Formula Satisfaction Testing

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Joint work with

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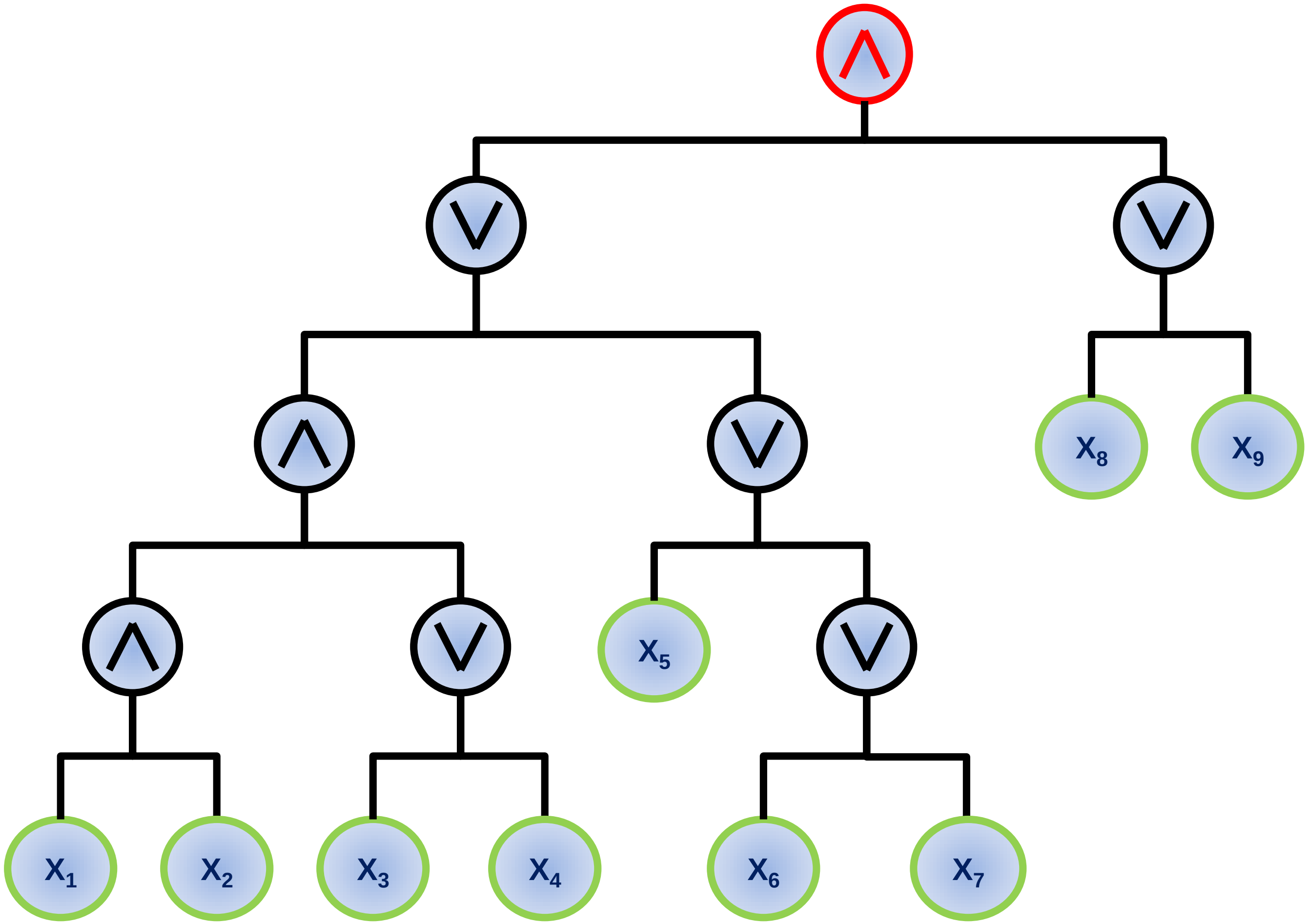
Monotone Binary Read-Once Boolean Formula Φ



<u>Goal</u>: Given full knowledge of the formula, $\varepsilon > 0$ and oracle access to an assignment to Φ, "evaluate" Φ

Related Results

- “*Some 3CNF properties are hard to test*”

  E. Ben-Sasson P. Harsha, S. Raskhodnikova (2003)

- *“Testing Properties of Constraint-Graphs”*

  S. Halevy, O. Lachish, I. Newman, D. Tsur (2007)

- *“Languages that can be accepted by Read-Once Bounded Width Branching Programs are Testable with a Constant Number of Queries”*

  I. Newman (2000)

- *“There exists a language that can be accepted by Read-Twice Bounded Width Branching Program whose query complexity is $\Omega(n)$”*

  E. Fischer, I. Newman, J. Sgall (2002)

- “*Regular Languages are Testable with a Constant Number of Queries*”.

  N. Alon, M. Krivelevich, I. Newman and M. Szegedy (1999)

Some of Our Results

- Read-*once,* binary, Boolean formula

  non-adaptive query complexity: quasi polynomial in $1/\varepsilon$

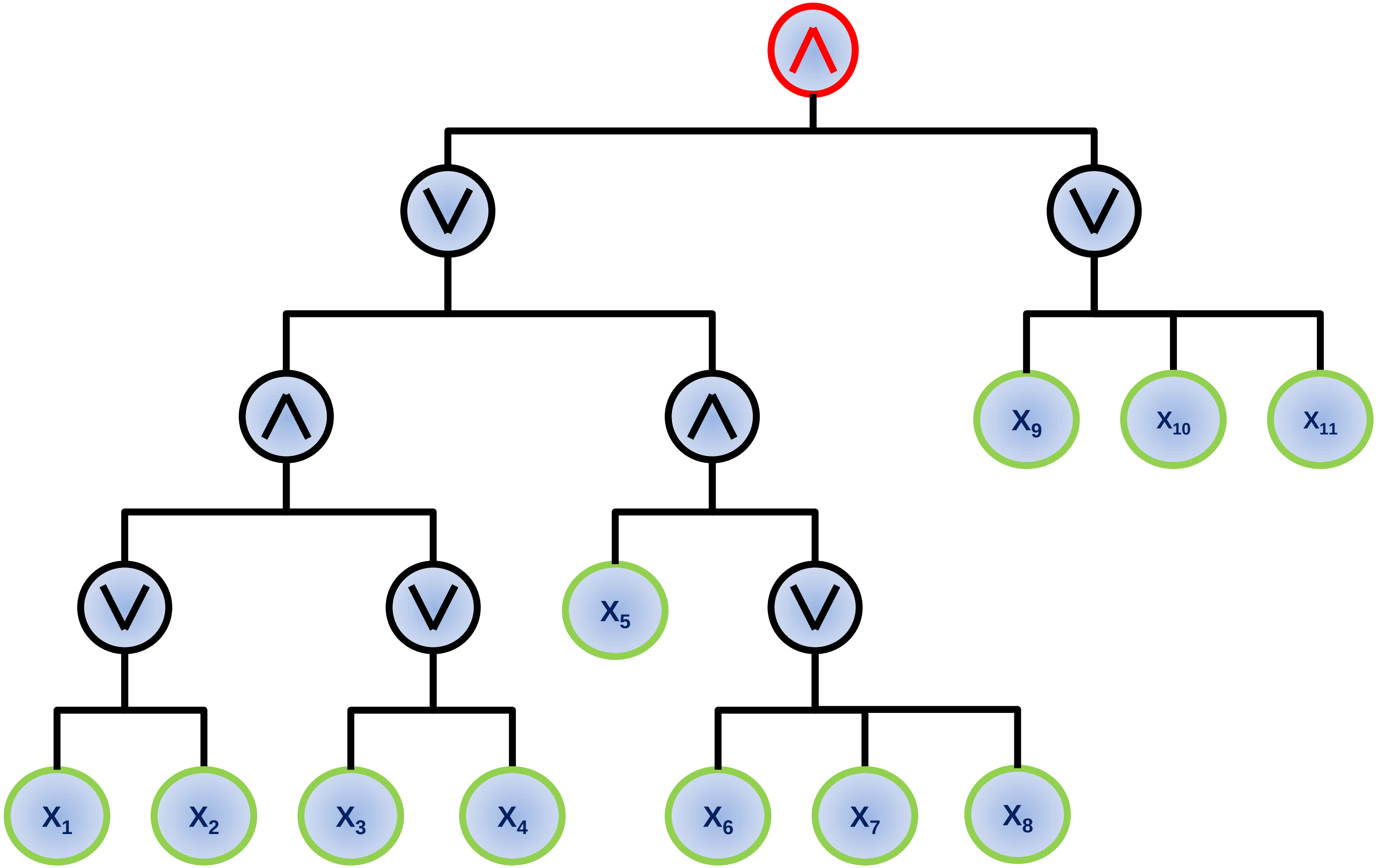
- Read-*k*-times, **monotone,** binary Boolean formula

  non-adaptive query complexity: quasi polynomial in k/ε

- Read-*once,* binary, formula over alphabet size four,

  query complexity may depend on formula size

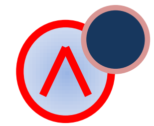
Testing satisfaction ("exponential algorithm")



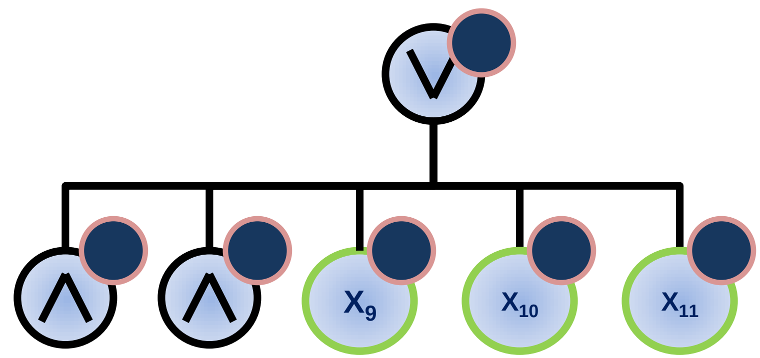
<u>Assumption</u>: formula consists of interleaved AND, OR layers

Token Phase

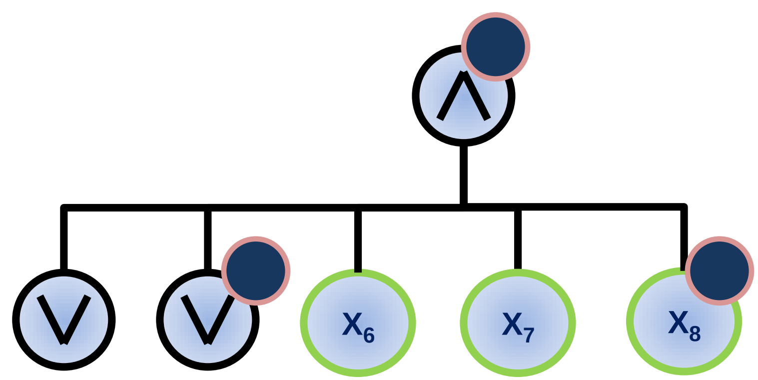
- Put a token on output



- For a gate *v* at depth at most *poly(1/ε)* that have a token
  - OR with at most *1/ε* children children, each one of its children is given a token.

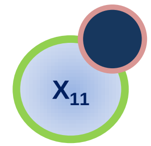


  - AND, select a child so that the probability of a child *u* being selected is *|u|/ |v|.* Repeat independently *O(1/ε)* times.

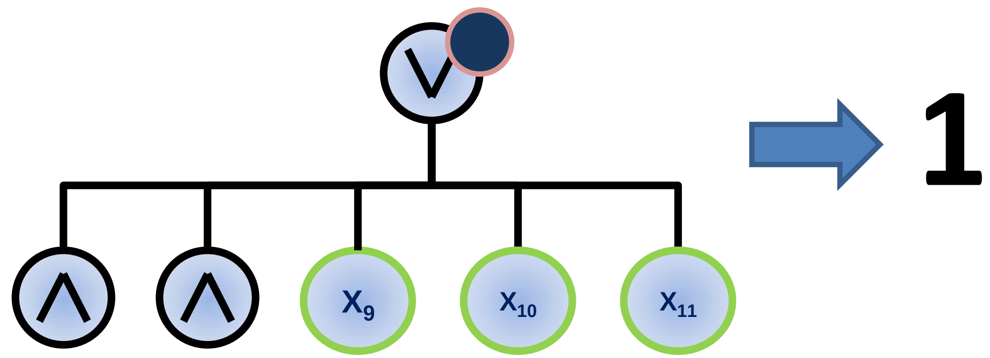


Evaluation Phase

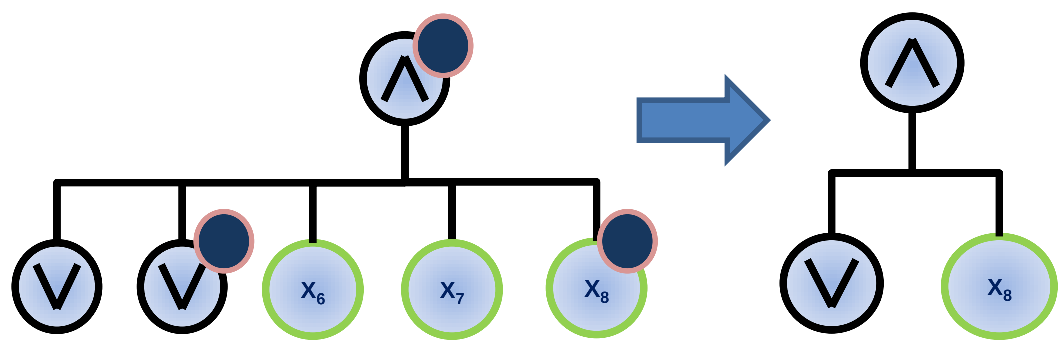
- Variable with token - Query



- Evaluate to *1 :*
  - Gate with a token and all of its children without a token,
  - OR with a child that has less than *ε* an *ε* fraction of its parents variables,



- AND with token yet all of its children without a token – ignore children without a token



The rest of the evaluation is done in the standard way. The result is the output.

Analysis

- Query complexity $\left(\frac{1}{\varepsilon}\right)^{O\left(\frac{1}{\varepsilon}\right)}$

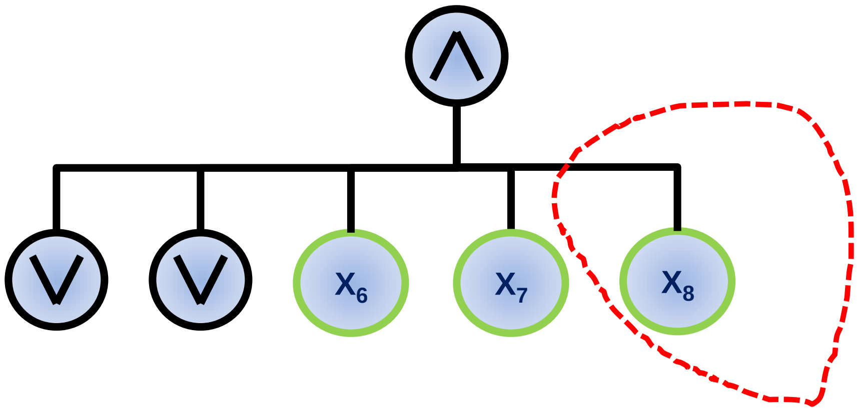
- A *0* evaluation, is a witness that the formula is not satisfied

- If formula satisfied then there is no *0*-witness

? Assignment is far from satisfying the formula then w.h.p a *0*-witness is found

Testing satisfaction, *0*-witness

Assume assignment is *ε–far* from satisfying the output



- Output AND, then with w.h.p. the assignment is *ε(1- ε /10)* from satisfying one of its children that got a token

- Output OR, then for each one of its children *u* the assignment is *ε(1+ ε)* far from satisfying u

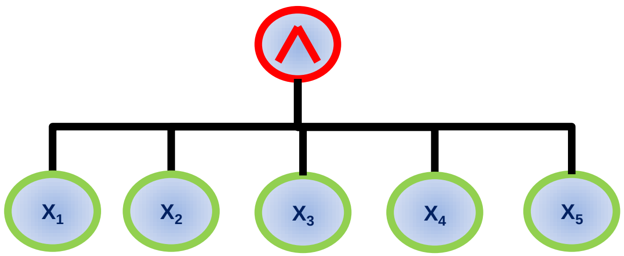
Testing satisfaction, *0*-witness

<u>Lemma</u>:

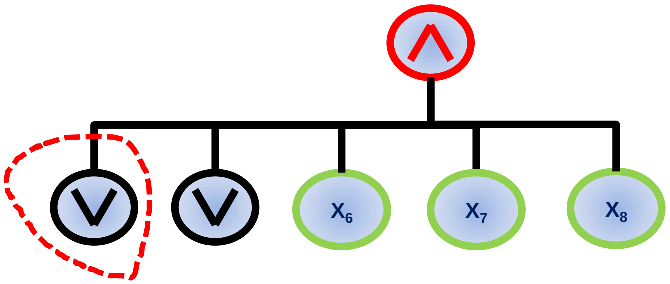
- If the assignment is ε–far from satisfying the formula then w.h.p a *0*-witness is found

Proof

- If $\varepsilon > ½$

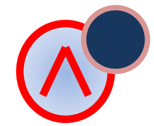


- Assume true for every $\varepsilon>\delta>1/2$
- Let $\varepsilon > \delta(1- \delta/2)$
  - If the output is an OR gate then for each one of its children *u* the assignment is δ far from satisfying u
  - If the output is an AND gate the assignment is $\varepsilon(1- \varepsilon/10)$ from satisfying one of its children that got a token. For that child we are back to the OR case (if it is a variable we are done)



Token Phase Revisit

- Put a token on output



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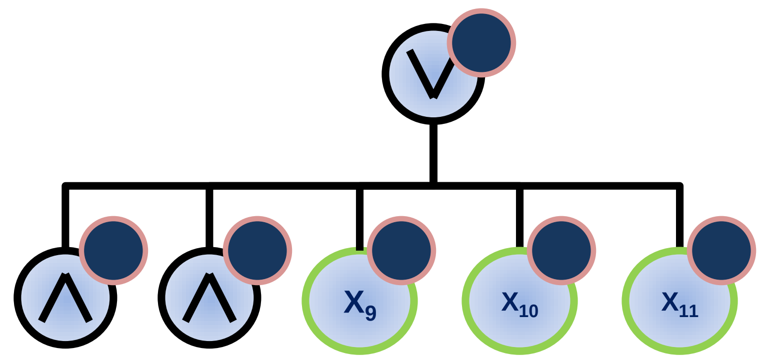

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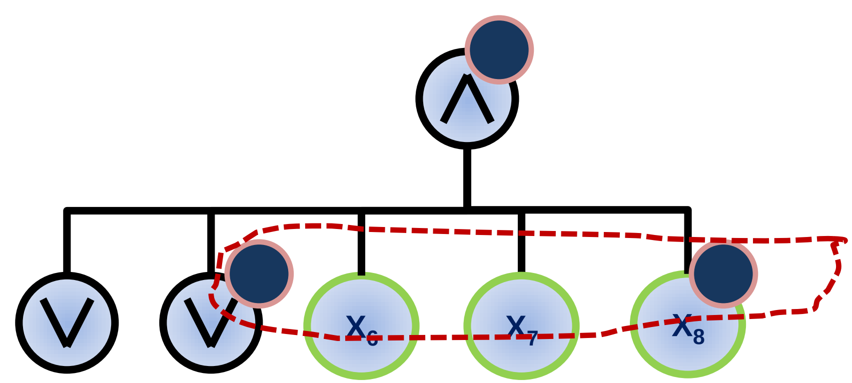
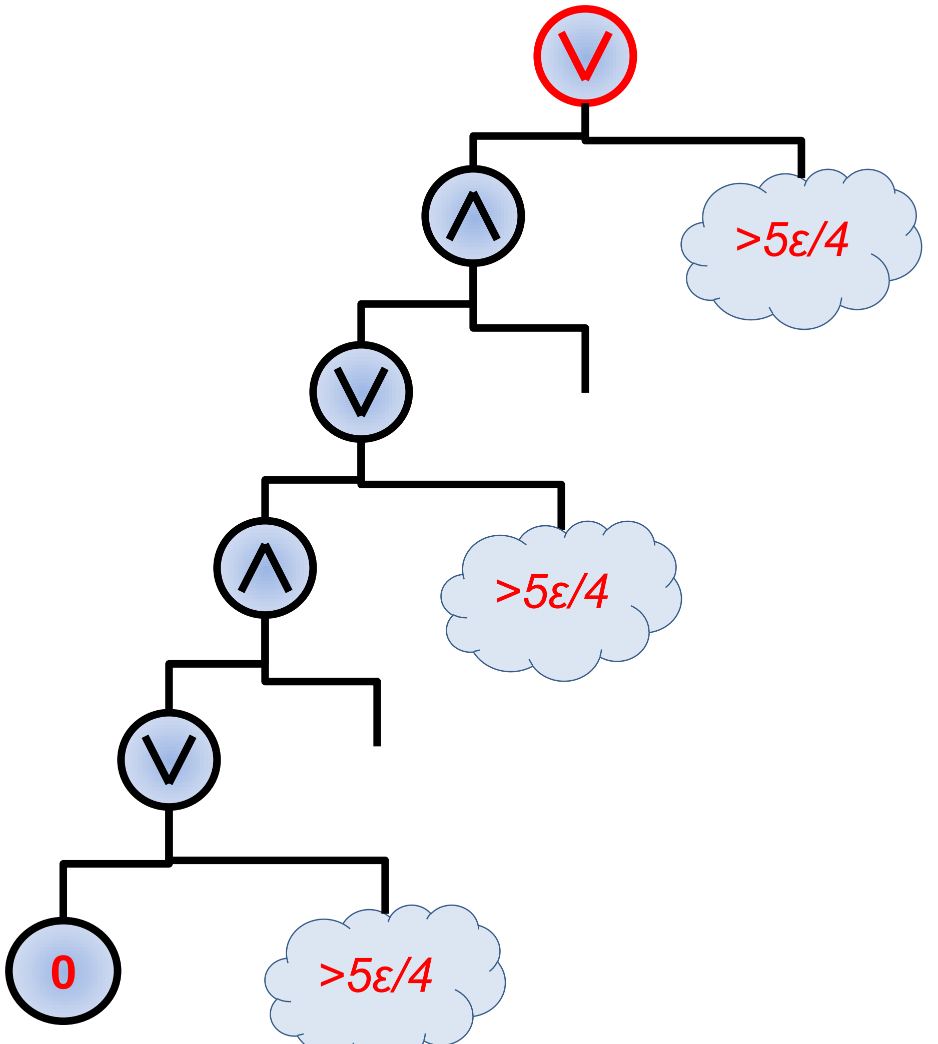


Quasi Polynomial test

Idea:

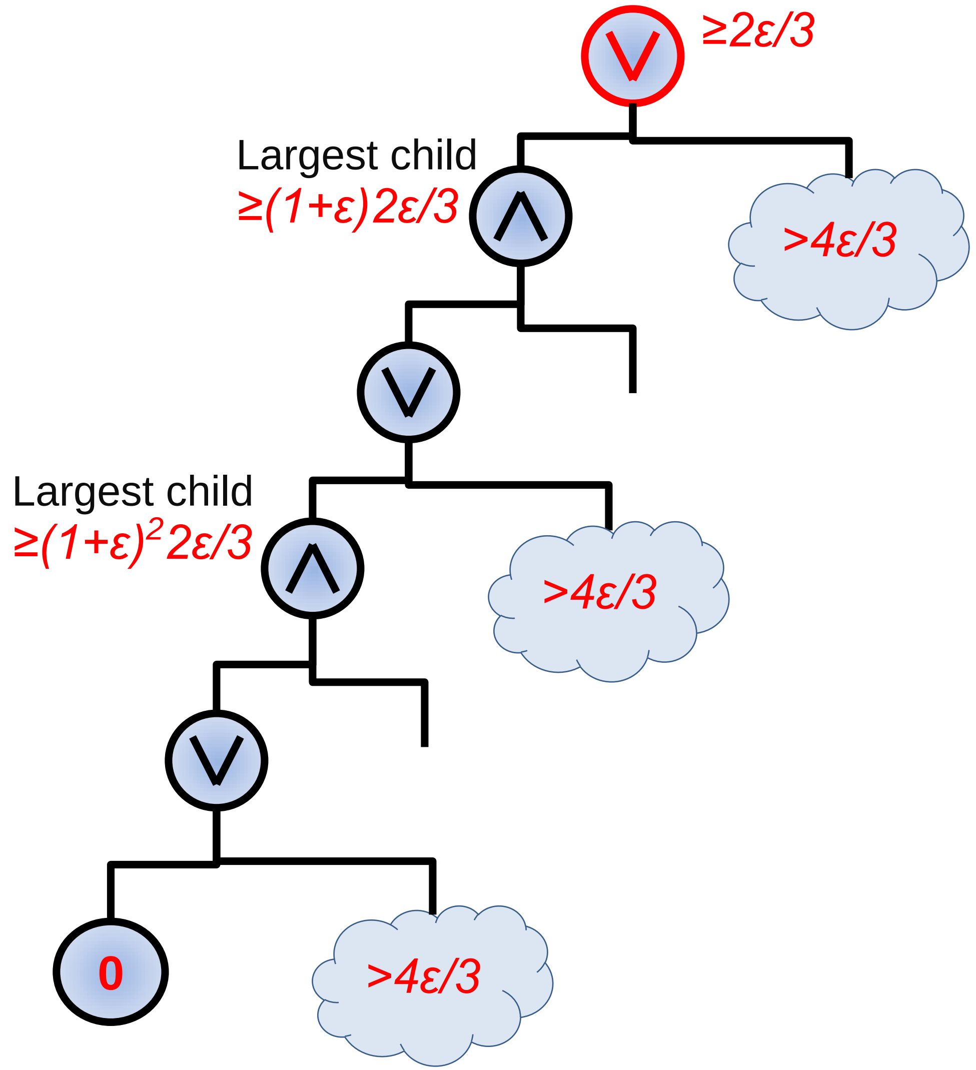
Critical variable



Quasi Polynomial test

Idea:

Critical variable exists



Enough Critical

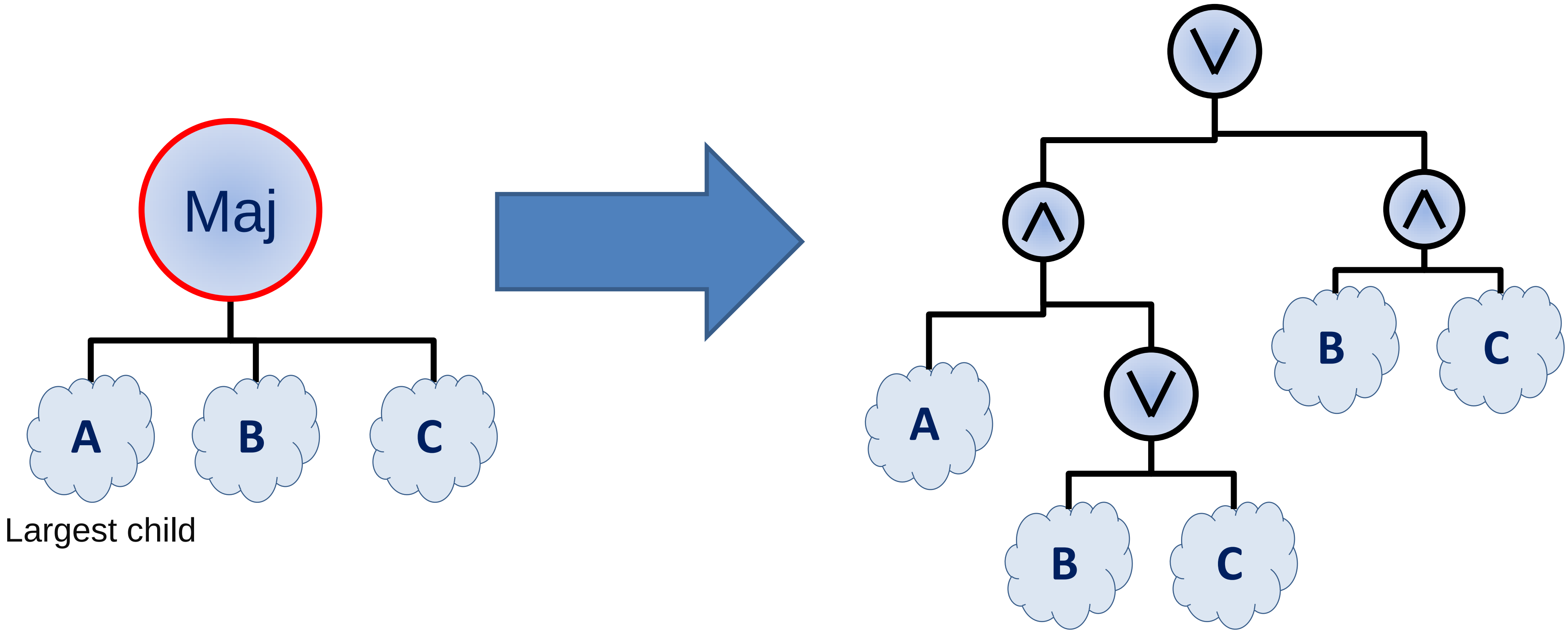
Lemma:

- If assignment is "$2\varepsilon/3$ - far" from satisfying Φ then there exists a critical variable for the assignment.

Observation:

- If the assignment is "ε - far" from satisfying Φ then $\varepsilon/3$ of the variables are critical for the assignment.

Monotone Ternary Gates



Maj(A,B,C) = (A∧(B∨C)) ∨(B ∧ C)

Open Problems

- Improve upper bound or prove lower bound (ideally *poly(1/ε)* upper bound)
- More type of gates in the non binary case
- Lower bound for ternary alphabet

Thank You