Brownian motion can feel the shape of a drum

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THE PROBLEM

Inspired by Benjamini, Hollander and Keane, we ask:

Q: Let $X_t$ be a stochastic process on $\mathbb{T}^d$, let $A$ be a set, and let $f$ be the indicator of $A$. Can $A$ be completely reconstructed from the (infinite) trace $f(X_t)$?

DISCRETE CASE

Color the cycle of length $\ell$ with two colors: $G$ and $O$.

If the steps have distribution $\gamma$ and the Fourier coefficients $\{\hat{\gamma}(k)\}_{k=0}^{\ell-1}$ are all distinct, then the coloring can be reconstructed from the trace.

[Matzinger and Lember, 2006]

CONTINUOUS CASE

Let $X_t$ be an infinitely divisible process on $\mathbb{T}^d$ whose increments $D_t$ satisfy

$$D_t(x) = \beta_t \delta(x) + (1 - \beta_t) \gamma_t(x),$$

where $\delta$ is the Dirac distribution and $\gamma$ is an $L^2$ probability density function.

**Theorem:** If the Fourier coefficients $\{\hat{\gamma}_{k_0}\}_{k_0 \in \mathbb{Z}^d}$ are all distinct and non-zero for some $t_0$, then $A$ can be reconstructed from $f(X_t)$.

PROOF SKETCH

From the known temporal correlations

$$T_n(t) = \mathbb{E}[f(X_0) f(X_{t_1}) \cdots f(X_{\sum_{i=1}^n t_i})],$$

we can learn the spatial correlations

$$S_n(y) = \int_{\mathbb{T}^d} f(x) \int_{\mathbb{T}^d} f(x + y_1) \cdots f(x + \sum_{i=1}^n y_i) dx$$

$\infty$ — VANDERMONDE

By applying Parseval's theorem on $T_n$, we can learn

$$\sum_{k \in \mathbb{Z}^d} \left( \prod_{i=1}^n \hat{\gamma}_{t_i}(k_i) \right) m S(k)$$

Now use the following lemma:

**Lemma:** Let $V_{ij} = z_j^i$, where $z_n \in \ell^2 \to 0$, $z_n \neq 0$.

If $x \in \ell^2$ s.t. $Vx = 0$, then $x = 0$.