that every sequence of states over Σ has a representative trace in the tableau. Finally, Theorem 8.8 states that a transition system satisfies an FO-LTL formula iff its product with the tableau of the negated formula has no fair traces.

Lemma 8.5. *In a fair trace $\pi = s_0, s_1, \ldots$ of T_A (over Σ_A), for every FO-LTL formula $\psi(\overline{x}) \in A$, for every assignment σ and for every index $i \in \mathbb{N}$, we have that $s_i, \sigma \models \mathrm{FO}\,[\psi(\overline{x})]$ iff $\pi^i, \sigma \models \psi(\overline{x})$.*

Lemma 8.6. *Every infinite sequence of states $\hat{s}_0, \hat{s}_1, \ldots$ over Σ can be extended to a fair trace $\pi = s_0, s_1, \ldots$ of T_A (over Σ_A) s.t. for every $i \in \mathbb{N}$, $s_i|_\Sigma = \hat{s}_i$.*

Definition 8.7 (Product system)**.** Given a transition system $T = (\Sigma, \Gamma, \iota, \tau)$, a closed FO-LTL formula φ over Σ, a finite set A of FO-LTL formulas over Σ closed under subformulas such that $\neg\varphi \in A$, we define the *product system* of T and $\neg\varphi$ over A as the first-order

transition system $T_P = (\Sigma_P, \Gamma_P, \iota_P, \tau_P, \Phi_P)$ given by:

$$\Sigma_P = \Sigma_A$$
$$\Gamma_P = \Gamma$$
$$\iota_P = \iota \wedge \mathrm{FO}\,[\neg\varphi]$$
$$\tau_P = \tau \wedge \tau_A$$
$$\Phi_P = \Phi_A$$

where $T_A = (\Sigma_A, \Gamma_A, \iota_A, \tau_A, \Phi_A)$ is the tableau for A.

Theorem 8.8. *Let T_P be the product system of T and $\neg\varphi$ over A as defined in Definition 8.7. Then $T \models \varphi$ iff T_P has no fair traces.*

Intuitively, the product system augments the states of T with temporal formulas from A, splitting each state into many (often infinitely many) states according to the future behavior of its outgoing traces. Note that Theorem 8.8 holds already when $A = \mathit{sub}(\neg\varphi)$. However, as we will see, taking a larger set A is useful for proving fair termination via the liveness-to-safety reduction.

8.3 Liveness-to-Safety Reduction with Temporal Prophecy

In this section we present our liveness proof approach using temporal prophecy and a liveness-to-safety reduction. As in earlier approaches, our reduction (i) uses a tableau construction to construct a product transition system equipped with fairness constraints such that the latter has no fair traces iff the temporal property holds of the original system, and (ii) defines a safety property over the product transition system such that safety implies that no fair traces exist (note that the opposite direction does not hold).

The gist of our reduction is that we augment the construction of the product transition system with two forms of prophecy detailed in below. We then apply the liveness-to-safety reduction of Chapter 7. We will see that the augmentation with temporal prophecy has the effect of "refining" the product system such that it eliminates spurious abstract cycles.

Our construction exploits both *temporal prophecy formulas* and *prophecy witnesses*, explained below. For the rest of this section we fix a first-order transition system $T = (\Sigma, \Gamma, \iota, \tau)$ and a closed FO-LTL formula φ over Σ, and consider proving that $T \models \varphi$.

8.3.1 Temporal Prophecy Formulas

First, given a set A of (not necessarily closed) FO-LTL formulas closed under subformula that contains $\neg\varphi$, we construct the product system $T_P = (\Sigma_P, \Gamma_P, \iota_P, \tau_P, \Phi_P)$ defined in Definition 8.7. By Theorem 8.8, $T \models \varphi$ iff T_P has no fair traces. Note that classical tableau constructions are defined with $A = sub(\neg\varphi)$, and we allow A to include more formulas. These formulas act as "temporal prophecy variables" in the sense that they split the states of T, according to the future behavior of outgoing traces.

While the construction is already sound with $A = sub(\neg\varphi)$, one of the chief observations of this chapter is that temporal prophecy formulas improve the precision of the liveness-to-safety reduction. The additional formulas in A split the states of T into more states in T_P, and they cause some non-determinism of the future trace to be "pulled backwards" (the outgoing traces contain less non-determinism). For example, if $r_{\Box\varphi}$ holds for some elements in the current state, then φ must continue to hold for these elements in the future of the trace. Similarly, for elements where $r_{\Box\varphi}$ does not hold, there will be some time in the future of the trace where φ would not hold for them.

This is exploited by the liveness-to-safety reduction in three ways, eliminating spurious abstract lassos. First, having more temporal formulas in A increases Φ_P. This refines the definition of a fair segment (Definition 7.7). As per Definition 7.8, this postpones the freeze point, making the abstraction defined by the footprint up to the freeze point more precise. For example, if $r_{\Box\varphi}$ does not hold for a ground formula φ in the initial state, then the freeze point would be postponed until after φ does not hold for the first time. Second, having more temporal formulas in A strengthens the requirement on the segment between the repeating states $s_{k_1} \dots s_{k_2}$ (see Definition 7.8), in a similar way. Third, the additional relations in $\Sigma_P = \Sigma_A$ are part of the state as considered by the reduction, and a difference in these relations (projected to the footprint up to the freeze point) is a valid difference that prevents a cycle. These three ways all play a role in the examples considered in our evaluation.

8.3.2 Temporal Prophecy Witnesses

The notion of an abstract lasso (Definition 7.8) considers a finite abstraction according to the footprint, which depends on the constant symbols in the vocabulary. To further increase the precision of the abstraction, we augment the vocabulary with fresh constant symbols that serve as *prophecy witnesses* for existential temporal properties.

To illustrate the idea, consider the formula $\psi(x) = \Diamond\Box p(x)$ where x is a free variable. If ψ holds for some element, it is useful to include in the vocabulary a constant c that serves as

a witness for $\psi(x)$, and whose interpretation will be taken into account by the abstraction (via the footprint). If ψ holds for some x, the interpretation of c will be taken as such an x. Otherwise, c will be allowed to take any value. In the notation of Hilbert's epsilon calculus, we are defining the constant c to be $\epsilon x\, \psi(x)$.

Temporal prophecy witnesses not only refine the abstraction, they can also be used in the inductive invariant. In some cases this allows to avoid quantifier alternation cycles in the verification conditions, leading to decidability of VC checking. This is demonstrated in the TLB Shootdown example considered in our evaluation, as explained in Section 8.5.2.3.

Formally, given a set $B \subseteq A$, we construct a transition system augmented with witness constants, $T_W = (\Sigma_W, \Gamma_W, \iota_W, \tau_W, \Phi_W)$, as follows. We extend Σ_P to Σ_W by adding fresh constant symbols $c_1^\psi, \ldots, c_n^\psi$ for every formula $\psi(x_1, \ldots, x_n) \in B$. We denote by C the set of new constants, i.e., $\Sigma_W = \Sigma_P \cup C$. We set $\Gamma_W = \Gamma_P$ and $\Phi_W = \Phi_P$. We extend the transition relation formula to keep the new constants unchanged by transitions, i.e. $\tau_W = \tau_P \wedge \bigwedge_{c \in C} c = c'$. We define ι_W by

$$\iota_W = \iota_P \wedge \bigwedge_{\psi(x_1,\ldots,x_n) \in B} \mathrm{FO}\left[(\exists x_1, \ldots, x_n . \psi(x_1, \ldots, x_n)) \rightarrow \psi(c_1, \ldots, c_n)\right]$$

Namely, for every $\psi(x_1, \ldots, x_n) \in B$, the constants $c_1^\psi, \ldots, c_n^\psi$ are required to serve as witnesses for $\psi(x_1, \ldots, x_n)$ in case it holds in the initial state for some elements, and otherwise they may get any interpretation at the initial state, after which their interpretation remains unchanged. Adding these fresh constants and their defining formulas to the initial state is a conservative extension, in the sense that every fair trace of T_P can be extended to a fair trace of T_W, and every fair trace of T_W can be projected to a fair trace of T_P. As such we have the following:

Lemma 8.9. *Let $T_P = (\Sigma_P, \Gamma_P, \iota_P, \tau_P, \Phi_P)$ and $T_W = (\Sigma_W, \Gamma_W, \iota_W, \tau_W, \Phi_W)$ be defined as above. Then, T_P has no fair traces iff T_W has no fair traces.*

We then apply the liveness-to-safety reduction of Chapter 7 (Section 7.3) to T_P, whose vocabulary includes the prophecy witness constant symbols, and therefore these constants will participate in the footprint and increase the precision of the liveness-to-safety reduction. The overall soundness of the liveness-to-safety reduction with temporal prophecy and prophecy witnesses is given by the following theorem.

Theorem 8.10 (Soundness)**.** *Consider a given first-order transition system T and a closed FO-LTL formula φ both over Σ, as well as a given set of temporal prophecy formulas A over*

Σ that contains $\neg\varphi$ and is closed under subformula, and a given set of temporal prophecy witness formulas $B \subseteq A$. Let T_W be defined as above, and let $\widehat{T}$ and $\widehat{P}$ be defined by applying the liveness-to-safety reduction to T_W, i.e., as in Theorem 7.14. If $\widehat{T}$ satisfies the safety property $\widehat{P}$, then $T \models \varphi$.

Proof. Assume $\widehat{T} \models \widehat{P}$. By Theorem 7.14, T_W has no fair traces. Then, by Lemma 8.9, T_P has no fair traces. Finally, by Theorem 8.8, $T \models \varphi$. □

8.3.3 Illustration on the Ticket Protocol with Task Queues

In this section we show in greater detail how prophecy increases the power of the liveness-to-safety reduction. As an illustration we return to the ticket protocol with task queues depicted in Figure 8.1. As we shall see in detail below, in this example the reduction without temporal prophecy fails, but it succeeds when adding prophecy.

To model the ticket example as a first-order transition system, we use a vocabulary with two sorts: thread and number. The first represents threads, and the second represents ticket values and counter values. The vocabulary also includes a static binary relation symbol $\leq$: number, number, with suitable first-order axioms to make it a total order (as in Chapter 3). The state of the system is modeled by unary relations for the program counter: *idle*, *wait*, *critical*, constant symbols of sort number for the global variables n, s, and functions from thread to number for the local variables m, c. The vocabulary also includes a unary relation *scheduled*, which holds the last scheduled thread.

Insufficiency of the reduction without temporal prophecy We illustrate the incompleteness of the reduction without temporal prophecy on the ticket example. Namely, we show that without temporal prophecy, an abstract lasso exists in the obtained transition system.

Suppose we do not augment the tableau with additional prophecy. That is, take A to contain just the subformulas of the temporal property:

$$(\forall x.\Box\Diamond \mathit{scheduled}(x)) \rightarrow \forall y.\Box\,(\mathit{wait}(y) \rightarrow \Diamond \mathit{critical}(y))$$

We observe that the reduction results in a system that contains an abstract lasso. This is regardless of the choice of B. To see this, consider the following trace with two threads denoted t_1 and t_2:

1. Process t_1 enters the wait state, obtaining $m = 0$ and increasing n to 1;

2. Process t_2 enters the wait state, obtaining $m = 1$ and increasing n to 2;
3. Process t_1 enters the critical section, setting its counter to 10;
4. Process t_1 is scheduled, decreasing its counter to 9 and staying in the critical section;
5. Process t_2 is scheduled, and stays in the wait state;
6. Process t_1 is scheduled, decreasing its counter to 8 and staying in the critical section;
7. Process t_2 is scheduled, and stays in the wait state...

The freeze point occurs after step 2, before t_1 enters the critical section (in this point both threads have already been scheduled, making the segment fair). Thus, the footprint contains only thread values $\{t_1, t_2\}$ and number values $0 \dots 2$. The projection to the footprint turns the function encoding the local variable c into a partial function that is undefined for t_1 (since the counter value is greater than 2). Because of this, steps 4 and 5 constitute a lasso, since they satisfy the fairness constraints on $\{t_1, t_2\}$, and the starting and ending states agree on the value of all relations (and functions) using the footprint.

Sufficiency of the reduction with temporal prophecy Next we show that when adding the temporal prophecy formula $\exists x.\Diamond\Box \mathit{critical}(x)$ to the tableau construction, no abstract lasso exists in the augmented transition system, hence the reduction succeeds to prove the property. Formally, in this case, A includes the following two formulas and their subformulas:

$$\neg\left((\exists x.\neg\Box\neg\Box\neg \mathit{scheduled}(x)) \vee \neg\exists x.\neg\Box\left(\neg \mathit{wait}(x) \vee \neg\Box\neg \mathit{critical}(x)\right)\right)$$
$$\exists x.\neg\Box\neg\Box \mathit{critical}(x)$$

And $B = \{\neg\Box\left(\neg \mathit{wait}(x) \vee \neg\Box\neg \mathit{critical}(x)\right),\ \neg\Box\neg\Box \mathit{critical}(x)\}$. Therefore, Σ_W extends the original vocabulary with the following 6 unary relations:

$$r_{\Box\neg \mathit{scheduled}(x)}, r_{\Box\neg\Box\neg \mathit{scheduled}(x)}, r_{\Box\neg \mathit{critical}(x)},$$
$$r_{\Box\neg \mathit{wait}(x)\vee\neg\Box\neg \mathit{critical}(x)}, r_{\Box \mathit{critical}(x)}, r_{\Box\neg\Box \mathit{critical}(x)}$$

as well as two constants for prophecy witnesses: c_1 for $\neg\Box\left(\neg \mathit{wait}(x) \vee \neg\Box\neg \mathit{critical}(x)\right)$, and c_2 for $\neg\Box\neg\Box \mathit{critical}(x)$.

We now explain why there is no abstract lasso. To do this, we show that the tableau construction, combined with the dynamic abstraction and the fair segment requirements,

result in the same reasoning that was presented informally in Section 8.1.

First, observe that from the definition of c_1 and the negation of the liveness property (both assumed by ι_W), for the initial state s_0 we have $s_0 \models \mathrm{FO}\left[\neg\Box\left(\neg \mathit{wait}(c_1) \vee \neg\Box\neg \mathit{critical}(c_1)\right)\right]$. For brevity, denote $p = (\neg \mathit{wait}(c_1) \vee \neg\Box\neg \mathit{critical}(c_1))$, so we have $s_0 \models \mathrm{FO}\left[\neg\Box p\right]$, i.e., $s_0 \models \neg r_{\Box p}$ (where $r_{\Box p}$ denotes $r_{\Box(\neg \mathit{wait}(c_1)\vee\neg\Box\neg \mathit{critical}(c_1))}$). Since c_1 is also in the footprint of the initial state, the fair segment requirement ensures that the freeze point can only happen after encountering a state satisfying: $\mathrm{FO}\left[(\Box p) \vee \neg p\right] \equiv r_{\Box p} \vee \mathrm{FO}\left[\neg p\right]$. Recall that the transition relation of the tableau (τ_A), ensures $(r_{\Box p}) \leftrightarrow (\mathrm{FO}\left[p\right] \wedge r_{\Box p}{}')$. Therefore, on update from a state satisfying $\neg r_{\Box p}$ to a state satisfying $r_{\Box p}$ can only happen if the pre-state satisfies $\mathrm{FO}\left[\neg p\right]$. Therefore, the freeze point must come after encountering a state that satisfies $\mathrm{FO}\left[\neg p\right] \equiv \mathit{wait}(c_1) \wedge r_{\Box\neg \mathit{critical}(c_1)}$. From the freeze point onward, τ_A will ensure both $r_{\Box\neg \mathit{critical}(c_1)}$ and $\neg \mathit{critical}(c_1)$ continue to hold, so c_1 will stay in *wait* (since the protocol does not allow to go from *wait* to anything but *critical*). So, we see that the mechanism of the tableau, combined with the temporal prophecy witness and the fair segment requirement, ensures that the freeze point happens after c_1 makes a request that is never granted. This will ensure that the footprint used for the dynamic abstraction will include all threads ahead of c_1 in line, i.e., those with smaller ticket numbers.

As for c_2, the initial state will either satisfy $\mathrm{FO}\left[\neg\Box\neg\Box \mathit{critical}(c_2)\right]$ or it would satisfy $\mathrm{FO}\left[\neg\exists x.\neg\Box\neg\Box \mathit{critical}(x)\right]$. In the first case, by an argument similar to the one used above for c_1, the freeze point will happen after c_2 enters the critical section and then stays in it. Therefore, the footprint used for the dynamic abstraction will include all numbers smaller than q of c_2 when it enters the critical section[1]. Since c_2 is required to be scheduled between the repeating states (again by the tableau construction and the fair segment requirement), its value for q will be decreased, and this will be visible in the dynamic abstraction. Thus, in this case, an abstract lasso is not possible.

In the second case the initial state satisfies $\mathrm{FO}\left[\neg\exists x.\neg\Box\neg\Box \mathit{critical}(x)\right]$. By a similar argument that combines the tableau with the fair segment requirement for the repeating states, we will obtain that between the repeating states, any thread in the footprint of the first repeating state, must both be scheduled and visit a state outside the critical section. In particular, this includes all threads that are ahead of c_1 in line. This entails a change to the program counter of one of them (the one that had a ticket number equal to the service number at the first repeating state), which will be visible in the abstraction. Thus, an abstract lasso is not possible in this case either.

[1] When modeling natural numbers in first-order logic, the footprint is adjusted to include all numbers lower than any constant (still being a finite set), as mentioned in Section 7.3.2.

8.4 Closure Under First-Order Reasoning

The reduction from temporal verification to safety verification enhanced by temporal prophecy, as summarized in Theorem 8.10, still involves an abstraction, and incurs a loss of precision. That is, for some systems and properties, liveness holds but the safety of the resulting system does not hold, no matter what temporal prophecy is used. (This is unavoidable for a reduction from arbitrary FO-LTL properties to safety properties, as mentioned in Chapter 7.) However, in this section, we show that the set of instances for which the reduction can be made precise (via temporal prophecy) is closed under first-order reasoning. This is unlike the reduction of Chapter 7. It shows that the use of temporal prophecy results in a particular kind of robustness.

We consider a proof system in which the reduction of Section 8.3, i.e., with temporal prophecy, is performed and the resulting safety property is checked by an oracle. That is, for a transition system T and a temporal property φ (a closed FO-LTL formula), we write $T \vdash \varphi$ if there exist finite sets of FO-LTL formulas A and B satisfying the conditions of Theorem 8.10, such that the resulting transition system is safe, i.e., T_W does not contain an abstract lasso.

We now show that the relation $\vdash$ satisfies a powerful closure property.

Theorem 8.11 (Closure under first-order reasoning)**.** *Let T be a transition system, and $\psi, \varphi_1, \ldots, \varphi_n$ be closed FO-LTL formulas, such that* $\mathrm{FO}[\varphi_1 \wedge \ldots \wedge \varphi_n] \models \mathrm{FO}[\psi]$*. If $T \vdash \varphi_i$ for all $1 \leq i \leq n$, then $T \vdash \psi$.*

The condition that $\mathrm{FO}[\varphi_1 \wedge \ldots \wedge \varphi_n] \models \mathrm{FO}[\psi]$ means that $\varphi_1 \wedge \ldots \wedge \varphi_n$ entails ψ when using only first-order reasoning, and treating temporal operators as uninterpreted. The theorem states that provability using the liveness-to-safety reduction with temporal prophecy and prophecy witnesses is closed under such reasoning. Two special cases of Theorem 8.11 given by the following corollaries:

Corollary 8.12 (Modus Ponens)**.** *If T is a transition system and φ and ψ are closed FO-LTL formulas such that $T \vdash \varphi$ and $T \vdash \varphi \rightarrow \psi$, then $T \vdash \psi$.*

Corollary 8.13 (Cut)**.** *If T is a transition system and φ and ψ are closed FO-LTL formulas such that $T \vdash \varphi \rightarrow \psi$ and $T \vdash \neg\varphi \rightarrow \psi$, then $T \vdash \psi$.*

Proof of Theorem 8.11. In the proof we use the notation $T_W(T, \varphi, A, B)$ to denote the transition system constructed for T and φ when using A, B as temporal prophecy formulas. Likewise, we refer to the vocabulary, initial states and transition relation formulas of

the transition system as $\Sigma_W(T, \varphi, A, B)$, $\iota_W(T, \varphi, A, B)$, and $\tau_W(T, \varphi, A, B)$, respectively. Let $(A_1, B_1), \ldots, (A_n, B_n)$ be such that $T_W(T, \varphi_i, A_i, B_i)$ has no abstract lasso, for every $1 \leq i \leq n$. Now, let $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{i=1}^{n} B_i$. We show that $T_W(T, \psi, A, B)$ has no abstract lasso. Assume to the contrary that $s_0, \ldots, s_i, \ldots, s_j, \ldots, s_k, \ldots, s_n$ is an abstract lasso for $T_W(T, \psi, A, B)$. Since $s_0 \models \iota_W(T, \psi, A, B)$, we know that $s_0 \models \neg \mathrm{FO}[\psi]$, and since $\mathrm{FO}[\varphi_1 \wedge \ldots \wedge \varphi_n] \models \mathrm{FO}[\psi]$, there must be some $1 \leq \ell \leq n$ s.t. $s_0 \models \neg \mathrm{FO}[\varphi_\ell]$. Denote $\Sigma' = \Sigma_W(T, \varphi_\ell, A_\ell, B_\ell)$. Now, $s_0|_{\Sigma'}, \ldots, s_i|_{\Sigma'}, \ldots, s_j|_{\Sigma'}, \ldots, s_k|_{\Sigma'}, \ldots, s_n|_{\Sigma'}$ is an abstract lasso of $T_W(T, \varphi_\ell, A_\ell, B_\ell)$, which is a contradiction. To see that, we first simplify the notation and denote $s_m|_{\Sigma'}$ by $\hat{s}_m$. Let $f(s_0, \ldots, s_i)$ denote the footprint of the trace $s_0, \ldots, s_i$, and similarly for $f(\hat{s}_0, \ldots, \hat{s}_i)$. The footprint $f(s_0, \ldots, s_i)$ contains more elements than the footprint $f(\hat{s}_0, \ldots, \hat{s}_i)$, since $\Sigma_W(T, \psi, A, B) \supseteq \Sigma_W(T, \varphi_\ell, A_\ell, B_\ell)$. Therefore, given that $s_j|_{f(s_0,\ldots,s_i)} = s_k|_{f(s_0,\ldots,s_i)}$, we have that $\hat{s}_j|_{f(\hat{s}_0,\ldots,\hat{s}_i)} = \hat{s}_k|_{f(\hat{s}_0,\ldots,\hat{s}_i)}$ as well. Moreover, the fairness constraints in $T_W(T, \varphi_\ell, A_\ell, B_\ell)$, determined by A_ℓ, are a subset of those in $T_W(T, \psi, A, B))$, determined by A, so the segments $[0, i]$ and $[j, k]$ are also fair in $T_W(T, \varphi_\ell, A_\ell, B_\ell)$. □

The proof of Theorem 8.11 sheds more light on the power of using temporal prophecy formulas that are not subformulas of the temporal property to prove. In particular, the theorem does not hold if A is restricted to subformulas of the temporal proof goal.

8.5 Implementation & Evaluation

We have implemented the temporal verification approach described in this chapter and integrated it into the Ivy deductive verification system. We have also evaluated the benefit of temporal prophecy on several challenging examples. Below we report on the scheme of integrating the liveness-to-safety reduction with temporal prophecy into Ivy, report on protocols we verified to evaluate it, and compare it to the nesting structure of Chapter 7.

8.5.1 Integration in Ivy

Our implementation in Ivy allows the user to model the transition system in the Ivy language (which internally translates into a first-order transition system), and express temporal properties directly in FO-LTL. In our implementation, the safety property that results from the liveness-to-safety reduction is proven by a suitable inductive invariant, provided by the user. Ivy internally constructs T_W, and composes it with a suitable monitor for the safety property, i.e., the absence of abstract lasso's in T_W, as described in Theorem 8.10. The user then provides an inductive invariant for T_W composed with the monitor. As explained

in Section 7.3.2, the monitor keeps track of the footprint and the fairness constraints, and non-deterministically selects the freeze point and repeated states of the abstract lasso. The monitor keeps a shadow copy of the "saved state", which is the first of the two repeated states. These are maintained via designated relation symbols (in addition to Σ_W). The user's inductive invariant must then prove that it is impossible for the monitor to detect an abstract lasso.

Mining temporal prophecy from the inductive invariant As presented in previous sections, our liveness-to-safety reduction is parameterized by sets of formulas A and B. In the implementation, these sets are implicit, and are extracted automatically from the inductive invariant provided by the user. The inductive invariant provided by the user contains temporal formulas, and also prophecy witness constants, where every temporal formula $\Box\varphi$ is a shorthand, and is internally rewritten to, $r_{\Box\varphi}$. The set A to be used in the construction is defined by all the temporal subformulas that appear in the inductive invariant (and all their subformulas), and the set B is defined according to the prophecy witness constants that are used in the inductive invariant.

In addition, the user's invariant may refer to the satisfaction of each fairness constraint $\mathrm{FO}\,[\Box\varphi \vee \neg\varphi]$ for $\Box\varphi \in A$, both before the freeze point and between the repeated states, via a convenient syntax provided by Ivy and illustrated in Section 8.5.2.1.

Interacting with Ivy If the user provides an inductive invariant that is not inductive, Ivy presents a graphical counterexample to induction. This guides the user to adjust the inductive invariant, which may also lead to new formulas being added to A or B, if the user adds new temporal formulas or prophecy witnesses to the inductive invariant. In this process, the user's mental image is of a liveness-to-safety reduction where A and B include all (countably many) FO-LTL formulas over the system's vocabulary, so the user is free to use any temporal formula, or prophecy witness for any formula. However, since the user's inductive invariant is a finite formula, the liveness-to-safety reduction needs only to be applied to finite A and B, and the infinite A and B are just a mental model.

8.5.2 Examples

We have used our implementation to prove liveness for several challenging examples, summarized in Figure 8.14. We focused here on examples that were beyond reach for the liveness-to-safety reduction of Chapter 7, unless a nesting structure is used. Our experience shows that with temporal prophecy, the invariants are simpler than with a nesting structure

Protocol	**# A**	**# B**	**# LOC**	**# C**	**FO-LTL**	**t** [sec]
Ticket w/ Task Queues	1	2	90	60	22%	9.4
Alternating Bit Protocol	4	1	143	70	40%	32
TLB Shootdown	6	3	468	102	49%	283

Figure 8.14: Protocols for which we verified liveness using temporal prophecy. For each protocol, **# A** reports the number of temporal prophecy formulas used. **# B** reports the number of prophecy witnesses used. **# LOC** reports the number of lines of code for the system model (without proof) in Ivy's modeling language. **# C** reports the number of conjectures used in the inductive invariant (a typical conjecture is one or few lines). **FO-LTL** reports the fraction of the conjectures that use temporal formulas. Finally, **t** reports the run time (in seconds) for checking the verification conditions using Ivy and Z3. The experiments were performed on a laptop running 64-bit Linux, with a Core-i7 1.8 GHz CPU, using Z3 version 4.6.0.

(for additional comparison with nesting structure see Section 8.5.3). For all examples we considered, the verification conditions are in the EPR decidable fragment, which is supported by Z3. Interestingly, for the TLB shootdown example, the proof obtained in Chapter 7 (using a nesting structure) required non-stratified quantifier alternation, which is eliminated here by the use of temporal prophecy witnesses (see Section 8.5.2.3 below). Due to the decidability of verification conditions, Z3 behaves predictably, and whenever the invariant is not inductive it produces a finite counterexample to induction, which Ivy presents graphically. Our experience shows that the graphical counterexamples provide valuable guidance towards finding an inductive invariant, and also for coming up with temporal prophecy formulas as needed.

Below we provide more detail on each example.

8.5.2.1 Ticket with Task Queues

The ticket with task queues example has been introduced in Section 8.1, and Section 8.3.3 presented more details about its proof with temporal prophecy, using a single temporal prophecy formula and two prophecy witness constants. To give a flavor of what the proof looks like in Ivy, we present a couple of the conjectures that make up the inductive invariant for the resulting system, in Ivy's syntax. In Ivy, the prefix `l2s` indicates symbols that are introduced by the liveness-to-safety reduction.

Some of the conjectures needed for the inductive invariant mention only properties of reachable states of the original system that are important for liveness. An example of this is:

```
forall K. n > K & s <= K -> exists T. m(T) = K & ~idle(T)
```

This conjecture states that for any ticket number between s and n, there is a thread holding

that ticket that is also not in the *idle* state (see Figure 8.1).

Some conjectures are needed to state that the footprint used in the dynamic abstraction contains enough elements. An example of such a conjecture is:

```
l2s_frozen & (globally critical(c2)) -> forall N.  N <= q(c2) -> l2s_a(N)
```

This conjecture states that after the freeze point (indicated by the special symbol `l2s_frozen`), if the prophecy witness `c2` (which is the prophecy witness defined for $\Diamond\Box critical(x)$) is globally in the critical section, then the finite domain of the frozen abstraction (stored in the unary relation `l2s_a`) contains all numbers up to `q(c2)`.

Other conjectures are needed to show that the current state is different from the saved state. One example is:

```
l2s_saved & (globally critical(c2)) & ~($l2s_w X. scheduled(X))(c2) ->
      q(c2) ~= ($l2s_s X. q(X))(c2)
```

The special operator `$l2s_w` lets the user query whether a fairness constraint has been encountered, and `$l2s_s` exposes to the user the saved state (both syntactically λ-like binders). This conjecture states that after we saved a shadow state (indicated by `l2s_saved`), if the prophecy witness `c2` is globally in the critical section, and if we have encountered the fairness constraints associated with $scheduled(x) \vee \Box\neg scheduled(x)$ instantiated for `c2` (which can only happen after `c2` has been scheduled), then the current value `c2` has for `q` is different from the same value in the shadow state.

8.5.2.2 Alternating Bit Protocol

The alternating bit protocol (also presented in Section 7.4.1) is a classic communication algorithm for transition of messages using lossy first-in-first-out (FIFO) channels. The protocol uses two channels: a data channel from the sender to the receiver, and an acknowledgment channel from the receiver to the sender. The sender and the receiver each have a state bit, and messages include a bit that functions as a "sequence number". We assume that the sender has an (infinite) array of values to send, which is filled by some independent process. The liveness property we wish to prove is that every value entered into the sender array is eventually received by the receiver.

The protocol is live under fair scheduling assumptions, as well as standard fairness constraints for the channels: if messages are infinitely often sent, then messages are infinitely often received. This makes the structure of the temporal property more involved. Formally,

the liveness property we prove is:

$$(\Box\Diamond \mathit{sender\_scheduled}) \wedge (\Box\Diamond \mathit{receiver\_scheduled}) \wedge$$
$$((\Box\Diamond \mathit{data\_sent}) \rightarrow (\Box\Diamond \mathit{data\_received})) \wedge ((\Box\Diamond \mathit{ack\_sent}) \rightarrow (\Box\Diamond \mathit{ack\_received})) \rightarrow$$
$$\forall x.\Box(\mathit{sender\_array}(x) \neq \bot \rightarrow \Diamond \mathit{receiver\_array}(x) \neq \bot))$$

As explained in Section 7.4.2, this property cannot be proven without temporal prophecy or a nesting structure. However, it can be proven using 4 temporal prophecy formulas:

$$\{\Diamond\Box\,(\mathit{sender\_bit} = s \wedge \mathit{receiver\_bit} = r) \mid s, r \in \{0, 1\}\}$$

Intuitively, these formulas make a distinction between traces in which the sender and receiver bits eventually become fixed, and traces in which they change infinitely often.

8.5.2.3 TLB Shootdown

The TLB shootdown algorithm [29], also considered in Section 7.6.1, is used (e.g., in the Mach operating system) to maintain consistency of Translation Look-aside Buffers (TLB) across processors. When some processor (dubbed the initiator) changes the page table, it interrupts all other processors currently using the page table (dubbed the responders) and waits for them to receive the interrupt before making changes.

The liveness property we prove is that no processor can become stuck either as an initiator or as a responder (formally, it will respond or initiate infinitely often). This liveness depends on fair scheduling assumptions, as well as strong fairness assumptions for the page table locks used by the protocol.

We use one witness for the process that does not satisfy the liveness property. Another witness is used for a pagemap that is never unlocked, if this exists. A third witness is used for a process that possibly gets stuck while holding the lock blocking the first process.

We use six prophecy formulas to case split on when some process may get stuck. Two of them are used for the two loops in the initiator to distinguish the cases whether the process that hogs the lock gets stuck there. They are of the form $\Diamond\Box \mathit{pc}(c_2) \in \{i_3, \ldots, i_8\}$. Two are used for the two lock instructions to indicate that the first process gets stuck: $\Diamond\Box \mathit{pc}(c_1) = i_2$. And two are used for the second and third witness to indicate whether such a witness exists, e.g., $\Diamond\Box \mathit{plock}(c_3)$. Compared to the proof of Chapter 7, our proof is simpler due to the temporal prophecy, and avoids non-stratified quantifier alternation, resulting in decidable verification conditions.

8.5.3 Comparison With Chapter 7

Comparing Figure 8.14 and Figure 7.23, we see that here we generally use more conjectures. The reason for this is that in the evaluation described in Section 7.6, temporal monitors were constructed manually, and contained some shortcuts. In contrast, the tableau construction (Section 8.2), which is made automatic in this chapter (and implemented in Ivy), results in systems that require slightly more verbose invariants. To illustrate this on the ticket protocol, note that FO $[\forall x. \Box\Diamond scheduled(x)]$ must be included in the inductive invariant to assert that it keeps holding. In contrast, in the evaluation of Section 7.6, $scheduled(x)$ was taken as an apriori fairness constraint, which is a manual optimization of the tableau construction. For the Alternating Bit Protocol, some conjectures related to fairness constraints repeat 4 times, for combinations of sender and receiver bit values. Similarly, in the TLB Shootdown example, further conjectures are needed to track properties of the prophecy formulas and the witness constants.

Comparing temporal prophecy to the nesting structure introduced in Section 7.4, we note that a nesting structure must be defined by the user (via first-order formulas), and has the effect of splitting the transition system into levels (analogous to nested loops) and proving each level separately. Temporal prophecy in contrast is more general, and in particular, any proof that is possible with a nesting structure, is also possible with temporal prophecy (by adding a temporal prophecy formula $\Diamond\Box\delta$ for every nesting level, defined by δ). However, our experience is that coming up with temporal prophecy is usually easier and more natural than coming up with a nesting structure. Moreover, the nesting structure does not admit cut elimination or closure under first-order reasoning, and is therefore less robust.

8.6 Related Work for Chapter 8

Prophecy variables were first introduced in [3], in the context of refinement mappings. There, prophecy variables are required to range over a finite domain to ensure soundness. Our notion of prophecy via first-order temporal formulas and witness constants does not meet this criterion, but is still sound as assured by Theorem 8.10. In [118], LTL formulas are used to define prophecy variables in a way that is similar to ours, but only to show refinement between finite-state processes. We use temporal prophecy defined by FO-LTL formulas in the context of infinite-state systems. Furthermore, we consider a liveness-to-safety reduction (rather than refinement mappings), which can be seen as a proof system for FO-LTL.

One effect of prophecy is to split cases in the proof on some aspect of the future. This

very general idea occurs in various approaches to liveness, particularly in the large body of work on lexicographic or disjunctive rankings for termination [17, 36, 48, 51, 52, 55, 81, 89, 95, 98, 128, 146, 200, 201, 226–228, 230]. In the work of [97], the partitioning of the space of potentially infinite executions is based on the *a priori* decomposition of regular expressions for iterated loop segments. Often the partitioning here amounts to a split according to a fairness condition ("command a is taken infinitely often or it is not"). The partitioning is constructed dynamically (and represented explicitly through a union of Büchi automata) in [101] (for termination), in [64] (for liveness), and in [73] (for liveness of parameterized systems). None of these works uses a temporal tableau construction to partition the space of futures, however.

Here, we use prophecy to, in effect, partially determinize a system by making nondeterministic choices earlier in an execution. This same effect was used for a different purpose in refining an abstraction from LTL to ACTL [50] and checking CTL* properties [53]. The prophecy in this case relates only to the next transition and is not expressed temporally. The method of "temporal case splitting" in [170] can also be seen as a way to introduce prophecy variables to increase the precision of an abstraction, though in that case the reduction was to finite-state liveness, not infinite-state safety. Moreover, it only introduces temporal witnesses.

We have considered only proof methods that reduce liveness to safety (which includes the classical ranking approach for while loops). There are approaches, however, which do *not* reduce liveness to safety. For example, the approaches in [8, 55, 229] are essentially forms of widening in a CTL-style backwards fixpoint iteration. It is not clear to what extent temporal prophecy might be useful in increasing the precision of such abstractions, but it may be an interesting topic for future research.

Chapter 9

Conclusion

The central theme explored in this thesis is using first-order logic, and the EPR decidable fragment, for deductive verification. As we have seen, this seemingly weak and inexpressive fragment can be made surprisingly powerful, by using appropriate encodings and abstractions, and by utilizing a technique for elimination of quantifier alternation cycles when they arise. We have also seen a technique for proving liveness and temporal properties that fits well with the scheme of first-order logic based verification, and does not require ranking functions nor reasoning about arithmetic.

From a practical perspective, this thesis provides a step towards increasing the productivity and practicality of deductive verification using automated theorem provers. The increase in productivity is obtained by using the automated solvers only on a decidable logical fragment, which leads to greater solver stability, and as a result to a more productive and enjoyable verification process. To obtain this, one must pay the cost of engineering the verification such that VCs are always in the decidable fragment. The initial experience presented in this thesis suggests that this is possible in many interesting cases, and that the effort is worthwhile.

The techniques developed in this thesis have been implemented in the Ivy deductive verification system, and successfully applied to verify several protocols, some formally verified for the first time. It is encouraging to note that EPR was powerful enough to verify protocols from the Paxos family including Stoppable Paxos, as well as other protocols such as the Chord protocol and the TLB Shootdown protocol. These protocols are quite challenging to verify, and the verification obtained in this thesis required significantly less effort compared to state-of-the-art techniques.

Aside from this thesis, the author has used Ivy in a verification workshop course in Tel Aviv University. Twenty undergraduate students participated in the workshop, and each

team of 2-3 students verified safety of a classic mutual exclusion protocol. The workshop was encouraging as all the students were able to use Ivy's modeling language to model protocols as transition systems in first-order logic, and were successful in finding comprehensible inductive invariants. While working with Ivy, the students greatly benefited from the fact that Z3 is a practically stable decision procedure for EPR. The finite counterexamples produced by Z3 and displayed by Ivy were key in enabling the students to find correct inductive invariants.

In the setting of using first-order logic for deductive verification, and with the VCs set in a decidable logic, this thesis also explored the question of invariant inference. We have seen several decidability and undecidability results for the invariant inference problem. While these results were developed and presented primarily in a theoretical mindset, the author hopes that some of the ideas presented can also be useful in designing practical heuristics for automated or semi-automated invariant inference. This thesis also presented a technique for finding universally quantified inductive invariants in a cooperative process between the user and well-defined decidable automated queries. While the technique presented here targets universally quantified invariants, the idea of a systematic interactive process for finding inductive invariants is general, and can hopefully be extended to other classes of inductive invariants.

For liveness proofs, we have seen a powerful proof technique that is based on transforming liveness verification to safety verification, by exploiting the representation of states as first-order structures. The transformation involves an abstraction, and we have seen two mechanisms for increasing the precision of the abstraction: a nesting structure that allows to split the proof into nested levels, analogous to nested loops; and the more powerful mechanism of temporal prophecy, which also leads to a cut elimination theorem for the proof method. Temporal prophecy also facilitates verification using EPR, as it allows to eliminate some quantifiers from inductive invariants by replacing them with a special kind of Skolem constants.

While the liveness verification techniques developed in this thesis were presented in the context of verification based on first-order logic, some of the ideas behind them are actually general, and can hopefully be useful in other contexts. The idea of dynamic abstraction, as a technique for proving liveness and temporal properties, does not depend on the use of first-order logic; and the parameterized formulation of dynamic abstraction developed here may be instantiated in other contexts. The idea of temporal prophecy is also general, and using prophecy variables defined by temporal formulas may also benefit other liveness and termination proofs techniques.

Finally, as this thesis developed several techniques for reducing safety and liveness verification of complex protocols to decidable EPR satisfiability queries, an interesting question that remains open is the following: what is it about these protocols that makes it possible to reduce their verification to EPR queries? The techniques developed in this thesis were successfully applied to a variety of protocols (e.g., several Paxos variants, Chord, TLB Shootdown). Furthermore, the techniques' structure is also general and influenced mostly by logical considerations (e.g., handling quantifiers), rather than protocol specific ones. However, it is clear that the developed techniques are not a complete proof system, and not every infinite-state system can be verified in EPR; in particular, this would contradict Gödel's incompleteness theorems [90]. Despite trying, the author has not been able to find any interesting case of a protocol that *provably* cannot be verified in EPR. (For an uninteresting case, construct a system whose safety or liveness depends on the truth of the Gödel sentence of ZFC.) It therefore remains for future research to obtain a better theoretical understanding of what systems can be proven by EPR.

Another perspective on this issue is that the provability of a system in EPR (i.e., by the techniques we have seen) exploits and uncovers some simplicity that exists in the system and its correctness argument. The simplicity exists even though the system is expressed in a Turing-complete formalism that does not make it apparent. This situation occurs across most of the field of program analysis and verification: we devise an algorithm or a method to analyze programs, which is known to be theoretically impossible; yet we expect that the specific programs encountered in practice will have some simplicity in them that will allow our method to succeed. (Even for classes in which verification is theoretically decidable, the theoretical time complexity is very high, yet we expect practical run time to be reasonable.) The author hopes that in the future, we will have a better theoretical understanding of this simplicity, which underlies our expectation that program analysis and verification methods would succeed in spite of the theoretical intractability of the problems they aim to solve.

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Appendix A

Paxos Variants

This appendix is based on the results published in [187, 188].

This appendix contains additional details about protocols from the Paxos family that have been successfully verified in this thesis, as presented in Chapter 4. For each protocol, we briefly explain the protocol, and present its modeling in first-order logic (along with RML code), its inductive invariant, and the transformation required to carry the verification in EPR.

A.1 Vertical Paxos

Vertical Paxos [141] is a variant of Paxos whose set of participating nodes and quorums (called a configuration) can be changed dynamically by an external reconfiguration master (master for short) assumed reliable. Vertical Paxos is important in practice because reconfiguration allows to replace failed nodes to achieve long-term reliability. Moreover, by appropriately choosing the quorums, Vertical Paxos can survive the failure of all but one node, compared to at most $\lfloor n/2 \rfloor$ nodes for Paxos, where n is the total number of nodes. Finally, the role of master in Vertical Paxos is limited to managing configurations and requires few resources. A reliable master can therefore be implemented cheaply using an independent replicated state machine. Vertical Paxos has two variants, Vertical Paxos I and Vertical Paxos II. We consider only Vertical Paxos I.

In Vertical Paxos I, a node starts a round r only when directed to by the master through a *configure-round* message that includes the configuration to use for round r. This allows the master to change the configuration by directing a node to start a new round. Since quorums are now different depending on the round, and quorums of different rounds may

not intersect at all, we require the owner of r to send start-round messages and to collect join-acknowledgment messages from at least one quorum in each round $r' < r$ (we say the the owner of round r accesses all rounds $r' < r$). However this may be costly in practice or simply impossible as nodes from old configurations may not stay reachable forever.

To limit the number of previous rounds that must be accessed when starting a new round, the master tracks the rounds r for which (a) no value is choosable at any $r' < r$, or (b) a value v has beed decided. Such a round r is said to be *complete*. Note that if r is complete, then rounds lower than r do not need to be accessed anymore and can safely be retired, because a quorum of join-acknowledgment messages from round r suffices to compute a value that is safe with respect to any decision that may have been made below round r.

To let the configuration master keep track of complete rounds, a node noticing that a round has become complete when it receives a quorum of join-acknowledgment messages notifies the master. In turn, when directing a node to start a new round, the master passes on the highest complete round it knows of along with the new configuration. A node starting a new round r then only accesses the rounds above the highest complete round r_c associated to the configuration of r, starting from r_c.

A.1.1 Protocol Model in First-Order Logic

Our RML model of Vertical Paxos appears in Figure A.1. Below wee highlight the main changes compared to the first-order logic model of Paxos (Figure 4.6).

Axiomatization of configurations We model configurations in first-order logic by introducing a new sort config, the relation $\mathit{quorum\_in} : \mathsf{quorum}, \mathsf{config}$, and the function $\mathit{complete\_of} : \mathsf{config} \to \mathsf{round}$ where $\mathit{quorum\_in}(c, q)$ means that quorum q is a quorum of configuration c and $\mathit{complete\_of}(c) = r$ means that r is the complete round that the master associated to configuration c. Note that the function $\mathit{complete\_of}$ never needs to be updated: when the master must associate a particular complete round r to a configuration c, we simply pick c such that $\mathit{complete\_of}(c) = r$ holds.

We change the intersection property of quorums to only require that quorums of the same configuration intersect, with the following axiom:

$$\forall c, q_1, q_2.\ \mathit{quorum\_in}(q_1, c) \wedge \mathit{quorum\_in}(q_2, c) \to \exists n.\ \mathit{member}(n, q_1) \wedge \mathit{member}(n, q_2)$$

State of the master The state of the master consists of a single individual $\mathit{master\_complete} : \mathsf{round}$ that represents the highest complete round that the master knows

```
1  sort node, quorum, round, value
2  sort config # configurations
3
4  # in join_ack_msg, ⊥ indicates the absence of a vote
5  individual ⊥: value
6
7  relation ≤ : round, round
8  axiom Γ_{total order}[≤]
9
10 relation member : node, quorum
11 relation quorum_in : quorum, config
12 axiom ∀c : config, q1, q2 : quorum.
13   quorum_in(q1, c) ∧ quorum_in(q2, c) →
14     ∃n : node. member(n, q1) ∧ member(n, q2)
15
16 relation start_round_msg : round, round
17 relation join_ack_msg : node, round, round, value
18 relation propose_msg : round, value
19 relation vote_msg : node, round, value
20 relation decision : node, round, value
21 relation configure_round_msg : round, config
22 individual complete_msg : round
23 function complete_of : config → round
24 # master state: highest round known to be complete
25 individual master_complete : round
26
27 init ∀r1, r2. ¬start_round_msg(r1, r2)
28 init ∀n, r1, r2, v. ¬join_ack_msg(n, r1, r2, v)
29 init ∀r, v. ¬propose_msg(r, v)
30 init ∀n, r, v. ¬vote_msg(n, r, v)
31 init ∀n, r, v. ¬decision(n, r, v)
32 init ∀r, c. ¬configure_round_msg(r, c)
33 init ∀r. ¬complete_msg(r)
34 # master_complete is initially set to the first round:
35 init ∀r. master_complete ≤ r
36
37 # master actions:
38 action CONFIGURE_ROUND(r : round, c : config) {
39  assume ∀c. ¬configure_round_msg(r, c)
40  assume master_complete ≤ r
41  assume complete_of(c) = master_complete
42  configure_round_msg(r, c) := true
43 }
44 action MARK_COMPLETE(r : round) {
45  # assume a node sent a "complete" message
46  assume complete_msg(r)
47  if (master_complete < r) {
48   master_complete := r
49  }
50 }
51
52 # node actions:
53 action START_ROUND(r : round, c : config) {
54  # receive a "configure-round" message:
55  assume configure_round_msg(r, c)
56  start_round_msg(r, R) := start_round_msg(r, R) ∨
57      (complete_of(c) ≤ R ∧ R < r)
58 }
59
60 action JOIN_ROUND(n : node, r : round, rp : round) {
61  assume start_round_msg(r, rp)
62  assume ¬∃r', r'', v. join_ack_msg(n, r', r'', v) ∧ r < r'
63  local v:value := *
64  if (∀v. ¬vote_msg(n, rp, v)) {
65   v := ⊥
66  } else {
67   assume vote_msg(r, rp, v)
68  }
69  join_ack_msg(n, r, rp, v) := true
70 }
71
72 # the function quorum_of_round is local to PROPOSE
73 function quorum_of_round : round → quorum
74 action PROPOSE(r : round, c : config) {
75  quorum_of_round := *
76  assume configure_round_msg(r, c)
77  assume ∀v. ¬propose_msg(r, v)
78  # rounds between the complete round and r must
79  # be configured:
80  assume ∀r'. complete_of(c) ≤ r' < r →
81    ∃c. configure_round_msg(r', c)
82  # quorum_of_round(r') is a quorum of the config. of r':
83  assume ∀r', c.
84    complete_of(c) ≤ r' < r ∧ configure_round_msg(r', c) →
85      quorum_in(quorum_of_round(r'), c)
86  # got messages from all quorums between complete_of(c) and r
87  assume ∀r', n.
88    complete_of(c) ≤ r' < r ∧ member(n, quorum_of_round(r')) →
89      ∃v. join_ack_msg(n, r, r', v)
90  # find the maximal maximal vote in the quorums:
91  local maxr, v := max{(r', v') | ∃n.
92    complete_of(c) ≤ r' ∧ r' < r ∧ member(n, quorum_of_round(r'))
93    ∧join_ack_msg(n, r, r', v') ∧ v' ≠ ⊥}
94  if (v = ⊥) {
95   v := * # set v to an arbitrary non-none value
96   assume v ≠ ⊥
97   # notify master that r is complete:
98   complete_msg(r) := true
99  }
100 propose_msg(r, v) := true # propose value v
101 }
102
103 action VOTE(n : node, r : round, v : value) {
104  assume v ≠ ⊥
105  assume propose_msg(r, v)
106  # never joined a higher round:
107  assume ¬∃r', r'', v. r' > r ∧ join_ack_msg(n, r', r'', v)
108  vote_msg(n, r, v) := true
109 }
110 action LEARN(n : node, r : round, c : config,
111              v : value, q : quorum) {
112  assume v ≠ ⊥
113  assume configure_round_msg(r, c)
114  assume quorum_in(q, c)
115  assume ∀n. member(n, q) → vote_msg(n, r, v)
116  decision(n, r, v) := true
117  complete_msg(r) := true
118 }
```

Figure A.1: RML model of Vertical Paxos.

of. The individual *master_complete* is initializedd to the first round.

Messages In addition to the messages used in Paxos, there are two kinds of messages exchanged between the nodes and the master. First, the master sends messages to instruct a round owner to start a round with a prescribed configuration and complete round. The relation *configure_round_msg* : round, config models configure-round messages from the master. A message *configure_round_msg*(r, c) is informs the owner of round r that it must start round r with configuration c and complete round *complete_of*(c). Second, nodes send messages to the master to notify it that a round has become complete. The relation *complete_msg* : round models these messages.

Finally, in contrast to Paxos, start-round messages are not sent to all nodes, but only to the nodes of the configuration of particular rounds. Therefore we make the *start_round_msg* : round, round relation binary, such that *start_round_msg*(r, r') models start-round messages from round r to the nodes in the configuration of round r'.

Master actions The action CONFIGURE_ROUND$(r$: round$, c$: config$)$ models the master sending a message to the owner of round r to inform it that it must start round r with configuration c and complete round *complete_of*(c). When this action happens, we say that the master configures round r. The master can perform this action when round r has not been configured yet and is strictly greater than the highest complete round known to the master. Moreover, the master picks a configuration c whose complete round equals its highest known complete round.

The action MARK_COMPLETE$(r$: round$)$ models the master receiving a notification from a node that round r has become complete. The master then updates its estimate of the highest complete round by updating *master_complete*.

Node actions As in Paxos, nodes perform five types of actions: START_ROUND, JOIN_ROUND, PROPOSE, VOTE, and LEARN. The major changes compared to Paxos is how the owner of a round starts a new round and determines what proposal to make, i.e., the actions START_ROUND, JOIN_ROUND, and PROPOSE.

In the action START_ROUND(r, c) the owner of round r starts r upon receiving a configure-round message from the master instructing it to start r with a configuration c and a complete round *complete_of*(c). The owner of round r broadcasts one join-round message to each configuration corresponding to a round r' such that *complete_of*$(c) \leq r' < r$.

The action JOIN_ROUND(n, r, rp) differs from Paxos in that it models node n responding

to a start-round message addressed specifically to round rp. Node n responds with a join-acknowledgment message $join\_ack\_msg(n, r, \text{rp}, v)$ indicating that n voted for v in rp. If n did not vote in rp, the value component of the join-acknowledgment message is set to $\perp$. Note that the meaning of a join-acknowledgment message is different from Paxos; in Paxos, a $join\_ack\_msg(n, r, \text{rmax}, v)$ message contains the highest round rmax lower than r in which n voted.

In the action PROPOSE, the owner of round r makes a proposal to the configuration c associated to r after receiving enough join-acknowledgment messages. The action requires that for each round r' such that $complete\_of(c) \leq r' < r$, the owner of r received join-acknowledgment messages from a quorum of the configuration of r' (which also requires that those rounds have a known configuration). This is modeled in first-order logic by fixing a function $quorum\_of\_round : \mathsf{round} \to \mathsf{quorum}$ and assuming that for every round r' such that $complete\_of(c) \leq r' < r$, the quorum $quorum\_of\_round(r')$ is a quorum of the configuration of r'. To determine its proposal, the owner of round r checks whether a vote was reported between $complete\_of(c)$ and r. If no vote was reported (i.e. all $join\_ack\_msg(n, r, r', v)$ have $v = \perp$), then the owner of r proposes an arbitrary value and notifies the master that r is complete, as no value can be choosable below r. If a vote was reported, then the owner of round r computes the maximal round $maxr$, with $complete\_of(c) \leq maxr < r$, in which a vote v was reported and proposes v (in this case v is may still be choosable at a lower round, so r is not complete).

The action VOTE(n, r, v), modeling node n casting a vote for v in round r remains the same as in Paxos. The action LEARN(n, r, c, v, q) differs from Paxos in that it must take into account the configuration of a round to determine the quorums that must vote for a value v for v to become decided. Moreover, a node making a decision notifies the master that round r has become complete, modeled by updating the *complete_msg* relation.

A.1.2 Inductive Invariant

As for Flexible Paxos and Fast Paxos, the safety property that we prove is the same as in Paxos.

The core property of the inductive invariant, on top of the relation between proposals and choosable values established for Paxos, is that a round r declared complete by a node is such that either (a) no value is choosable in any round $r' < r$, or (b) there is a value v decided in r.

As for Paxos, we start with rather mundane properties that are required for the in-

ductiveness. The first such property is that there is at most one proposal per round:

$$\forall r : \mathsf{round}, v_1, v_2 : \mathsf{value}.\ \mathit{propose\_msg}(r, v_1) \wedge \mathit{propose\_msg}(r, v_2) \rightarrow v_1 = v_2. \tag{A.2}$$

There is a most one configuration assigned to a round:

$$\forall r : \mathsf{round}, c_1, c_2 : \mathsf{config}.\ \mathit{configure\_round\_msg}(r, c_1) \wedge \mathit{configure\_round\_msg}(r, c_2) \rightarrow c_1 = c_2. \tag{A.3}$$

A start-round message is sent only upon receiving a configure-round message from the master:

$$\forall r_1, r_2 : \mathsf{round}.\ \mathit{start\_round\_msg}(r_1, r_2) \rightarrow \exists c : \mathsf{config}.\ \mathit{configure\_round\_msg}(r_1, c). \tag{A.4}$$

A start-round message starting round r_1 is sent only to round strictly lower than r_1:

$$\forall r_1, r_2 : \mathsf{round}.\ \mathit{start\_round\_msg}(r_1, r_2) \rightarrow r_2 < r_1. \tag{A.5}$$

A node votes only for a proposal:

$$\forall r : \mathsf{round}, n, v : \mathsf{value}.\ \mathit{vote\_msg}(n, r, v) \rightarrow \mathit{propose\_msg}(r, v) \tag{A.6}$$

A proposal is made only when the configuration of lower rounds is known:

$$\forall r_1, r_2 : \mathsf{round}, v : \mathsf{value}.\ \mathit{propose\_msg}(r_2, v) \wedge r_1 < r_2 \rightarrow \exists c : \mathsf{config}.\ \mathit{configure\_round\_msg}(r_1, c). \tag{A.7}$$

The special round $\perp$ is never used for deciding a value:

$$\forall r : \mathsf{round}, n : \mathsf{node}.\ \neg \mathit{propose\_msg}(r, \perp) \wedge \neg \mathit{vote\_msg}(n, r, \perp) \wedge \neg \mathit{decision}(n, r, \perp). \tag{A.8}$$

A join-acknowledgment message is sent only in response to a start-round message:

$$\forall n, r_1, r_2, v.\ \mathit{join\_ack\_msg}(n, r_1, r_2, v) \rightarrow \mathit{start\_round\_msg}(r_1, r_2). \tag{A.9}$$

The join-acknowledgment messages faithfully represent votes:

$$\forall n, r_1, r_2, v.\ \mathit{join\_ack\_msg}(n, r_1, r_2, \perp) \rightarrow \neg \mathit{vote\_msg}(n, r_2, v) \tag{A.10}$$

$$\forall n, r_1, r_2, v.\ \mathit{join\_ack\_msg}(n, r_1, r_2, v) \wedge v \neq \perp \rightarrow \mathit{vote\_msg}(n, r_2, v). \tag{A.11}$$

A configure-round message configuring round r contains a complete round lower than r and lower than the highest complete round known to the master:

$$\forall r, c, \mathrm{cr}.\ \mathit{configure\_round\_msg}(r, c) \wedge \mathit{complete\_of}(c, \mathrm{cr}) \rightarrow \mathrm{cr} \leq r \wedge \mathrm{cr} \leq \mathit{master\_complete}. \tag{A.12}$$

Any round lower than a round appearing in a complete messages has been configured:

$$\forall r_1, r_2.\ \mathit{complete\_msg}(r_2) \wedge r_1 \leq r_2 \rightarrow \exists c.\ \mathit{configure\_round\_msg}(r_1, c). \tag{A.13}$$

The highest complete round known to the master or the complete round that the master assigns to a configuration has been marked complete by a node or is the first round:

$$\forall r_1, r_2, r_3, c.\ (r_2 = \mathit{master\_complete} \vee (\mathit{configure\_round\_msg}(r_3, c) \wedge \mathit{complete\_of}(c) = r_2) \wedge r_1 < r_2 \rightarrow \mathit{complete\_msg}(r_2). \tag{A.14}$$

A decisions comes from a quorum of votes from a configured round:

$$\begin{aligned} &\forall r, v.\ (\exists n.\ \mathit{decision}(n, r, v)) \rightarrow \\ &\quad \exists c : \mathsf{config}, q.\ \mathit{configure\_round\_msg}(r, c) \wedge \mathit{quorum\_in}(q, c) \wedge \\ &\qquad (\forall n. \mathit{member}(n, q) \rightarrow \mathit{vote\_msg}(n, rv)). \end{aligned} \tag{A.15}$$

Note that the number of the invariants above may seem overwhelming, but those invariants are easy to infer from the counterexamples to induction displayed graphically by Ivy when they are missing.

Finally, the two crucial invariants of Vertical Paxos I relate, first, proposals to choosable

values (similarly to Paxos), and, second, relate rounds declared complete to choosable values:

$$\forall r_1, r_2, v_1, v_2, q, c.$$
$$r_1 < r_2 \wedge \mathit{propose\_msg}(r_2, v_2) \wedge v_1 \neq v_2 \wedge \mathit{configure\_round\_msg}(r_1, c) \wedge \mathit{quorum\_in}(q, c) \rightarrow$$
$$\exists n, r_3, r_4, v.\ \mathit{member}(n, q) \wedge r_1 < r_3 \wedge \mathit{join\_ack\_msg}(n, r_3, r_4, v) \wedge \neg \mathit{vote\_msg}(n, r_1, v_1) \tag{A.16}$$

$$\forall r_1, r_2, c, q.$$
$$\mathit{complete\_msg}(r_2) \wedge r_1 < r_2 \wedge \mathit{configure\_round\_msg}(r_1, c) \wedge \mathit{quorum\_in}(q, c)) \wedge$$
$$\neg(\exists n, r_3, r_4, v. \mathit{member}(n, q) \wedge r_1 < r_3 \wedge \mathit{join\_ack\_msg}(n, r_3, r_4, v) \wedge \neg \mathit{vote\_msg}(n, r, v)) \rightarrow$$
$$\exists n. \mathit{decision}(n, r, v) \tag{A.17}$$

A.1.3 Transformation to EPR

As in Paxos, we start by introducing the derived relation $\mathit{left\_round}(n, r)$ with representation invariant

$$\forall n, r.\ \mathit{left\_round}(n, r) \leftrightarrow \exists r' > r, \mathrm{rp}, v.\ \mathit{join\_ack\_msg}(n, r', \mathrm{rp}, v) \tag{A.18}$$

and we rewrite the JOIN_ROUND and PROPOSE accordingly.

Then, we notice that the function $\mathit{complete\_of} : \mathsf{config} \rightarrow \mathsf{round}$, together with eqs. (A.4), (A.7), (A.13) and (A.15), creates a cycle in the quantifier alternation graph involving the sort config and round. We eliminate this cycle by introducing the derived relation $\mathit{complete\_of}(c : \mathsf{config}, r : \mathsf{round})$ (here we overload *complete_of* to represent both the function and the relation) with representation invariant

$$\forall c, r.\ \mathit{complete\_of}(c) = r \leftrightarrow \mathit{complete\_of}(c, r) \tag{A.19}$$

We rewrite the CONFIGURE_ROUND, START_ROUND, and PROPOSE actions to use the *complete_of* in its relation form instead of the function. The function *complete_of* is now unused in the model and we remove it, thereby eliminating the cycle in the quantifier alternation graph involving the sorts config and round.

After this step we observe that the quantifier alternation graph is acyclic, and Ivy successfully verifies that the invariant is inductive.

Surprisingly, the transformation of the Vertical Paxos I model to EPR is simpler than the

transformation of the Paxos model to EPR because we do not need to introduce the derived relation *joined_round* and, in consequence, to rewrite the propose action to use *vote_msg* instead of *join_ack_msg*. The derived relation *joined_round* is not needed because, compared to Paxos, the assume condition of the PROPOSE action contains the formula

$$\forall r' n.\ \mathrm{cr} \leq r' \wedge r' < r \wedge member(n, quorum\_of\_round(c)) \rightarrow \exists v : \mathsf{value}. join\_ack\_msg(n, r, r', v) \tag{A.20}$$

instead of the formula

$$\forall n : \mathsf{node}.\ member(n, q) \rightarrow \exists r' : \mathsf{round},\ v : \mathsf{value}.\ join\_ack\_msg(n, r, r', v). \tag{A.21}$$

Notice how eq. (A.20) introduces an edge in the quantifier alternation graph from sort node to value only, whereas eq. (A.21) introduces an edge from sort round to sorts value and round. In combination with the conjunct of the inductive invariant that expresses the relation between choosable values and proposals, eq. (A.21) introduces a cycle in the quantifier alternation graph, whereas eq. (A.20) does not.

The full model of Vertical Paxos in EPR appears in Figure A.22

A.2 Fast Paxos

When a unique node starts a round, a value sent by a client to that node is decided by Paxos in at most 3 times the worst-case message latency (one message to deliver the proposal to the owner of the round, and one round trip for the owner of the round to complete phase 2). Fast Paxos [138] reduces this latency to twice the worst-case message latency under the assumption that messages broadcast by the nodes are received in the same order by all nodes (a realistic assumption in some settings, e.g., in an Ethernet network). When this assumption is violated, the cost of Fast Paxos increases depending on the conflict-recovery mechanism employed.

In Fast Paxos, rounds are split into a set of fast rounds and a set of classic rounds (e.g. even rounds are fast and odd ones are classic). In a classic round, nodes vote for the proposal that the owner of the round sends and a value is decided when a quorum of nodes vote for it, as in Paxos. However, in a fast round r, the owner can send an "any" message instead of a proposal, and a node receiving it is allowed to vote for any value of its choice in r (but it cannot change its mind after having voted). This allows clients to broadcast their proposals without first sending it to the leader, saving on message delay. But, as a result, different

```
1  sort node, quorum, round, value
2  sort config # configurations
3
4  # in join_ack_msg, ⊥ indicates the absence of a vote
5  individual ⊥: value
6
7  relation ≤ : round, round
8  axiom Γ_{total order}[≤]
9
10 relation member : node, quorum
11 relation quorum_in : quorum, config
12 axiom ∀c : config, q1, q2 : quorum.
13   quorum_in(q1, c) ∧ quorum_in(q2, c) →
14     ∃n : node. member(n, q1) ∧ member(n, q2)
15
16 relation start_round_msg : round, round
17 relation join_ack_msg : node, round, round, value
18 relation propose_msg : round, value
19 relation vote_msg : node, round, value
20 relation decision : node, round, value
21 relation configure_round_msg : round, config
22 individual complete_msg : round
23 # relation replacing the function:
24 relation complete_of : config, round
25 # immutable, so we can use axiom:
26 axiom ∀c, r1, r2.
27   complete_of(c, r1) ∧ complete_of(c, r2) → r1 = r2
28 # master state: highest round known to be complete
29 individual master_complete : round
30 relation left_round : node, round
31
32 init ∀r1, r2. ¬start_round_msg(r1, r2)
33 init ∀n, r1, r2, v. ¬join_ack_msg(n, r1, r2, v)
34 init ∀r, v. ¬propose_msg(r, v)
35 init ∀n, r, v. ¬vote_msg(n, r, v)
36 init ∀n, r, v. ¬decision(n, r, v)
37 init ∀r, c. ¬configure_round_msg(r, c)
38 init ∀r. ¬complete_msg(r)
39 # master_complete is initially set to the first round:
40 init ∀r. master_complete ≤ r
41
42 # master actions:
43 action CONFIGURE_ROUND(r : round, c : config) {
44  assume ∀c. ¬configure_round_msg(r, c)
45  assume master_complete ≤ r
46  assume complete_of(c) = master_complete
47  configure_round_msg(r, c) := true
48 }
49 action MARK_COMPLETE(r : round) {
50  # assume a node sent a "complete" message
51  assume complete_msg(r)
52  if (master_complete < r) {
53   master_complete := r
54  }
55 }
56
57 # node actions:
58 action START_ROUND(r : round, c : config, cr : round) {
59  # receive a "configure-round" message:
60  assume configure_round_msg(r, c)
61  # get the complete round sent with the config.:
62  assume complete_of(c, cr)
63  start_round_msg(r, R) := start_round_msg(r, R) ∨
64      (cr ≤ R ∧ R < r)
65 }
66 action JOIN_ROUND(n : node, r : round, rp : round) {
67  assume start_round_msg(r, rp)
68  assume ¬left_round(n, r) # rewritten
69  local v:value := *
70  if (∀v. ¬vote_msg(n, rp, v)) {
71   v := ⊥
72  } else {
73   assume vote_msg(r, rp, v)
74  }
75  join_ack_msg(n, r, rp, v) := true
76  left_round(n, R) := left_round(n, R) ∨ R < r
77 }
78
79 # the function quorum_of_round is local to PROPOSE
80 function quorum_of_round : round → quorum
81 action PROPOSE(r : round, c : config, cr : round) {
82  quorum_of_round := *
83  assume configure_round_msg(r, c)
84  assume complete_of(c, cr)
85  assume ∀v. ¬propose_msg(r, v)
86  # rounds between the complete round and r must
87  # be configured:
88  assume ∀r'. cr ≤ r' < r →
89   ∃c.configure_round_msg(r', c)
90  # quorum_of_round(r') is a quorum of the config. of r':
91  assume ∀r', c.
92    cr ≤ r' < r ∧ configure_round_msg(r', c) →
93      quorum_in(quorum_of_round(r'), c)
94  # got messages from all quorums between cr and r
95  assume ∀r', n.
96    cr ≤ r' < r ∧ member(n, quorum_of_round(r')) →
97      ∃v. join_ack_msg(n, r, r', v)
98
99  # find the maximal maximal vote in the quorums:
100 local maxr, v := max{(r', v') | ∃n.
101   cr ≤ r' ∧ r' < r ∧ member(n, quorum_of_round(r'))
102   ∧join_ack_msg(n, r, r', v') ∧ v' ≠ ⊥}
103 if (v = ⊥) {
104  v := * # set v to an arbitrary non-none value
105  assume v ≠ ⊥
106  # notify master that r is complete:
107  complete_msg(r) := true
108 }
109 propose_msg(r, v) := true # propose value v
110 }
111
112 action VOTE(n : node, r : round, v : value) {
113  assume v ≠ ⊥
114  assume propose_msg(r, v)
115  # never joined a higher round:
116  assume ¬left_round(n, r) # rewritten
117  vote_msg(n, r, v) := true
118 }
119
120 action LEARN(n : node, r : round, c : config,
121              v : value, q : quorum) {
122  assume v ≠ ⊥
123  assume configure_round_msg(r, c)
124  assume quorum_in(q, c)
125  assume ∀n. member(n, q) → vote_msg(n, r, v)
126  decision(n, r, v) := true
127  complete_msg(r) := true
128 }
129
130
```

Figure A.22: RML model of Vertical Paxos in EPR.

values may be voted for in the same (fast) round, a situation that cannot arise in Paxos.

Now that different values may be voted for in the same (fast) round, the rule used in the propose action of Paxos to determine a value to propose does not work anymore, as there may be different votes in the highest reported round. Remember that this rule ensured that the owner p of round r would not miss any value choosable in a lower round $r' < r$. With different values being voted for in fast round r', how is node p to determine which value is choosable in r'? This problem is solved by requiring that any 2 fast quorums and one classic quorum have a common node, and by modifying the way p determines which value is choosable at r', as follows.

As in Paxos, the owner p waits for a classic quorum q of nodes to have joined its round r. If the highest reported round $maxr$ is a fast round and there are different votes reported in $maxr$, then p checks whether there exists a fast quorum f such that all nodes in $q \cap f$ voted for the same value v in $maxr$; if there is such a fast quorum f and value v, then p proposes v, and otherwise it can propose any value. By the intersection property of quorums there can be at most one value v satisfying those conditions. Moreover, if v is choosable at r', then there will be a fast quorum f such that all nodes in $q \cap f$ voted for v at $maxr$. If there is only a single value v reported voted for in $maxr$, then p proposes v, as in Paxos.

In Paxos, a round may not decide a value if its owner is not able to contact a quorum of nodes before they leave the round (e.g. because the owner crashed, is slow, or because of message losses in the network). In Fast Paxos there is one more cause for a round not deciding a value: a fast round r may not decide a value if the votes cast in r are such that, no matter what new votes are cast from this point on, no value can be voted for by a fast quorum (e.g. when every nodes voted for a different value). In this case we say that a conflict has occured. Fast Paxos has three different conflict-recovery mechanisms: starting a new round, coordinated recovery, and uncoordinated recovery. Our model does not include coordinated or uncoordinated recovery, but starting a new round is of course part of the model.

A.2.1 FOL model of Fast Paxos

The FOL model of Fast Paxos appears in fig. A.47. We now highlight the main changes compared to the FOL model of Paxos.

Quorums We axiomatize the properties of fast quorums and classic quorums in first-order logic by defining two different sorts $\mathsf{quorum_c}$ and $\mathsf{quorum_f}$ for classic and fast quorums, and a separate membership relation for each. The intersection properties of quorums are expressed

as follows:

$$\forall q_1, q_2 : \mathsf{quorum_c}.\exists n : \mathsf{node}.c\_member(n, q_1) \wedge c\_member(n, q_2) \tag{A.23}$$

$$\forall q : \mathsf{quorum_c}, f_1, f_2 : \mathsf{quorum_f}.\exists n : \mathsf{node}.c\_member(n, q) \wedge f\_member(n, f_1) \wedge f\_member(n, f_2) \tag{A.24}$$

New relations To identify fast rounds we add a unary relation *fast* : round which contains all fast rounds. We add a relation *any_msg* to model the "any" messages sent by the owners of fast rounds.

Actions The START_ROUND, VOTE, and JOIN_ROUND actions remain the same as in Paxos. The PROPOSE action is modified to reflect the new rule that the owner of a round r uses to determine what command to propose. We start by assuming the owner has received join-acknowledgment messages from a quorum q (line 63), and we compute the maximal reported round $maxr$ and pick a vote v reported at $maxr$ (line 69). Note that there may be different votes reported in $maxr$ if $maxr$ is a fast round.

If at least one vote was reported ($maxr \neq \bot$), then the owner propose the value v' chosen as follows (line 75): if there exists a fast quorum f such that all nodes in $f \cap q$ reported voting for v' at $maxr$, then the owner proposes v'. Otherwise, if no such fast quorum exists, it proposes v (as defined above). Formally, we assume

$$\begin{aligned}&(\exists f.\ \forall n.\ f\_member(n, f) \wedge c\_member(n, q) \rightarrow join\_ack\_msg(n, r, maxr, v')) \\ &\vee\ (v' = v \wedge \forall v'', f'.\ \exists n.\ f\_member(n, f') \wedge c\_member(n, q) \wedge \neg join\_ack\_msg(n, r, maxr, v''))\end{aligned} \tag{A.25}$$

If no vote was reported ($maxr = \bot$) then, if the round being started is a fast round then the owner sends an "any" message, and otherwise the owner proposes an arbitrary value.

Finally, we replace the LEARN action by two actions LEARN_C and LEARN_F that update the *decision* relation when a classic (for LEARN_C) or fast (for LEARN_F) quorum has voted for the same value in the same round.

A.2.2 Inductive Invariant

The safety property we prove is the same as in Paxos, namely that all decisions are for the same value regardless of the node making them and of the round in which they are made:

$$\forall n_1, n_2, r_1, r_2, v.\ decision(n_1, r_1, v_1) \wedge decision(n_2, r_2, v_2) \rightarrow v_1 = v_2 \tag{A.26}$$

The inductive invariant is similar to the one of Paxos with cases to distinguish classic and fast rounds. The main addition is that no value can be choosable at a round $r' < r$ when there is an "any" message in r.

We start by expressing rather mundane facts about the algorithm, but which are necessary for inductiveness. There is at most one proposal per round:

$$\forall r : \mathsf{round}, v_1, v_2 : \mathsf{value}.\ \neg \mathit{fast}(r) \wedge \mathit{propose\_msg}(r, v_1) \wedge \mathit{propose\_msg}(r, v_2) \rightarrow v_1 = v_2. \tag{A.27}$$

In a classic round, nodes voted for a value v only if v was proposed:

$$\forall r : \mathsf{round}, n, v : \mathsf{value}.\ \neg \mathit{fast}(r) \wedge \mathit{vote\_msg}(n, r, v) \rightarrow \mathit{propose\_msg}(r, v). \tag{A.28}$$

An "any" message can occur only in a fast round:

$$\forall r : \mathsf{round}.\ \mathit{any\_msg}(r) \rightarrow \mathit{fast}(r). \tag{A.29}$$

In a fast round, a node votes for a value v only if v was proposed or if an "any" message was sent in the round:

$$\forall r : \mathsf{round}, n, v.\ \mathit{fast}(r) \wedge \mathit{vote\_msg}(n, r, v) \rightarrow (\mathit{propose\_msg}(r, v) \vee \mathit{any\_msg}(r)). \tag{A.30}$$

There cannot be both a proposal and an any message in the same round:

$$\forall r : \mathsf{round}, v.\ \neg(\mathit{propose\_msg}(r, v) \wedge \mathit{any\_msg}(r)). \tag{A.31}$$

A node votes only once per round:

$$\forall n : \mathsf{node}, r : \mathsf{round}, v_1, v_2 : \mathsf{value}.\ \mathit{vote\_msg}(n, r, v_1) \wedge \mathit{vote\_msg}(n, r, v_2) \rightarrow v_1 = v_2. \tag{A.32}$$

There is no vote in the round $\bot$:

$$\forall n : \mathsf{node}, v : \mathsf{value}.\ \neg \mathit{vote\_msg}(n, \bot, v). \tag{A.33}$$

Decisions come from fast quorums in fast rounds and classic quorums in classic rounds:

$$\forall r : \mathsf{round}, v : \mathsf{value}.\ \neg \mathit{fast}(r) \wedge (\exists n : \mathsf{node}.\mathit{decision}(n, r, v)) \rightarrow \exists q : \mathsf{quorum_c}.\forall n : \mathsf{node}.\mathit{c\_member}(n, q) \rightarrow \mathit{vote\_msg}(n, r, v) \tag{A.34}$$

$$\forall r : \mathsf{round}, v : \mathsf{value}.\ \mathit{fast}(r) \wedge (\exists n : \mathsf{node}.\mathit{decision}(n, r, v)) \rightarrow \exists f : \mathsf{quorum_f}.\forall n : \mathsf{node}.\mathit{f\_member}(n, f) \rightarrow \mathit{vote\_msg}(n, r, v). \tag{A.35}$$

Then we express the fact that join-acknowledgment messages faithfully represent the node votes (exactly as in Paxos).

$$\forall n : \mathsf{node}, r, r' : \mathsf{round},\ v, v' : \mathsf{value}.\ \mathit{join\_ack\_msg}(n, r, \bot, v) \wedge r' < r \rightarrow \neg \mathit{vote\_msg}(n, r', v') \tag{A.36}$$

$$\forall n : \mathsf{node}, r, r' : \mathsf{round},\ v : \mathsf{value}.\ \mathit{join\_ack\_msg}(n, r, r', v) \wedge r' \neq \bot \rightarrow \tag{A.37}$$

$$r' < r \wedge \mathit{vote\_msg}(n, r', v) \tag{A.38}$$

$$\forall n : \mathsf{node}, r, r', r'' : \mathsf{round},\ v, v' : \mathsf{value}.\ \mathit{join\_ack\_msg}(n, r, r', v) \wedge r' \neq \bot \wedge r' < r'' < r \rightarrow \neg \mathit{vote\_msg}(n, r'', v'). \tag{A.39}$$

Finally, we express that if v is choosable at round r, then only v can be proposed at a round $r' > r$, and that there cannot be an "any" message at a round $r' > r$. We differentiate the case of a value choosable in a classic round and in a fast round.

$$\forall r_1, r_2, v_1, v_2, q : \mathsf{quorum_c}.\ \neg \mathit{fast}(r_1) \wedge ((\mathit{propose\_msg}(r_2, v_2) \wedge v_1 \neq v_2) \vee \mathit{any\_msg}(r_2)) \wedge r_1 < r_2 \rightarrow \exists n : \mathsf{node}, r', r'' : \mathsf{round}, v : \mathsf{value}.\ \mathit{c\_member}(n, q) \wedge \neg \mathit{vote\_msg}(n, r_1, v_1) \wedge r' > r_1 \wedge \mathit{join\_ack\_msg}(n, r', r'', v) \tag{A.40}$$

$$\forall r_1, r_2, v_1, v_2, f : \mathsf{quorum_f}.\ \mathit{fast}(r_1) \wedge ((\mathit{propose\_msg}(r_2, v_2) \wedge v_1 \neq v_2) \vee \mathit{any\_msg}(r_2)) \wedge r_1 < r_2 \rightarrow \exists n : \mathsf{node}, r', r'' : \mathsf{round}, v : \mathsf{value}.\ \mathit{f\_member}(n, f) \wedge \neg \mathit{vote\_msg}(n, r_1, v_1) \wedge r' > r_1 \wedge \mathit{join\_ack\_msg}(n, r', r'', v) \tag{A.41}$$

A.2.3 Transformation to EPR

To transform the Fast Paxos model to EPR we introduce the same derived relations as in Paxos (i.e. *left_round* and *joined_round*) and we rewrite as explained in section 4.4. However the verification of the second rewrite step is not as simple as in Paxos. This step consists in rewriting the PROPOSE action by considering directly votes instead of join-acknowledgment messages. In Paxos, we verified the rewrite using the auxiliary invariant I_{aux}, and the verification condition is in EPR. We employ the same method for Fast Paxos, using eqs. (A.27) to (A.39) as auxiliary invariant. However, for Fast Paxos, the verification condition of the rewrite does not fall in EPR when done naively.

Verifying the rewrite of the statement at line line 69, from

$$\max\{(r', v') \mid \exists n.\ c\_member(n, q) \wedge join\_ack\_msg(n, r, r', v') \wedge r' \neq \bot\} \tag{A.42}$$

to

$$\max\{(r', v') \mid \exists n.\ c\_member(n, q) \wedge vote\_msg(n, r, v') \wedge r' \neq \bot \wedge r' < r\} \tag{A.43}$$

is exactly as in Paxos and poses no problem.

However, we also need to verify the rewrite of the assume statement at line 75 from assuming

$$\begin{aligned}&(\exists f.\ \forall n.\ f\_member(n, f) \wedge c\_member(n, q) \rightarrow join\_ack\_msg(n, r, maxr, v'))\\ &\quad \vee\ (v' = v \wedge \forall v'', f'.\ \exists n.\ f\_member(n, f') \wedge c\_member(n, q) \wedge \neg join\_ack\_msg(n, r, maxr, v''))\end{aligned} \tag{A.44}$$

to assuming

$$\begin{aligned}&(\exists f.\ \forall n.\ f\_member(n, f) \wedge c\_member(n, q) \rightarrow vote\_msg(n, maxr, v'))\\ &\quad \vee\ (v' = v \wedge \forall v'', f'.\ \exists n.\ f\_member(n, f') \wedge c\_member(n, q) \wedge \neg vote\_msg(n, maxr, v''))\end{aligned} \tag{A.45}$$

With the assume statement at the beginning of the PROPOSE action (without which the equivalence does not hold)

$$\forall n.\ c\_member(n, q) \rightarrow \exists r', v.\ join\_ack\_msg(n, r, r', v) \tag{A.46}$$

we get a verification condition that is not stratified: there is a cycle involving the sorts node and value in the quantifier alternation graph. However, as explained in section 4.1.2, we observe that we only have to verify the rewrite of the sub-formula $c\_member(n, q) \wedge \neg join\_ack\_msg(n, r, maxr, v'')$ to $c\_member(n, q) \wedge \neg vote\_msg(n, maxr, v'')$. Assuming the auxiliary invariant and the conditions of the assume statements of the PROPOSE action before line 75, this equivalence check falls in EPR and successfully verifies.

The EPR model of Fast Paxos appears in fig. A.48.

A.3 Flexible Paxos

Flexible Paxos [107] extends Paxos based on the observation that in Paxos it is only necessary that every phase-1 quorum intersects with every phase-2 quorum (quorums of the same phase do not have to intersect). Thus nodes may use different sets of quorums for phase 1 (to compute which value may be choosable in lower rounds) and for phase 2 (to get a value decided in the current round) as long as every phase-1 quorum intersects every phase-2 quorum. For example, in a system with a large number of nodes, one may choose that a value is decided if any set of 3 nodes (a phase-2 quorum) votes for it, and at the same time require that a node wait for $n-2$ nodes (a phase-1 quorum) to join its round when starting a new round, where n is the total number of nodes. The opposite configuration is also possible (i.e. phase-1 quorums of size 3 and phase-2 quorums of size $n-2$). This approach allows a trade-off between the cost of starting a new round and the cost of deciding on a value. Note that no inconsistency may arise due to the fact that same-phase quorums do not intersect because the leader of a round proposes a unique value.

To model Flexible Paxos in IVY we introduce two sorts for the two different types of quorums, *quorum_1* and *quorum_2*, and we modify the actions to use quorums of sort *quorum_1* in phase 1 and of sort *quorum_2* in phase 2. We also adapt the quorum intersection axiom:

$$\forall q_1 : \mathsf{quorum}_1, q_2 : \mathsf{quorum}_2.\ \exists n : \mathsf{node}.\ member_1(n, q_1) \wedge member_2(n, q_2).$$

The derived relations and rewriting steps are the same as in Paxos, and the safety property and inductive invariant is also the same as in Paxos except that the quorums are taken from the sort *quorum_2* in eq. (4.13) and eq. (4.18).

The FOL model of Flexible Paxos appears in fig. A.49.

```
1  sort node, round, value
2  sort quorum_c # classic quorums
3  sort quorum_f # fast quorums
4
5  relation ≤ : round, round
6  axiom Γ_total order[≤]
7  individual ⊥ : round
8
9  relation fast : round # identifies fast rounds
10
11 relation c_member : node, quorum_c
12 relation f_member : node, quorum_f
13
14 # classic quorums intersect
15 axiom ∀q1, q2 : quorum_c. ∃n.
16  c_member(n, q1) ∧ c_member(n, q2)
17
18 # a classic quorum and a two fast quorums intersect
19 axiom ∀q1 : quorum_c, q2, q3 : quorum_f. ∃n.
20  c_member(n, q1) ∧ f_member(n, q2) ∧ f_member(n, q3)
21
22 relation start_round_msg : round
23 relation join_ack_msg : node, round, round, value
24 relation propose_msg : round, value
25 relation vote_msg : node, round, value
26 relation decision : node, round, value
27 relation any_msg : round # the "any" messages
28
29 init ∀r. ¬start_round_msg(r)
30 init ∀n, r1, r2, v. ¬join_ack_msg(n, r1, r2, v)
31 init ∀r, v. ¬propose_msg(r, v)
32 init ∀n, r, v. ¬vote_msg(n, r, v)
33 init ∀n, r, v. ¬decision(n, r, v)
34 init ∀r. ¬any_msg(r)
35
36 action START_ROUND(r : round) {
37  assume r ≠ ⊥
38  start_round_msg(r) := true
39 }
40 action JOIN_ROUND(n : node, r : round) {
41  assume r ≠ ⊥
42  assume start_round_msg(r)
43  assume ¬∃r', r'', v. r' > r ∧ join_ack_msg(n, r', r'', v)
44  # find the maximal round in which n voted, and
45  # the corresponding vote; maxr = ⊥ and v is
46  # arbitrary when n never voted.
47  local maxr, v := max{(r', v') | vote_msg(n, r', v') ∧
48                                  r' < r}
49  join_ack_msg(n, r, maxr, v) := true
50 }
51 action VOTE(n : node, r : round, v : value) {
52  # either vote for a proposal, or vote arbitrarily if
53  # there is an "any" message
54  assume r ≠ ⊥
55  assume ¬∃r', r'', v. join_ack_msg(n, r', r'', v) ∧ r < r'
56  assume ∀v. ¬vote_msg(n, r, v)
57  assume propose_msg(r, v) ∨ any_msg(r)
58  vote_msg(n, r, v) := true
59 }
60 action PROPOSE(r : round, q : quorum_c) {
61  assume r ≠ ⊥
62  assume ∀v. ¬propose_msg(r, v)
63  assume ∀n. c_member(n, q) → ∃r', v.
64                                join_ack_msg(n, r, r', v)
65  # find the maximal round in which a node in the
66  # quorum reported voting, and pick a corresponding
67  # vote (there may be several); v is arbitrary when
68  # no such node voted.
69  local maxr, v := max{(r', v') | ∃n. c_member(n, q) ∧
70                      join_ack_msg(n, r, r', v') ∧ r' ≠ ⊥}
71  if (maxr ≠ ⊥) {
72   # a vote was reported in round maxr, and there
73   # are no votes in higher rounds.
74   local vp := * # the proposal the node will make
75   assume
76    (∃f. ∀n. f_member(n, f) ∧ c_member(n, q) →
77     join_ack_msg(n, r, maxr, vp)) ∨
78    (vp = v ∧ ∀v'', f'. ∃n. f_member(n, f') ∧
79     c_member(n, q) ∧ ¬join_ack_msg(n, r, maxr, v''))
80   propose_msg(r, vp) := true
81  } else { # no vote was reported at all.
82   if fast(r) { # fast round, send "any" message
83    any_msg(r) := true
84   } else { # classic round, propose arbitrary value
85    propose_msg(r, v) := true
86   }
87  }
88 }
89 action LEARN_C(n : node, r : round, v : value,
90                q : quorum_c) {
91  assume r ≠ ⊥
92  assume ∀n. c_member(n, q) → vote_msg(n, r, v)
93  decision(n, r, v) := true
94 }
95 action LEARN_F(n : node, r : round, v : value,
96                q : quorum_f) {
97  assume r ≠ ⊥
98  assume ∀n. f_member(n, q) → vote_msg(n, r, v)
99  decision(n, r, v) := true
100 }
```

Figure A.47: RML model of Fast Paxos.

```
1  sort node, round, value, quorum_c, quorum_f
2
3  relation ≤ : round, round
4  axiom Γ_{total order}[≤]
5  individual ⊥ : round
6
7  relation fast : round # identifies fast rounds
8
9  relation c_member : node, quorum_c
10 relation f_member : node, quorum_f
11
12 # classic quorums intersect
13 axiom ∀q_1, q_2 : quorum_c. ∃n.
14  c_member(n, q_1) ∧ c_member(n, q_2)
15
16 # a classic quorum and a two fast quorums intersect
17 axiom ∀q_1 : quorum_c, q_2, q_3 : quorum_f. ∃n.
18  c_member(n, q_1) ∧ f_member(n, q_2) ∧ f_member(n, q_3)
19
20 relation start_round_msg : round
21 relation join_ack_msg : node, round, round, value
22 relation propose_msg : round, value
23 relation vote_msg : node, round, value
24 relation decision : node, round, value
25 relation any_msg : round # the ''any'' messages
26 relation left_round : node, round
27 relation joined_round : node, round
28
29 init ∀r. ¬start_round_msg(r)
30 init ∀n, r_1, r_2, v. ¬join_ack_msg(n, r_1, r_2, v)
31 init ∀r, v. ¬propose_msg(r, v)
32 init ∀n, r, v. ¬vote_msg(n, r, v)
33 init ∀n, r, v. ¬decision(n, r, v)
34 init ∀r. ¬any_msg(r)
35 init ∀n, r. ¬left_round(n, r)
36 init ∀n, r. ¬joined_round(n, r)
37
38 action START_ROUND(r : round) {
39  assume r ≠ ⊥
40  start_round_msg(r) := true
41 }
42 action JOIN_ROUND(n : node, r : round) {
43  assume r ≠ ⊥
44  assume start_round_msg(r)
45  assume ¬left_round(n, r) # rewritten
46  local maxr, v := max{(r', v') | vote_msg(n, r', v') ∧
47                                  r' < r}
48  join_ack_msg(n, r, maxr, v) := true
49  # generated update code for derived relations:
50  left_round(n, R) := left_round(n, R) ∨ R < r
51  joined_round(n, r) := true
52 }
53 action VOTE(n : node, r : round, v : value) {
54  # either vote for a proposal, or vote arbitrarily if
55  # there is an ''any'' message
56  assume r ≠ ⊥
57  assume ¬∃r', r'', v. join_ack_msg(n, r', r'', v) ∧ r < r'
58  assume ∀v. ¬vote_msg(n, r, v)
59  assume propose_msg(r, v) ∨ any_msg(r)
60  vote_msg(n, r, v) := true
61 }
62 action PROPOSE(r : round, q : quorum_c) {
63  assume r ≠ ⊥
64  assume ∀v. ¬propose_msg(r, v)
65  # rewriten:
66  assume ∀n. c_member(n, q) → joined_round(n, r)
67  # find the maximal round in which a node in the
68  # quorum reported voting, and pick a corresponding
69  # vote (there may be several); v is arbitrary when
70  # no such node voted.
71  local maxr, v := max{(r', v') | ∃n. c_member(n, q) ∧
72     vote_msg(n, r', v') ∧ r' ≠ ⊥ ∧ r' < r} # rewritten
73  if (maxr ≠ ⊥) {
74   # a vote was reported in round maxr, and there
75   # are no votes in higher rounds.
76   local vp := * # the proposal the node will make
77   # rewritten:
78   assume
79    (∃f. ∀n. f_member(n, f) ∧ c_member(n, q) →
80     vote_msg(n, maxr, vp)) ∨
81    (vp = v ∧ ∀v'', f'. ∃n. f_member(n, f') ∧
82     c_member(n, q) ∧ ¬vote_msg(n, maxr, v''))
83   propose_msg(r, vp) := true
84  } else { # no vote was reported at all.
85   if fast(r) { # fast round, send ''any'' message
86    any_msg(r) := true
87  } else { # classic round, propose arbitrary value
88    propose_msg(r, v) := true
89  }
90 }
91 }
92 action LEARN_C(n : node, r : round, v : value,
93                q : quorum_c) {
94  assume r ≠ ⊥
95  assume ∀n. c_member(n, q) → vote_msg(n, r, v)
96  decision(n, r, v) := true
97 }
98 action LEARN_F(n : node, r : round, v : value,
99                q : quorum_f) {
100 assume r ≠ ⊥
101 assume ∀n. f_member(n, q) → vote_msg(n, r, v)
102 decision(n, r, v) := true
103 }
104
```

Figure A.48: RML model of Fast Paxos in EPR.

```
1  sort node, round, value
2  sort quorum_1 # phase 1 quorums
3  sort quorum_2 # phase 2 quorums
4
5  relation ≤ : round, round
6  axiom Γ_{total order}[≤]
7  individual ⊥ : round
8
9  # every phase 1 quorum and phase 2 quorum intersect
10 relation member_1 : node, quorum_1
11 relation member_2 : node, quorum_2
12 axiom ∀q_1 : quorum_1, q_2 : quorum_2. ∃n : node. member_1(n, q_1) ∧ member_2(n, q_2)
13
14 relation start_round_msg : round
15 relation join_ack_msg : node, round, round, value
16 relation propose_msg : round, value
17 relation vote_msg : node, round, value
18 relation decision : node, round, value
19
20 init ∀r. ¬start_round_msg(r)
21 init ∀n, r_1, r_2, v. ¬join_ack_msg(n, r_1, r_2, v)
22 init ∀r, v. ¬propose_msg(r, v)
23 init ∀n, r, v. ¬vote_msg(n, r, v)
24 init ∀n, r, v. ¬decision(n, r, v)
25
26 action START_ROUND(r : round) {
27     assume r ≠ ⊥
28     start_round_msg(r) := true
29 }
30 action JOIN_ROUND(n : node, r : round) {
31     assume r ≠ ⊥
32     assume start_round_msg(r)
33     assume ¬∃r', r'', v. r' > r ∧ join_ack_msg(n, r', r'', v)
34     # find the maximal round in which n voted, and the corresponding vote;
35     # maxr = ⊥ and v is arbitrary when n never voted.
36     local maxr, v := max{(r', v') | vote_msg(n, r', v') ∧ r' < r}
37     join_ack_msg(n, r, maxr, v) := true
38 }
39 action PROPOSE(r : round, q : quorum_1) {
40     assume r ≠ ⊥
41     assume ∀v. ¬propose_msg(r, v)
42     # 1b from a phase 1 quorum q:
43     assume ∀n. member_1(n, q) → ∃r', v. join_ack_msg(n, r, r', v)
44     # find the maximal round in which a node in the quorum reported voting, and the corresponding vote;
45     # v is arbitrary when no such node voted.
46     local maxr, v := max{(r', v') | ∃n. member_1(n, q) ∧ join_ack_msg(n, r, r', v') ∧ r' ≠ ⊥}
47     propose_msg(r, v) := true # propose value v
48 }
49 action VOTE(n : node, r : round, v : value) {
50     assume r ≠ ⊥
51     assume propose_msg(r, v)
52     assume ¬∃r', r'', v. r' > r ∧ join_ack_msg(n, r', r'', v)
53     vote_msg(n, r, v) := true
54 }
55 action LEARN(n : node, r : round, v : value, q : quorum_2) {
56     assume r ≠ ⊥
57     # 2b from a phase 2 quorum q:
58     assume ∀n. member_2(n, q) → vote_msg(n, r, v)
59     decision(n, r, v) := true
60 }
```

Figure A.49: RML model of Flexible Paxos.

A.4 Stoppable Paxos

Stoppable Paxos [140] extends Multi-Paxos with the ability for a node to propose a special stop command in order to stop the algorithm, with the guarantee that if the stop command is decided in instance i, then no command is ever decided at an instance $j > i$. Stoppable Paxos therefore enables Virtually Synchronous system reconfiguration [28, 47]: Stoppable Paxos stops in a state known to all participants, which can then start a new instance of Stoppable Paxos in a new configuration (e.g., in which participants have been added or removed); moreover, no pending commands can leak from a configuration to the next, as only the final state of the command sequence is transfered from one configuration to the next.

Stoppable Paxos may be the most intricate algorithm in the Paxos family: as acknowledged by Lamport et al. [140], "getting the details right was not easy". The main algorithmic difficulty in Stoppable Paxos is to ensure that no command may be decided after a stop command while at the same time allowing a node to propose new commands without waiting, when earlier commands are still in flight (which is important for performance). In contrast, in the traditional approach to reconfigurable SMR [142], a node that has c outstanding command proposals may cause up to c commands to be decided after a stop command is decided; those commands need to be passed-on to the next configuration and may contain other stop commands, adding to the complexity of the reconfiguration system.

Before proposing a command in an instance in Stoppable Paxos, a node must check if other instances have seen stop commands proposed and in which round. This creates a non-trivial dependency between rounds and instances, which are mostly orthogonal concepts in other variants of Paxos. This manifest as the instance sort having no incoming edge in the quantifier alternation graph in other variants, while such edges appear in Stoppable Paxos. Interestingly, the rule given by Lamport et al. to propose commands results in verification conditions that are not in EPR, and rewriting seems difficult. However, we found an alternative rule which results in EPR verification conditions. As explained below, this alternative rule soundly overapproximates the original rule (and introduces new behaviors), and, as we have verified (in EPR), it also maintains safety.

Stoppable Paxos uses the same messages and actions as Multi-Paxos, and a special command value *stop*. In addition to the usual consensus property of Paxos, Stoppable Paxos ensures the following safety property, ensuring that nothing is ever decided after a *stop* value

is decided:

$$\forall i_1, i_2 : \mathsf{instance}, n_1, n_2 : \mathsf{node}, r_1, r_2 : \mathsf{round}, v : \mathsf{value}. \\ \mathit{decision}(n_1, i_1, r_1, \mathit{stop}) \wedge i_2 > i_1 \rightarrow \neg \mathit{decision}(n_2, i_2, r_2, v) \tag{A.50}$$

In order to obtain this property, Stoppable Paxos uses an intricate condition in the INSTATE_ROUND action, to ensure that if a *stop* value is choosable at instance i_1, then no value is be proposed for any instance $i_2 > i_1$. Recall that in Multi-Paxos, the owner of round r takes the INSTATE_ROUND action once it has received join-acknowledgment messages from a quorum of nodes. These messages allow to compute the maximal vote (by round number) in each instance, by any node in the quorum. Let m denote the votemap representing these voted (as computed in Figure 4.23 line 45). In Multi-Paxos, these votes are simply re-proposed for round r. However, in Stoppable Paxos we must take special care for *stop* commands. Suppose that for some instance i_1, we have $\mathit{roundof}(m, i_1) \neq \bot \wedge \mathit{valueof}(m, i_1) = \mathit{stop}$. Naively, this suggests we should not propose any value for instances larger than i_1; otherwise, if the stop value is eventually decided at i_i, we will violate the safety property that requires that no value be decided after a stop command. However, it could be that for some $i_2 > i_1$, we also find $\mathit{roundof}(m, i_2) \neq \bot$. Here we face a dilemma: if the stop value at i_1 is eventually decided, proposing at i_2 may lead to a safety violation; but if the stop value at i_1 is in fact not choosable and the value at i_2 is eventually decided, then re-proposing the stop value at i_1 may lead to a safety violation too. As explained in [140], there is a way to ensure that either the stop value at i_1 is choosable or the other value at i_2 is choosable, but not both, and to know which one is choosable. The solution depends on whether $\mathit{roundof}(m, i_2) > \mathit{roundof}(m, i_1)$, in which case the *stop* command for i_1 cannot be choosable and is *voided* by treating it as if $\mathit{roundof}(m, i_1) = \bot$. Otherwise, the value at i_2 cannot be choosable and the owner should propose the *stop* command for i_1, and not propose any other values for instances larger than i_1.

The rule described in [140] is to first compute which *stop* commands are voided, and then to propose all commands except those made impossible by a non-voided *stop* command (i.e., a non-voided stop command at a lower instance). Formalizing this introduces cyclic quantifier alternation over instances, since the condition for voiding a *stop* command involves an existential quantifier over instances. Formally, let m_L denote the votemap obtained from m by voiding according to the rule of [140] (where it is called *sval2a*). m_L is given by:

$$\forall i : \mathsf{instance}.\ (\mathit{valueof}(m_L, i) = \mathit{valueof}(m, i)) \wedge (\varphi_{\mathit{void}} \rightarrow \mathit{roundof}(m_L, i) = \bot) \wedge (\neg\varphi_{\mathit{void}} \rightarrow \mathit{roundof}(m_L, i) = \mathit{roundof}(m, i))$$

where:

$$\varphi_{\mathit{void}} = \exists i' : \mathsf{instance}.\ i' > i \wedge \mathit{roundof}(m, i') \neq \bot \wedge \mathit{roundof}(m, i') > \mathit{roundof}(m, i)$$

Then, the rule in [140] makes proposals for instances i that satisfy the condition condition:

$$\mathit{roundof}(m_L, i) \neq \bot \wedge \forall i' : \mathsf{instance}.\ i' < i \wedge \mathit{valueof}(m_L, i') = \mathit{stop} \rightarrow \mathit{roundof}(m_L, i') = \bot$$

This introduces yet another quantifier alternation cycle, since it must be applied with universal quantification to all instances.

To avoid this cyclic quantifier alternation, we observe that a relaxed rule can be used, and verify a realization of Stoppable Paxos based on our relaxed rule. The relaxed rule avoids the quantifier alternation, and it also provides an overapproximation of the rule of [140]. The relaxed rule is to first find the *stop* command with the highest round in m, and then check if it is voided (by a value at a higher instance and higher round). If it is not voided, then we void all other stop commands, and all proposals at higher instances. If the maximal stop is voided, we void all stop commands, as well as all proposals with higher instances and lower rounds (compared to the *stop* command with the highest round). This rule does not lead to any quantifie alternation cycles.

A.4.1 Model of the Protocol

Our model of Stoppable Paxos in first-order logic appears in Figure A.51. The only actions that differ from Multi-Paxos (Figure 4.23) are INSTATE_ROUND and PROPOSE. The rule described above is implemented in the INSTATE_ROUND action. Line 38 computes m as in Multi-Paxos. Then, line 41 checks if there are any *stop* commands reported in m. If there are no *stop* commands, then line 42 proposes exactly as in Multi-Paxos (Figure 4.23 line 48). If there are *stop* commands in m, then line 45 finds $\mathrm{i_s}$, the instance of the stop command with the highest round present in m. Line 47 then checks if this *stop* command is voided by a value present in m at a higher instance and with a higher round. If so, then line 48 proposes all values from m that are not *stop* commands, excluding those which are voided by the *stop* at $\mathrm{i_s}$, i.e., those at a higher instance and lower round. In case the *stop* at $\mathrm{i_s}$ is not voided, line 51 proposes all non-stop commands until $\mathrm{i_s}$, as well as a *stop* at $\mathrm{i_s}$, and does

```
1  sort node, quorum, round, value, instance, votemap
2
3  relation ≤_r : round, round
4  relation ≤_i : instance, instance
5  axiom Γ_{total order}[≤_r]
6  axiom Γ_{total order}[≤_i]
7  individual ⊥ : round
8  individual stop : instance # the special "stop" command
9  relation member : node, quorum
10 axiom ∀q1, q2 : quorum. ∃n : node. member(n, q1) ∧ member(n, q2)
11
12 relation start_round_msg : round
13 relation join_ack_msg : node, round, votemap
14 relation propose_msg : instance, round, value
15 relation active : round
16 relation vote_msg : node, instance, round, value
17 relation decision : node, instance, round, value
18 function roundof : votemap, instance → round
19 function valueof : votemap, instance → value
20
21 init ∀r. ¬start_round_msg(r)
22 init ∀n, r, m. ¬join_ack_msg(n, r, m)
23 init ∀i, r, v. ¬propose_msg(i, r, v)
24 init ∀r. ¬active(r)
25 init ∀n, i, r, v. ¬vote_msg(n, i, r, v)
26 init ∀r, v. ¬decision(r, v)
27
28 action START_ROUND   # same as Multi-Paxos (Figure 4.23 line 30)
29 action JOIN_ROUND    # same as Multi-Paxos (Figure 4.23 line 34)
30 action VOTE          # same as Multi-Paxos (Figure 4.23 line 55)
31 action LEARN         # same as Multi-Paxos (Figure 4.23 line 59)
32
33 action INSTATE_ROUND(r : round, q : quorum) {
34     assume r ≠ ⊥
35     assume ¬active(r)
36     assume ∀n. member(n, q) → ∃m. join_ack_msg(n, r, m)
37     local m : votemap := *
38     assume ∀i. (roundof(m, i), valueof(m, i)) = max {(r', v') | ∃n, m'. member(n, q) ∧ join_ack_msg(n, r, m') ∧
39                                                                  r' = roundof(m', i) ∧ v' = valueof(m', i) ∧ r' ≠ ⊥ }
40     active(r) := true
41     if (∀i.roundof(m, i) ≠ ⊥ → valueof(m, i) ≠ stop) {  # no stops in m
42         propose_msg(I, r, V) := propose_msg(I, r, V) ∨ (roundof(m, I) ≠ ⊥ ∧ V = valueof(m, I))
43     } else {  # find maximal stop in m and propose accordingly
44         local i_s : instance := *
45         assume roundof(m, i_s) ≠ ⊥ ∧ valueof(m, i_s) = stop ∧
46                 ∀i. (roundof(m, i) ≠ ⊥ ∧ valueof(m, i) = stop) → roundof(m, i) ≤_r roundof(m, i_s)
47         if (∃i. i >_i i_s ∧ roundof(m, i) ≠ ⊥ ∧ roundof(m, i) >_r roundof(m, i_s)) { # maximal stop is voided
48             propose_msg(I, r, V) := propose_msg(I, r, V) ∨
49                 (roundof(m, I) ≠ ⊥ ∧ V = valueof(m, I) ∧ V ≠ stop ∧ (I >_i i_s → roundof(m, I) >_r roundof(m, i_s)))
50         } else { # maximal stop not voided
51             propose_msg(I, r, V) := propose_msg(I, r, V) ∨
52                 (roundof(m, I) ≠ ⊥ ∧ V = valueof(m, I) ∧ I ≤_i i_s ∧ (V = stop → I = i_s))
53         }
54     }
55 }
56 action PROPOSE_NEW_VALUE(r : round, i : instance, v : value) {
57     assume r ≠ ⊥
58     assume active(r) ∧ ∀v. ¬propose_msg(i, r, v)
59     assume ∀i. i ≤_i i → ¬propose_msg(i, r, stop))
60     assume v = stop → ∀i, v. i ≤_i i → ¬propose_msg(i, r, v)
61     propose_msg(r, v) := true
62 }
```

Figure A.51: RML model of Stoppable Paxos.

not propose anything for higher instances.

The PROPOSE action is similar to Multi-Paxos, but contains a few changes as described in [140]. First, we may not propose any value if we have proposed a *stop* command at a lower instance (line 59). Second, we may only propose a stop value at an instance such that we have not already proposed anything at higher instances (line 60).

Our rule for Stoppable Paxos provides an overapproximation of the rule of [140], in that it forces less proposals in the INSTATE_ROUND action. Thus, any behaviour of [140] can be simulated by an INSTATE_ROUND action followed by several PROPOSE actions to produce the missing proposals. We have also verified this using Ivy and Z3, albeit the verification conditions were outside of EPR. Nevertheless, Z3 was able to verify this in under 2 seconds. This is in contrast to verifying the inductive invariant for the version of [140], for which Z3 diverged.

A.4.2 Inductive Invariant

The inductive invariant for Stoppable Paxos contains the inductive invariant for Multi-Paxos (Section 4.5.2), and in addition includes conjuncts that capture the special meaning of *stop* commands, and ensure the additional safety property of eq. (A.50). Below we list these additional conjuncts.

First, the inductive invariant asserts that a *stop* command proposed at some instance forbids proposals of other commands in higher instances *in the same round*:

$$\forall i_1, i_2 : \mathsf{instance}, r : \mathsf{round}, v : \mathsf{value}.\ \mathit{propose\_msg}(i_1, r, \mathit{stop}) \wedge i_2 >_i i_1 \rightarrow \neg \mathit{propose\_msg}(i_2, r, v) \tag{A.52}$$

Next, the inductive invariant connects different rounds via the *choosable* concept (see Section 4.3.2, eq. (4.18)). If any value is proposed, then *stop* command cannot be choosable at lower instances and lower rounds:

$$\begin{aligned}\forall i_1, i_2 : \mathsf{instance}, r_1, r_2 : \mathsf{round}, v : \mathsf{value}, q : \mathsf{quorum}.\ & \mathit{propose\_msg}(i_2, r_2, v) \wedge r_1 <_r r_2 \wedge i_1 <_i i_2 \rightarrow \\ & \exists n : \mathsf{node}, r', r'' : \mathsf{round}, v' : \mathsf{value}. \\ & \quad \mathit{member}(n, q) \wedge \neg \mathit{vote\_msg}(n, r_1, \mathit{stop}) \wedge r' > r_1 \wedge \mathit{join\_ack\_msg}(n, r', r'', v')\end{aligned} \tag{A.53}$$

And, in addition, if *stop* is proposed, than nothing can be choosable at lower rounds and

higher instances:

$$\forall i_1, i_2 : \mathsf{instance}, r_1, r_2 : \mathsf{round}, v : \mathsf{value}, q : \mathsf{quorum}.\ propose\_msg(i_1, r_2, stop) \wedge r_1 <_r r_2 \wedge i_1 <_i i_2 \rightarrow$$
$$\exists n : \mathsf{node}, r', r'' : \mathsf{round}, v' : \mathsf{value}.$$
$$member(n, q) \wedge \neg vote\_msg(n, r_1, v) \wedge r' > r_1 \wedge join\_ack\_msg(n, r', r'', v') \tag{A.54}$$

A.4.3 Transformation to EPR

The quantifier alternation structure of Figure A.51 and the inductive invariant described above is the same as the one obtained for Multi-Paxos. Notice that eqs. (A.53) and (A.54) introduce a quantifier alternation cycle identical to the one introduced by eq. (4.27) for Multi-Paxos (and by eq. (4.18) for single decree Paxos). Thus, the transformation of the Stoppable Paxos model to EPR is identical to that of Multi-Paxos (Section 4.5.3), using the same derived relations *joined_round* and *left_round*, when *left_round* is used to rewrite eqs. (A.53) and (A.54) in the inductive invariant in the same way it was used in Section 4.4. This again shows the reusability of the derived relations across many variants of Paxos.

פרסומים

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מספק הזדמנות ייחודית להוכחת תכונות טמפורליות באמצעות טכניקה חדשה שפותחה בתזה זו, ומוצגת בפרקים 7 ו־8 .

הטכניקה מנצלת את הגמישות של ייצוג מצבים כמבנים של לוגיקה מסדר ראשון, ומאפשרת למנף טכניקות אימות בטיחות קיימות, ואת הטכניקות האחרות שפותחו בעבודה זו, כדי לאמת את החיות ואת המאפיינים הזמניים של פרוטוקולים מבוזרים. אמנם טכניקה שכזו לא יכולה להיות שלמה מסיבות של תורת החישוביות, אבל חיבור זה מציג מספר הוכחות של תכונות טמפורליות של מספר פרוטוקולים מאתגרים, כולל פרוטוקולים שעבורם מחקר זה השיג את ההוכחה הטמפורלית הממוכנת הראשונה.

אימות של אלגוריתם Paxos ואלגוריתמים דומים

עבודה זו מציגה אימות דדוקטיבי באמצעות לוגיקה מסדר ראשון ו־ EPR למספר פרוטוקולים במשפחת Paxos . באופן מפתיע למדי, עבודה זו מראה כי פרוטוקולים מורכבים שכאלה אכן ניתנין לאימות בפרגמנט הכריע.

המחקר המוצג כאן השיג את האימות הראשון של אלגוריתם Paxos באמצעות לוגיקה כריעה. עבור כמה גרסאות, ובהן Vertical Paxos , Fast Paxos , ו־ Stoppable Paxos , עבודה זו מציגה את האימות ממוכן הראשון.

הסקה אינטראקטיבית של אינווראנטים מכומתים אוניברסאלית

כאשר הסקת אינווריאנטים אינה כריעה, וכן כאשר היא כריעה באופן תיאורטי אך טעניקות מעשיות (עדיין) אינן יעילות מספיק כדי לאפשר אוטומציה מלאה, המשתמש חייב להיות מעורב בתהליך של מציאת אינווריאנטים אינדוקטיביים. אפילו במקרים כאלה, עבודה זו מראה כי השימוש בפרגמנט כריע של לוגיקה מסדר ראשון מביא יתרונות נוספים מעבר ליכולת למצוא דוגמאות נגד.

מציאה ידנית של אינווריאנטים אינדוקטיביים היא אחד החלקים היצירתיים והמאתגרים ביותר של אימות דדוקטיבי. עבור המקרה המיוחד של אינווריאנטים אינדוקטיביים מכומתים אוניברסאלית, כלומר בעלי קידומת כמתים $\forall^*$ בצורה פרנקסית (NORMAL FORM PRENEX), עבודה זו מפתחת תהליך אינטראקטיבי המאפשר למשתמש למצוא אינווריאנט אינדוקטיבי. האינטראקציה בתהליך מבוססת על ייצוג גרפי הן של אינווריאנטים מכומתים אוניברסאלית והן של דוגמאות־נגד.

מתודולוגיה זו מוסברת בפירוט בפרק 6 , ומיושמת במערכת Ivy לאימות דדוקטיבי.

אימות של תכונות טמפורליות

תכונות בטיחות (SAFETY PROPERTIES) ניתן להוכיח באמצאות אינווריאנטים אינדוקטיביים. לעומת זאת, תכונות חיות (LIVENESS PROPERTIES) ותכונות טמפורליות (TEMPORAL PROPERTIES) כלליות של מערכות בעלות אינסוף מצבים, מוכחות על פי רוב באמצאות פונקציות דירוג (RANKING FUNCTIONS) לתוך קבוצה סדורה היטב. לעומת זאת, לוגיקה מסדר ראשון ללא תיאוריות לא יכולה לבטא פונקציות דירוג, או את המושג של קבוצה סדורה היטב. לכן, על פניו נראה כי אימות של תכונות טמפורליות אינו בר ביצוע בלוגיקה מסדר ראשון.

תזה זו מתמודת עם אתגר זה ומראה כי להיפך, הפורמליזם של לוגיקה מסדר ראשון

בהקשר כללי, מפתח הגבלות שמספיקות להבטיח שהבעיה כריעה, וגם משיגה תוצאות אי־כריעות שמראות שההגבלות הכרחיות לצורך כריעות. אבחנה מרכזית היא כי בשל הגבלה של אינווריאנטים אינדוקטיביים אפשריים לשפה מסוימת, בעיית ההסקה אינווריאנטים אינדוקטיביים שונה מבעיית אימות תכונות בטיחו, ולכן, עשויה להיות כריעה גם במקרים שבהם אימות בטיחות אינו כריע.

בבעיית ההסקה של אינווריאנטים אינדוקטיביים, התוצאה הצפויה אינה "בטוח / לא בטוח", אלא "אינווריאנט אינדוקטיבי קיים בשפה הנתונה / לא קיים אינווריאנט אינדוקטיבי בשפה הנתונה".

חקירת הכריעות של בעיית ההסקה של אינווריאנטים אינדוקטיביים חשובה כדי לפתח הבנה יסודית טובה יותר של שיטות קיימות להסקת אינווריאנטים ולאימות בטיחות, כדוגמת פירוש מופשט (INTERPRÉTATION ABSTRAITE) [56, 57] ו־ IC3 / PDR [34, 117], שכן שיטות כאלה יכולות רק כדי להסיק אינווריאנט אינדוקטיבי בשפה מסוימת, ולכן הבעיה הבסיסית שהם פותרים היא למעשה לא אימות בטיחות, אלא הסקת אינווריאנטים אינדוקטיביים בשפה נתונה. לכן, אם בעיית ההסקה של אינווריאנטים אינדוקטיביים בהקשר מסוים היא בלתי כריעה, אז כלי אימות אינו יכול להיות שלם אפילו עבור השפה שבה הוא מחפש; לעומת זאת, כאשר הבעיה היא כריעה, ניתן לשאוף לפתח כלים שמהווים אלגוריתם שלם עבור השפה המוגבלת שבה הם מחפשים.

חיבור זה משיג מספר תוצאות כריעות ואי־כריעות עבור בעיית ההסקה של אינווריאנטים אינדוקטיביים, במיוחד עבור שפות של אינווריאנטים אינדוקטיביים מכומתים אוניברסאלית. התוצאות מופיעות בפירוט בפרק 5.

בצורה פרנקסית (PRENEX NORMAL FORM).

בעוד מחלקת ה EPR הקלאסית אינה מאפשרת כל סימני פונקציה או כימות למעט $\exists^*\forall^*$, בעבודה זו אנו משתמשים בהרחבה של המחלקה ללוגיקה מסדר ראשון מרובת סוגים. בפרגמנט מוכלל זה מתאפשרים סימני פונקציה וחילופי כמתים כל עוד הם אינם יוצרים מעגלים.

המגבלה של הפרגמנט המורחב לסימני פונקציה וחילופי כמתים ללא מעגלים בלבד נראית מגבלה די חמורה. אכן, עבור פרוטוקולים רבים שנדונים בחיבור זה, קידוד טבעי בלוגיקה מסדר ראשון מוביל לתנאי אימות המכילים חילופי כמתים מעגליים. כדי להתגבר על כך, עבודה זו מפתחת דרך שיטתית לסלק את המעגלים בצורה נאותה, תוך שימוש ביחסים נגזרים ושכתוב נוסחאות.

טכניקות אלה, המוסברות ביתר פירוט בפרק 4, מאפשרות למשתמש להפוך את המערכת המקורית למערכת שניתן לאמת בפרגמנט הכריע. את הנאותות של השכתוב גם ניתן לבדוק בעזרת הפרגמנט הכריע. לכן, למעשה בעיית האימות המקורית מפוצלת למספר תת־בעיות, שכל אחת מהן ניתנת להבעה בפרגמנט הכריע.

כריעות של הסקת אינווראנטים אינדוקטיביים

בעת שימוש בפרגמנט כריע לצורך מידול מערכות והאינווריאנטים האינדוקטיביים שלהן, הבדיקה האם אינווריאנט הוא אינקודטיבי היא כריעה. לכן, טבעי לחקור את הכריעות של בעיית ההסקה של אינווריאנטים אינדוקטיביים; כלומר, הבעיה של מציאת אינווריאנט אינדוקטיבי מתאים כדי להוכיח כי מערכת נתונה מספקת תכונת נכונות (בטיחות) נתונה. בעיה זו היא פרמטרית בשפה מסדר ראשון (כלומר קבוצה אינסופית של נוסחאות בלוגיקה מסדר ראשון), שמהווה את מרחב החיפוש של אינווריאנטים אינדוקטיביים אפשריים.

חיבור זה בוחן את הכריעות של בעיית ההסקה של אינווריאנטים אינדוקטיביים

ובכל המקרים ראינו שהמאפיינים הדרושים כדי לאמת את הפרוטוקול הם למעשה ברי ביטוי בלוגיקה מסדר ראשון. רעיון זה מפשט מאוד את האימות הדדוקטיבי של פרוטוקולים מבוזרים, שכן הוא מפריד באופן נקי את ההוכחה לחלק שתלוי בגדלים של קבוצות, ולשאר הנכונות הלוגית של הפרוטוקול. זה מאפשר ניסוח נקי של הפרוטוקול והאינווראיאנטים האינדוקטיביים שלו, כמו גם את תנאי האימות, בלוגיקה מסדר ראשון, מה שמאוד מפשט ומקל על תהליך האימות.

טכניקה זו מפורטת בפרק 3.4.

מודלי רשת שונים פרוטוקולים מבוזרים הפועלים באמצעות העברת הודעות עשויים להניח מספר מודלי רשת שונים. דוגמה נפוצה אחת היא של רשת שיכולה לאבד, לשכפל ולשנות סדר של הודעות. דוגמה נוספת היא רשת שיכולה לאבד ולסדר מחדש הודעות, אבל לא יכול לשכפל הודעות. דוגמה נפוצה נוספת היא רשת שיכולה לאבד הודעות, אך ההודעות שכן נמסרות, נמסרות בסדר בו נשלחו.

עבודה זו מנצלת את העובדה כי כל המודלים האלה להתנהגות הרשת ניתנים להבעה בלוגיק מסדר ראשון בדרך טבעית, כפי שאנו מפתחים בפרק 3.5.

סילוק חילופי כמתים מעגליים

בחיבור זה אנו מפתחים השימוש בפרגמנט כריע של לוגיקה מסדר ראשון לצורך אימות דדוקטיבי. הפרגמנט בו אנו משתמשים הוא הכללה של המחלקה הקלאסית הידועה בשם מחלקת Bernays Schönfinkel Ramsey, ומכונה גם effectively propositional reasoning — EPR.

המחלקה הקלאסית מוגבלת לאוצר מילים רלציוני, כלומר נוסחאות מכילות סימני קבוע וסימני יחס אבל לא סימני פונקציה, ולנוסחאות בעלות קידומת כמתים מהצורה $\exists^*\forall^*$

להלן נסקור את הגישה המפותחת בעבודה זו להתמודדות עם אתגר כוח הביטוי של לוגיקה מסדר ראשון. גישה זו מופיעה ביתר פירוט בפרק 3 .

מסלולים דטרמיניסטיים אחד המכשולים העיקריים לשימוש בלוגיקה מסדר ראשון באימות, היא העובדה שהיא אינה יכולה לבטא סגור טרנזיטיבי. כמה עבודות קודמות [224, 225, 196, 195, 134, 114–112] הראו כי מסלולים במבני נתונים מקושרים יכולת הנגישות עבור מבני נתונים מקושרים כגון רשימות מקושרות ועצים יכולים להיות מקודדים בפרגמנט כריע של לוגיקה מסדר ראשון. בחיבור זה אנו מיישמים את אותם רעיונות למידול של מסלולים וסגור טרנזיטיבי בהקשר של אימות של פרוטוקולים מבוזרים.

פרטים נוספים על קידוד זה, וכן תוצאות הנוגעות לשלמות ולנאותות של הקידוד, מופיעים בפרק 3.3.

גדלי קבוצות וקבוצות רוב (QUORUMS) פרוטוקולים מבוזרים רבים משתמשים בקבוצות רוב (QUORUMS) וקבוצות נוספות המוגדרות על פי גודל הקבוצה. לדוגמה, פרוטוקול עשוי לחכות לפחות להודעות מ $\frac{N}{2}$ צמתים כדי לאשר הצעה לפני ביצוע פקודה, כאשר N הוא המספר הכולל של צמתים. זה משמש לעתים קרובות כדי להבטיח עקביות. במודלים של כישלון ביזנטי, סף נפוץ הוא $\frac{2N}{3}$, שבו לכל היותר שליש מהצמתים עלולים להיות ביזנטיים.

לוגיקה מסדר ראשון לא יכולה ללכוד לחלוטין את אילוצים על גדלים של קבוצות. עם זאת, בחיבור זה ננצל את העובדה כי נכונות הפרוטוקול מסתמכת על מאפיינים פשוטים למדי המשתמעים מהספים על גדלי הקבוצות. מאפיינים פשוטים אלה ניתנים להבעה בלוגיקה מסדר ראשון. הרעיון הוא להשתמש בגרסה של קידוד סטנדרטי עבור לוגיקה מסדר שני בלוגיקה מסדר ראשון, בתוספת אקסיומות שמבטאות את המאפיינים שנובעים מספים על גדלי הקבוצות.

עבודה זו מנצלת רעיון זה כדי לאמת פרוטוקולים מבוזרים המשתמשים במגוון ספים,

תרומות עיקריות

להלן נסקור את התרומות העיקריות המוצגות בחיבור זה. תרומות אלה פורסמו ב [186–191]. הטכניקות המפותחות גם ממומשות כחלק ממערכת Ivy [172, 187] — מערכת אימות דדוקטיבי שזמינה בקוד פתוח תחת רישיון MIT [170].

מידול בפרגמנט כריע של לוגיקה מסדר ראשון

עבודה זו מפתחת את השימוש בפרגמנט כריע של לוגיקה מסדר ראשון עבור אימות דדוקטיבי. כדי להשיג זאת, עלינו להיות מסוגלים להביע פרוטוקולים, תכונות נכונות, ואינווריאנטים אינדוקטיביים, בפרגמנט הכריע. אנו משיגים זאת בשני שלבים.

ראשית, עבודה זו מראה שניתן לבטא פרוטוקולים, תכונות נכונות, ואינווראינטים אינדוקטיביים בלוגיקה מסדר ראשון. בשלב זה עולים מספר אתגרים משמעותיים, וחיבור זה מפתח טכניקות להתמודדות איתם.

שנית, אנו מתמודדים עם המגבלות של הפרגמנט הכריע לגבי חילופי כמתים, וכיצד להתגבר על מגבלות אלה.

מידול בלוגיקה מסדר ראשון

הצעד הראשון שלנו הוא להביע פרוטוקולים, תכונות נכונות, ואינווריאנטים אינדוקטיביים, בלוגיקה מסדר ראשון. זה אולי נראה בלתי אפשרי, שכן לוגיקה מסדר ראשון אינה מסוגלת להביע מושגים רבים הנחוצים להוכחה. לדוגמה, לוגיקה מסדר ראשון אינה מסוגלת להביע סגור טרנזיטיבי, אשר חיוני להביע תכונות שקשורות בטופולוגיות רשת (למשל, רשת בטופולוגיה של טבעת). לוגיקה מסדר ראשון גם אינה מסוגלת להביא תכונות של גדלים של קבוצות אשר חיוניים לניתוח של פרוטוקולי הסכמה מבוזרת (CONSENSUS PROTOCOLS).

לוגיקה כריעה עבור אימות דדוקטיבי

חיבור זה בוחן את השימוש בפרגמנט כריע של לוגיקה מסדר ראשון עבור אימות דדוקטיבי. כלומר, תנאי האימות הם מוכחים על־ידי פותרן אוטומטי, ותהליך האימות מתוכנן כך שתנאי האימות יהיו מבוטאים בפרגמנט כריע שנתמך על־ידי הפותרן. השימוש בפרגמנט כריע של לוגיקה מסדר ראשון מביא הן יתרונות תיאורטיים והן יתרונות מעשיים, וכמו כן מספר אתגרים שיש להתמודד איתם.

מנקודת מבט תיאורטית, אם האימות יכול להיות מובנה כך שתנאי האימות מבוטאים בפרגמנט כריע, אז מובטח שהפותרן האוטומטי יצליח בזמן סופי להגיע לתשובה ברורה לשאלה "האם הטענות המקומיות שהוערו בתוכנית מוכיחות את הנכונות הכוללת של התוכנית או לא?".

לפרגמנט שנידון בחיבור זה יש גם תכונת מודל סופי. משמעות הדבר היא כי בכל פעם שהטענות המוערות אינן נכונות או אינן מספקות, יש מבנה סופי (של לוגיקה מסדר ראשון) שמהווה תוגמת נגד לתוקף של הטענות המוערות. פותרן אוטומטי יכול למצוא מודל כזה, ועל פיו ניתן להציג דוגמת נגד קונקרטית למשתמש.

מנקודת מבט מעשית, השימוש בפרגמנט כריע של לוגיקה מסדר ראשון יכול לשפר את הפרודוקטיביות של תהליך האימות. עבור הפרגמנט והיישומים שנידונים בחיבור זה, ניסיון ראשוני מצביע על כך שחוסר היציבות של הפותרן האוטומטי נעלם לחלוטין.

כתוצאה מכך תהליך האימות הוא יותר פרודוקטיבי (ומהנה) באופן משמעותי, שכן המשתמש תמיד מקבל משוב מועיל מהפותרן האוטומטי: או אישור כי הטענות תקפות, או דוגמת נגד מוחשית שמראה ההיפך. דוגמאות הנגד המוחשיות מדריכות ומנחות את המשתמש בכיוון של תיקון הטענות המוערות ומציאת טענות תקפות. המשוב המהיר מוביל למחזור התקדמות מהיר.

את תנאי האימות, אז התוכנית מוכחת כנכונה והאימות מצליח. עם זאת, הפרעות קלות, כגון שינויים בזרע האקראי (RANDOM SEED) שבו נעשה שימוש ביוריסטיקות של הפותרן, יכולים לגרום לפותרן להיכשל באופן בלתי צפוי, והאימות נכשל. ציטוט מ [75] ממחיש את הבעיה מנקודת המבט של מפתחים של מערכות מאומתות:

> הבעיה החוזרת המתסכלת ביותר הייתה חוסר יציבות בהוכחה. [...] אפילו לאחר שתוקנה, ההוכחה יכולה להיכשל שוב עקב הפרעות קלות. חמור מכך, שינויים קלים יכולים לגרום לכשלונות בהוכחות לא קשורות לכאורה.

העובדה שבעיית ההכרעה של תקיפות של תנאי אימות אינה כריעה, משמעה גם שאין אפשרות תיאורטית להבטיח כי פיצול האימות לבעיות קטנות יותר (למשל על־ידי פיצול הקוד למספר מודולים) ישפר את הביצועים של הפותרן. חמור מכך, כאשר הפותרן לא מסוגל להוכיח את התקפות של תנאי האימות, המשתמש נשאר ללא תשובה ברורה האם הטענות שסופקו על־ידו אינן נכונות או מספיקות, או האם הפותרן פשוט לא מצליח להוכיח אותם.

במקרים כאלה הברירה היחידה של המשתמש הנחוש היא לנתח את ריצת היוריסטיקות של הפותרן כדי להבין מה השתבש. זו משימה שרק משתמשים מעטים מסוגלים לה, ואפילו עבורם זה מייגע ודורש זמן רב.

מצב זה חמור אף יותר כשלוקחים בחשבון את האופי ההדרגתי והחזרתי של תהליך פיתוח תוכנה. כלומר, אימות לא יכול להיות מאמץ ענק וחד־פעמי, והוא חייב להיות רציף והמשכי, עם מאמץ סביר לתחזוקה ועדכון כשם שקוד המקור מתפתח.

הוכחה פורמלית.

מוכיחי משפטים אוטומטיים

מוכיחי משפטים אוטומטיים (AUTOMATED THEOREM PROVERS) ופותרני ספיקות ביחס לתיאוריה (SATISFIABILITY MODULO THEORIES – SMT) בדרך כלל תומכים בפורמליזם לוגי מבוסס על לוגיקה מסדר ראשון עם תיאוריות. תיאוריות יכולות לכלול סוגים של אריתמטיקה (שלמים, ממשיים, וקטורי סיביות, ליניארית או לא ליניארית), וכן תיאוריות המאפשרות ניתוח לוגי של מבני נתונים, כגון מערכים, רשימות, מחרוזות, טיפוסי נתונים אלגבריים ועוד.

כלי אימות המבוססים על מוכיחי משפטים אוטומתיים ופותרני ספיקות ביחס לתיאוריה הפכו מוצלחים מאוד בקהילת האימות ושפות התכנות. בעת השימוש בכלים כאלה, המשתמש צריך רק להוסיף טענות לתוכנית, והוכחת התקפות של תנאי האימות המתקבלים מבוצעת באופן אוטומטי לחלוטין, מה שיכול להפוך את ביצוע האימות לקל יותר מעשית לעומת מוכיחי משפטים אינטראקטיביים. בתחום היישום של מערכות מבוזרות, פרוייקט IRONFLEET [101] השתמש בשיטות אלה כדי לפתח יישום מאומת של פרוטוקול MULTI-PAXOS, שנידון ומאומת גם בחיבור זה.

אי־יציבות של מוכיחי משפטים אוטומטיים באימות דדוקטיבי

השימוש במוכיחי משפטים אוטומטיים ופותרני ספיקות ביחס לתיאוריה מפחית באופן משמעותי את נטל ההוכחה המוטל על המשתמש. עם זאת, השימוש בפותרנים אוטומטיים מכניס אי־יציבות רבה לתהליך, שכן האימות הופך להיות רגיש ליוריסטיקות של הפותרן האוטומטי. לאחרונה, פרוייקטים שמשתמשים בפותרנים אוטומטיים לצורך אימותת דדוקטיבי של מערכות מבוזרות כגון IRONFLEET ו־KOMODO [75] זיהו את חוסר היציבות של הפותרנים האוטומטיים שמכשול גדול לאימוץ מעשי של אימות דדוקטיבי. אם הפותרן מצליח לוכיח

1. מוכיחי משפטים אינטראקטיביים (INTERACTIVE THEOREM PROVERS), כלים שבהם ההוכחה נכתבת בעיקר באופן ידני והמחשב בודק כל צעד בהוכחה כדי להבטיח את נכונותה.

2. מוכיחי משפטים אוטומטיים (AUTOMATED THEOREM PROVERS), כלים שמשתמשים ביוריסטיקות כדי לנסות לפתור באופן אוטומטי בעיה בלתי כריעה.

מוכיחי משפטים אינטראקטיביים

מוכיחי משפטים אינטראקטיביים, הידועים גם כעוזרי הוכחה (PROOF ASSISTANTS), בדרך כלל תומכים בפורמליזם לוגי עשיר מאוד מבוסס על תורת טיפוס תלוי (TYPE THEORY DEPENDENT) או לוגיקה מסדר גבוה לוגיקה, אשר יכול לבטא את רוב המתמטיקה. בעוד כלים אלה תומכים בפורמליזם לוגי מאוד אקספרסיבי, הם בדרך כלל דורשים מאמץ ידני גדול. בניית הוכחות מבוססת על המשתמש, בהפעלה אינטראקטיבית של טקטיקות, הפחתת מטרת ההוכחה הנוכחית למטרות פשוטות יותר, והפעלת משפטים ולמות שהוכחו בעבר. זה בעצם דורש מהמשתמש לספק הוכחה מפורטת מאוד, בעוד הכלי מבטיח את הנאותות של ההוכחה. התוצאה היא כי אימות מעשי של מערכות הוא מייגע מאוד ודורש מאמץ ידני עצום.

לדוגמה, פרוייקט seL4 [120] פיתח מיקרו ליבת מערכת הפעלה מאומתת. פרויקט זה דרש מאמץ של 20 שנות אדם, וכולל כ־10,000 שורות קוד של יישום, ולמעלה מחצי מיליון שורות של הוכחה פורמלית.

קרוב יותר לתחום היישום של חיבור זה הוא פרוייקט Verdi [236,238]. אשר אימת מערכת שמיישמת את פרוטוקול Raft [184], פרוטוקול מבוזר מאותה משפחה של Paxos שנידונה גם בחיבור זה. פרויקט Verdi כולל 530 שורות קוד של יישום, ומעל 50,000 שורות

מתאימה לנוסחת המעבר

$$x'=x+1$$

.

שלשת הור (HOARE TRIPLE) מהצורה

$$\{P\}C\{Q\}$$

, קובעת כי אם פקודה C מבוצע במצב בו מתקיימת טענה P אזי המצב לאחר הביצוע יקיים את טענה Q. תוכנית מוערת עם טענות לוגיות מיתרגמת באופן טבעי לאוסף של שלשות הור, אשר נועדו להיות תקפות. תקיפות של שלשות הור ניתנת להפחתה לתקפות של נוסחאות לוגיות, תוך שימוש במשמעות הלוגית של פקודות התוכנית. לכן, לאחר שהתוכנית מוערת בטענות לוגיות, את הבעיה של בדיקת טענות אלה ניתן לתרגם מכנית לבעיה של בדיקת תוקפן של נוסחאות לוגיות, הנקראות תנאי אימות.

קביעת תוקפן של נוסחאות לוגיות גם ניתן למיכון, באמצעות כלים הנקראים מוכיחי משפטים (THEOREM PROVERS), שיכולים להיות אינטראקטיביים או אוטומטיים. גישה זו נודעת בשם אימות תוכנה דדוקטיבי, והיא מאפשרת להוכיח נכונות של תוכניות במהימנות גבוהה יותר מאשר הוכחות ידניות, או "הוכחות נייר".

הבעיה של קביעת התקיפות של תנאי האימות הלוגיים נותרת קשה מבחינה חישובית. הקושי של בעיה זו תלוי בפורמליזם הלוגי שבשימוש. עבור מערכות תוכנה, הטענות הדרושות כדי להוכיח את נכונות התוכנה על פי רוב דורשות פורמליזם לוגי עשיר הכולל כימות, אריתמטיקה, גדלים של קבוצות, סגור טרנזיטיבי, ועוד. כתוצאה מכך, בדיקת תוקף של תנאי האימות היא בדרך כלל אינה כריעה, וברוב המקרים אף אינה ניתנת למניה רקורסיבית. בקהילת האימות, מתמודדים עם מצב זה בשתי דרכים (והשילוב ביניהן):

תקציר

אימות תוכנה הוא אחת הבעיות הבסיסיות במדעי המחשב. באימות תוכנה אנו מבקשים להוכיח, עם ודאות מתמטית, כי תוכנית (או באופן כללי יותר מערכת ממוחשבת) מתנהגתת בצורה נכונה ביחס למפרט נתון. אלן טיורינג התייחס לנושא כבר בשנת 1949 [176, 226], והציע למעשה את מה שמכונה כיום אימות דדוקטיבי, או אימות בשיטת פלויד־הור (Hoare־Floyd), כפי שהוא פותח בעבודות המשפיעות של של פלויד [80] והור [106]. באימות בשיטת פלויד־הור, תוכנית מוערת בעזרת טענות לוגיות מקומיות (assertions) שמהן נובעת הנכונות הגלובלית של התוכנית. כפי שכתב טיורינג במאמרו משנת 1949 "בדיקת שגרה גדולה" [226]:

> איך אפשר לבדוק את השגרה במובן של לוודא שהיא נכונה? בכדי שלמי שיבדוק לא תהיה משימה קשה מדי, המתכנת צריך לטעון מספר טענות ברורות אשר ניתן לבדוק כל אחת מהן בנפרד, ושמהן הנכונות של כל התוכנית נובעת בקלות.

אימות דדוקטיבי מבוסס לוגיקה

באימות דדוקטיבי הטענות מבוטאות בנוסחאות בפורמליזם לוגי המתייחס למצב של התוכנית. לדוגמה, הטענה $x>0$ פירושה שהערך של משתנה התוכנית x חיובי. פקודות התוכנית מקבלות גם הן משמעות לוגית, או באמצעות התחשיב, של דייקסטרה לתנאי מקדים חלש ביותר (Dijkstra's weakest precondition) או לחילופין באמצעות נוסחאות מעבר באוצר מילים כפול המביעות את מעברי המצב בתוכנית. לדוגמה, הפקודה

$$x := x + 1$$

תמצית

אלגוריתמים מבוזרים ומערכות מבוזרות מהווים מרכיב קריטי בתשתיות המחשוב המודרניות. עם זאת, הבטחת נכונות של אלגוריתמים ומערכות כאלה תחת כל התרחישים האפשריים מהווה אתגר משמעותי. אימות פורמלי מספק דרך מתמטית להוכיח כי אלגוריתמים ומערכות הינם נכונים, על־ידי שימוש במחשב כדי לבדוק את כל התרחישים האפשריים ומקרי־הפינה. כך מתקבלת נכונות ברמת וודאות גבוהה יותר משל בדיקות תוכנה, וכן משל הוכחות מתמטיות ידניות. עם זאת, שימוש באימות פורמלי עבור אלגוריתמים מבוזרים ומערכות מבוזרות אינו עניין של מה בכך, שכן בדרך כלל יש להם מרחב מצבים אינסופי, ובאופן כללי בדיקת נכונות אוטומטית אינה כריעה.

חיבור זה מציג חקירה של אימות פורמלי של מערכות בעלות מרחב מצבים אינסופי, ובפרט אלגוריתמים מבוזרים, תוך שימוש בפרגמנט כריע של לוגיקה מסדר ראשון. הרעיון העיקרי הוא לבטא מערכות אינסופיות, תכונות נכונות, ואינווארינרטים אידוקטיביים, בפרגמט כריע של לוגיקה מסדר ראשון. כך, בעיית האימות הבלתי כריעה מפוצלת לתת־בעיות כריעות, שניתנות לפתירה על־ידי פותרנים אוטומטיים קיימים ומבוססים.

תרומות תיאורטיות מפותחות במספר היבטים, כולל כוח הביטוי המפתיע של הפרגמנט הכריע, שאלת הכריעות של בעיית ההסקה של אינווארינאנטים אינדוקטיביים, ושימוש בלוגיקה מסדר ראשון להוכחת תכונות טמפורליות. בהתבסס על רעיונות אלה, מפותחת גם מתודולוגיה מעשית לאימות של תכונות בטיחות ותכונות טמפורליות של מערכות אינסופיות. המתודולוגיה מופעלת לצורך אימות של כמה אלגוריתמים מבוזרים מאתגרים, כולל וריאנטים של אלגורתים Paxos ומספר אלגוריתמים מבוזרים שמאומתים פורמאלית בפעם הראשונה.

אוניברסיטת תל אביב
TEL AVIV UNIVERSITY

הפקולטה למדעים מדוייקים ע"ש ריימונד ובברלי סאקלר

בית הספר למדעי המחשב ע"ש בלבטניק

אימות דדוקטיבי של פרוטוקולים מבוזרים בלוגיקה מסדר ראשון

חיבור לשם קבלת תואר דוקטור לפילוסופיה
מאת

עודד צבי פדון־קורן

עבודה זו הוכנה בהנחייתו של:

פרופסור מולי שגיב

ובהדרכתה של:

דוקטור שרון שוהם

הוגש לסנאט של אוניברסיטת תל אביב

דצמבר 2018

טבת תשע"ט