Submodular Maximization in a Data Streaming Setting

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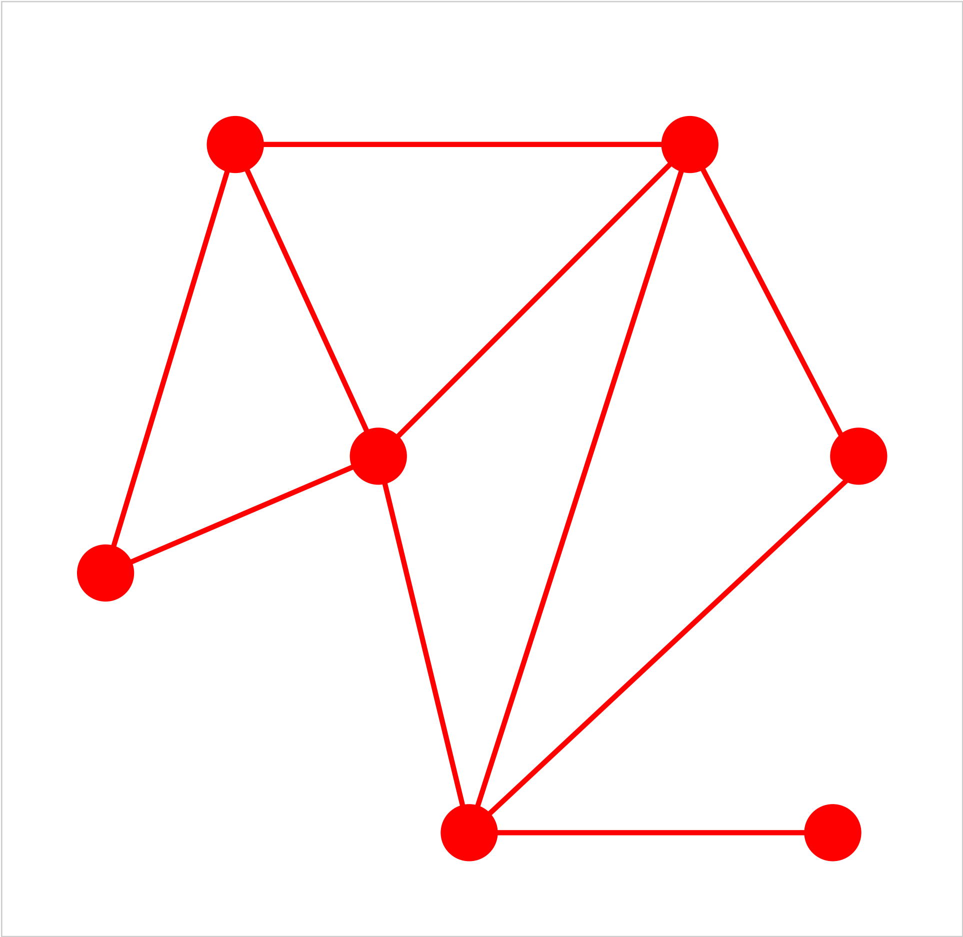
Based on joint work with Sagar Kale

Sublinear Algorithms Workshop
Bertinoro, May 2014

Maximum Matching



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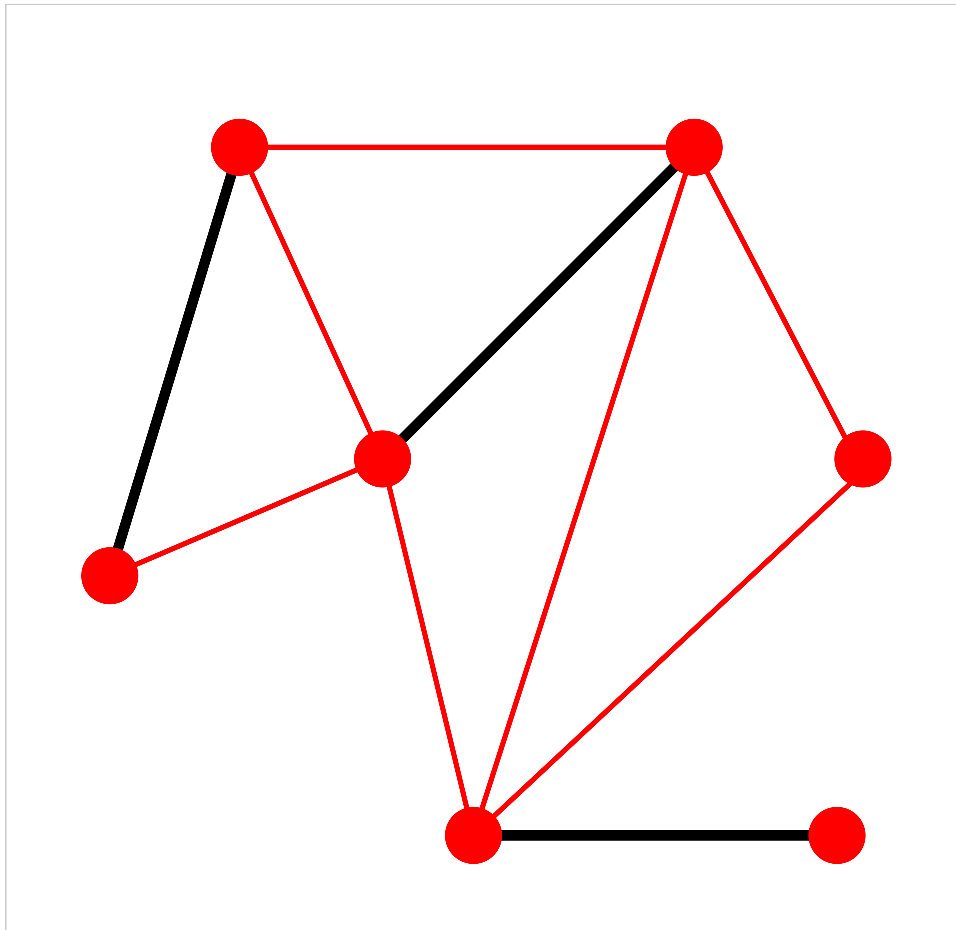

The cardinality version

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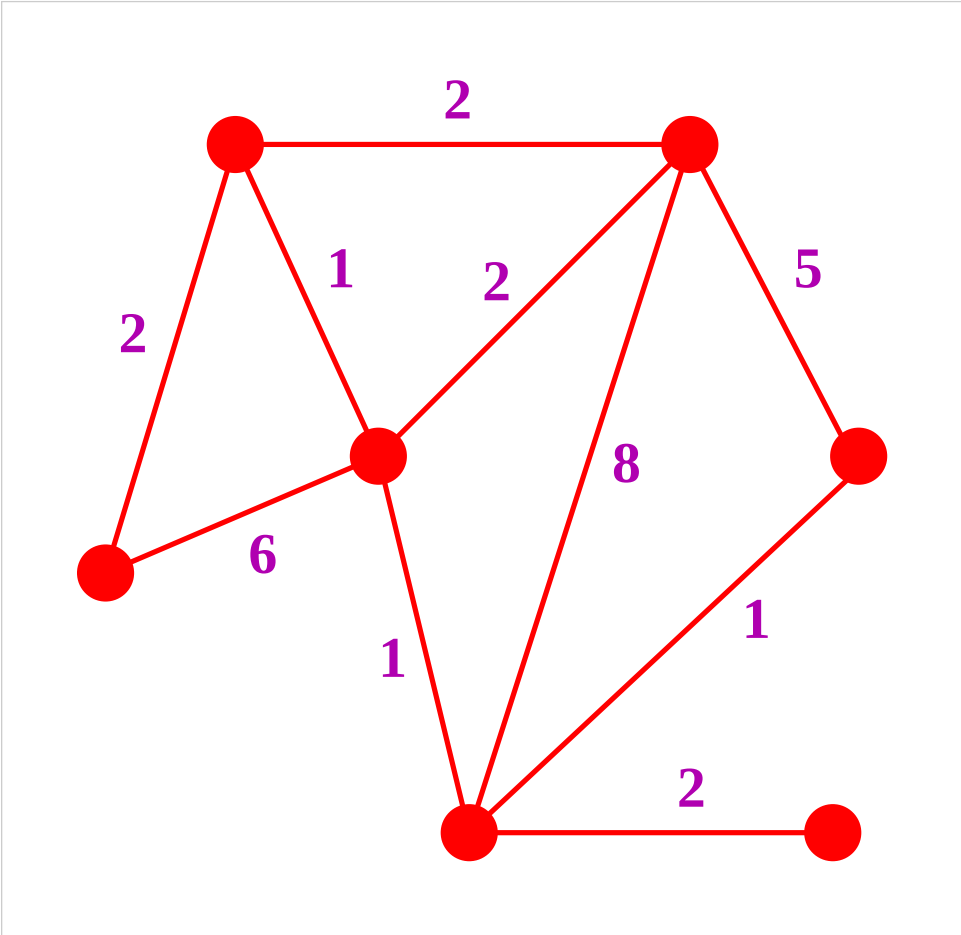

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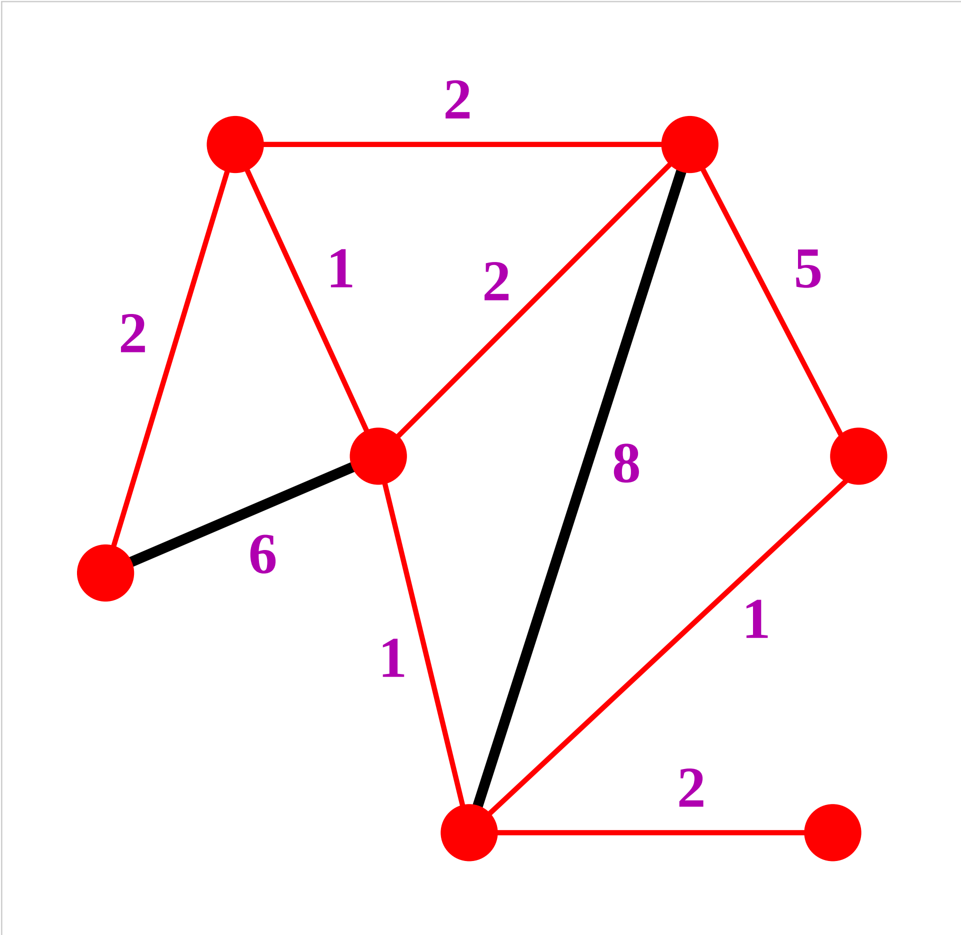

The weighted version

Maximum Matching in a Graph Stream

Maximum cardinality matching (MCM)

- Input: stream of edges $(u, v) \in [n] \times [n]$
- Describes graph $G = (V, E)$: n vertices, m edges, undirected, simple
- Each edge appears exactly once in stream
- Goal
  - Output a matching $M \subseteq E$, with $|M|$ maximal

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- Goal
  - Output a matching $M \subseteq E$, with $|M|$ maximal
  - Use sublinear (in m) working memory
  - Ideally $O(n \operatorname{polylog} n)$ ... "semi-streaming"
  - Need $\Omega(n \log n)$ to store M

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Maximum weight matching (MWM)

- Input: stream of weighted edges $(u, v, w_{uv}) \in [n] \times [n] \times \mathbb{R}^+$
- Goal: output matching $M \subseteq E$, with $w(M) = \sum_{e \in M} w(e)$ maximal

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Maximum submodular-function matching (MSM) $\leftarrow$ this talk

- Input: unweighted edges (u, v), plus submodular $f : 2^E \to \mathbb{R}^+$
- Goal: output matching $M \subseteq E$, with $f(M)$ maximal

Maximum Submodular Matching

Input

- Stream of edges $\sigma = \langle e_1, e_2, \ldots, e_m \rangle$
- Valuation function $f : 2^E \to \mathbb{R}^+$
  - Submodular, i.e., $\forall\, X \subseteq Y \subseteq E\ \forall\, e \in E$

$$f(X + e) - f(X) \geq f(Y + e) - f(Y)$$

  - Monotone, i.e., $X \subseteq Y \implies f(X) \leq f(Y)$
  - Normalized, i.e., $f(\varnothing) = 0$
- Oracle access to f: query at $X \subseteq E$, get $f(X)$
  - May only query at $X \subseteq$ (stream so far)

Goal

- Output matching $M \subseteq E$, with $f(M)$ ~~maximal~~ "large"
- Store $O(n)$ edges and f-values

Our Results

Can't solve MSM exactly

- MCM, approx $< e/(e-1) \implies$ space $\omega(n \operatorname{polylog} n)$ [Kapralov'13]
- Offline MSM, approx $< e/(e-1) \implies n^{\omega(1)}$ oracle calls [this work]
  - Via cardinality-constrained submodular max [Nemhauser-Wolsey'78]

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Our results, using $O(n)$ storage:

Theorem 1 MSM, one pass: 7.75-approx

Theorem 2 MSM, $(3+\varepsilon)$-approx in $O(e^{-3})$ passes

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More importantly:

Meta-Thm 1 Every compliant MWM approx alg $\rightarrow$ MSM approx alg

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More importantly:

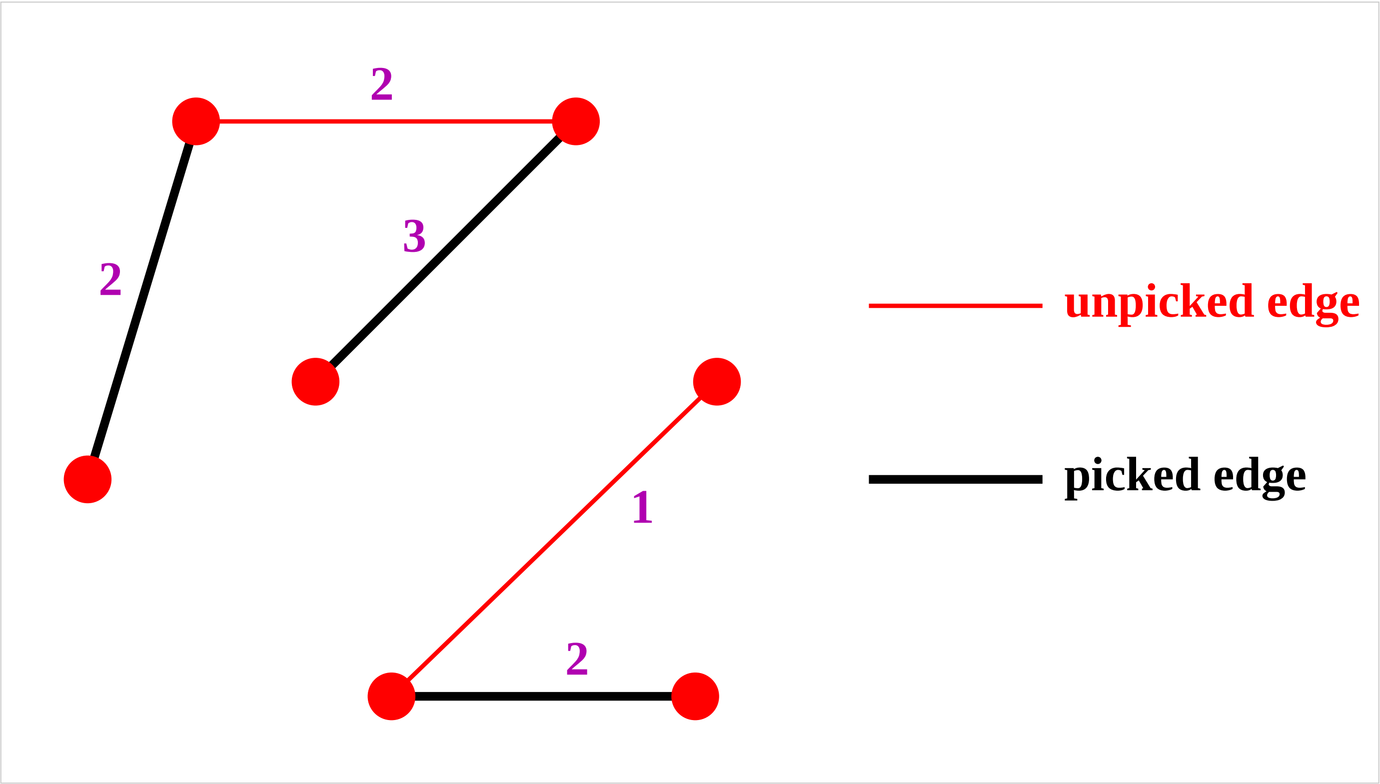
Meta-Thm 1 Every compliant MWM approx alg $\to$ MSM approx alg

Meta-Thm 2 Similarly, max weight independent set (MWIS) $\to$ MSIS

Some Previous Work on MWM

Greedy maximal matching: 2-approx for MCM, useless for MWM

Maintain "current solution" M, update if new edge improves it



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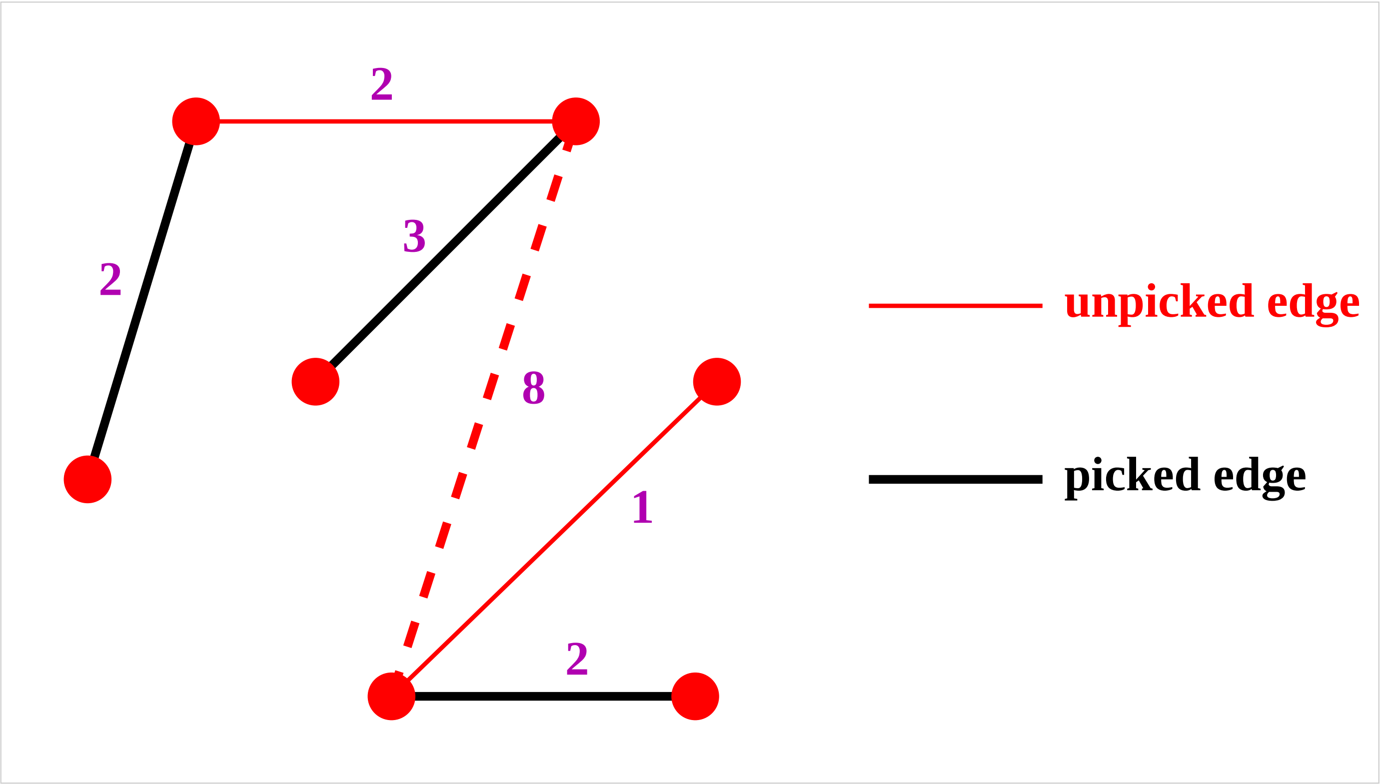

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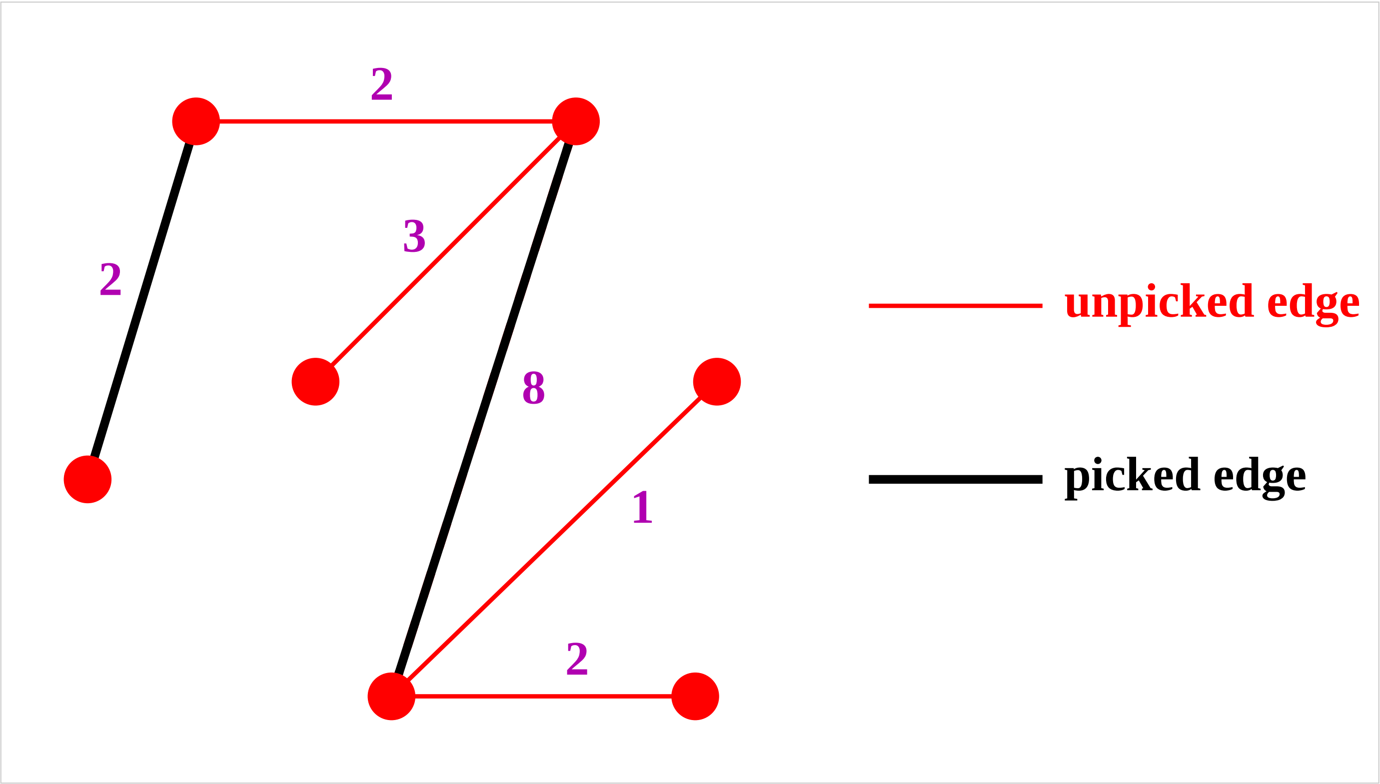

Compliant Algorithms for MWM

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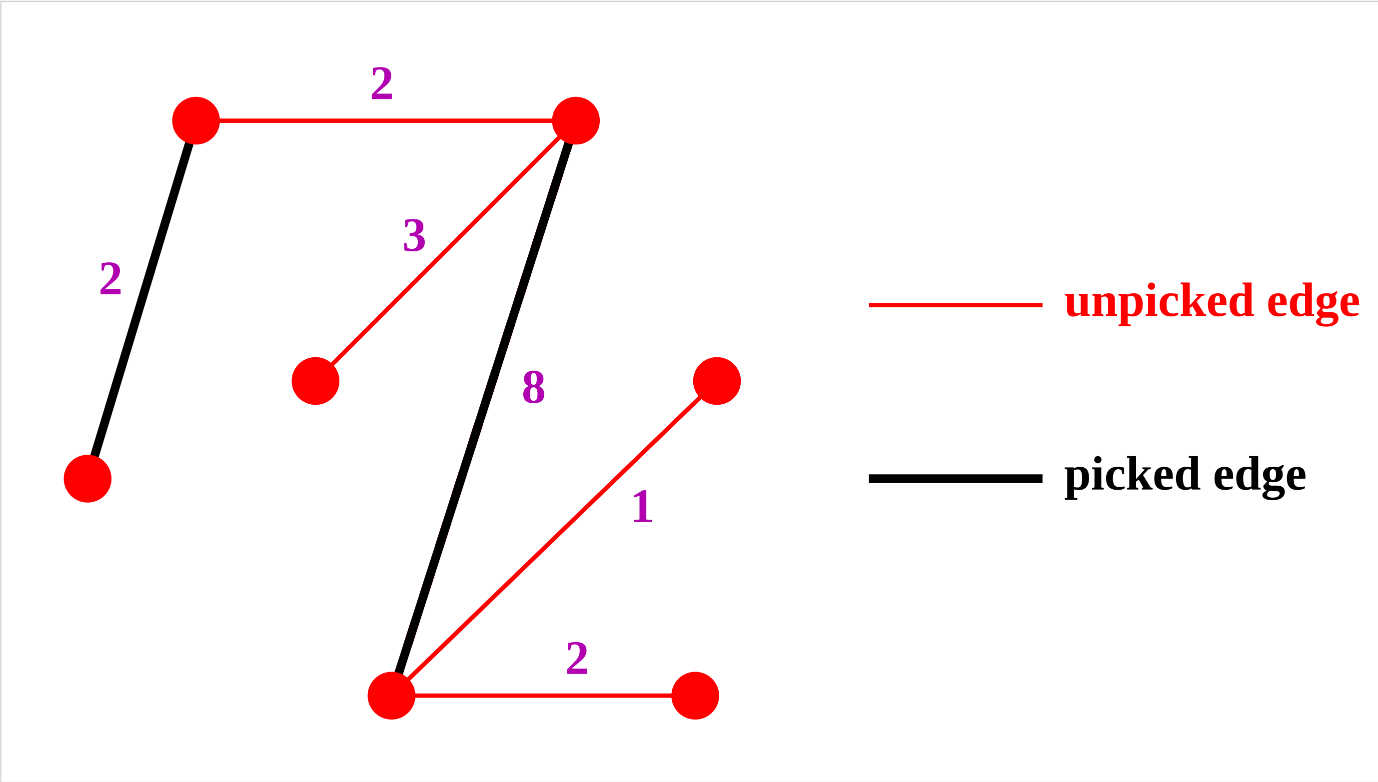

What if input is path with edge weights $1+\varepsilon, 1+2\varepsilon, 1+3\varepsilon, \ldots$?

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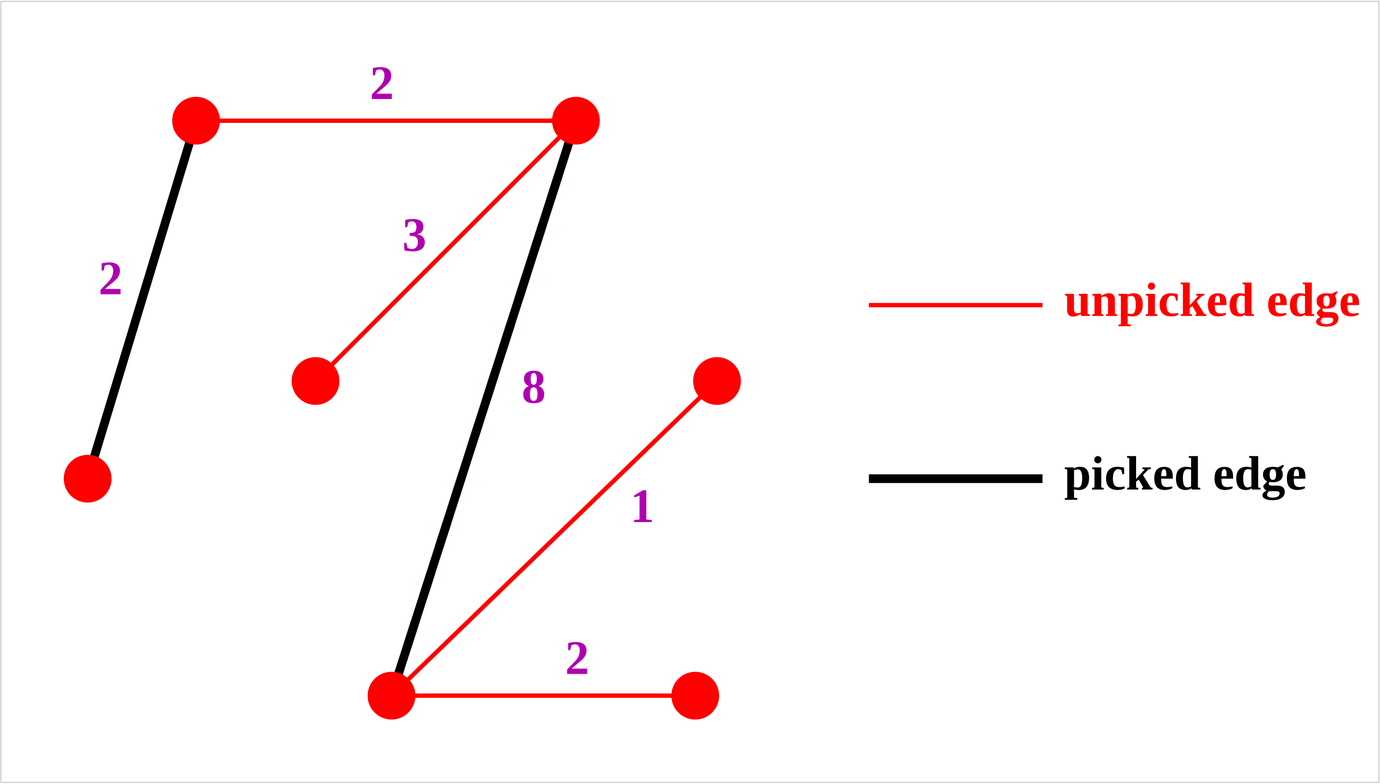

What if input is path with edge weights $1+\varepsilon, 1+2\varepsilon, 1+3\varepsilon, \ldots$?

Update M only upon <u>sufficient</u> improvement

Examples of Compliant Algorithms for MWM

Update of "current solution" M

- Given new edge e, pick "augmenting pair" (A, J)
  - $A \leftarrow \{e\}$
  - $J \leftarrow M \Cap A$ ... edges in M that conflict with A
  - Ensure $w(A) \geq (1+\gamma)w(J)$
- Update $M \leftarrow (M \setminus J) \cup A$

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Choice of gain parameter

- $\gamma = 1$, approx factor 6 [Feigenbaum-K-M-S-Z'05]
- $\gamma = 1/\sqrt{2}$, approx factor 5.828 [McGregor'05]

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- $\gamma = 1.717$, approx factor 5.585 [Zelke'08]

Examples of Compliant Algorithms for MWM

Update of "current solution" M + pool of "shadow edges" S

- Given new edge e, pick "augmenting pair" (A, J)
  - ~~$A \leftarrow \{e\}$~~ $A \leftarrow$ "best" subset of 3-neighbourhood of e
  - $J \leftarrow M \Cap A$ ... edges in M that conflict with A
  - Ensure $w(A) \geq (1 + \gamma)w(J)$
- Update $M \leftarrow (M \setminus J) \cup A$
- Update $S \leftarrow$ appropriate subset of $(S \setminus A) \cup J$

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Generic Compliant Algorithm and f-Extension for MSM

procedure PROCESS-EDGE(e, M, S, γ)

  $(A, J) \leftarrow$ a well-chosen augmenting pair for M
    with $A \subseteq M \cup S + e$, $w(A) \geq (1+\gamma)w(J)$

  $M \leftarrow (M \setminus J) \cup A$

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MWM alg $\mathcal{A}$ + submodular f $\rightarrow$ MSM alg $\mathcal{A}^f$ (the f-extension of $\mathcal{A}$)

Generic Compliant Algorithm and f-Extension for MSM

procedure $\textsc{Process-Edge}(e, M, S, \gamma)$
    $w(e) \leftarrow f(M \cup S + e) - f(M \cup S)$
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Generic Compliant Algorithm and f-Extension for MSM

function IMPROVE-SOLUTION(σ, M^0, γ)
    $M \leftarrow \varnothing$, $S \leftarrow \varnothing$
    foreach $e \in M^0$ in arbit order **do** $w(e) \leftarrow f(M+e) - f(M)$, $M \leftarrow M + e$
    foreach $e \in \sigma \setminus M^0$ in the σ order **do** PROCESS-EDGE(e, M, S, γ)
    return M

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MWIS (arbitrary ground set E, independent sets $\mathcal{I} \subseteq 2^E$) + f $\rightarrow$ MSIS

Generalize: Submodular Maximization (MWIS, MSIS)

function Improve-Solution(σ, I^0, γ)
 $I \leftarrow \varnothing$, $S \leftarrow \varnothing$
 foreach $e \in I^0$ in arbit order **do** $w(e) \leftarrow f(I+e) - f(I)$, $I \leftarrow I + e$
 foreach $e \in \sigma \setminus I^0$ in the σ order **do** Process-Element(e, I, S, γ)
 return I

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MWIS (arbitrary ground set E, independent sets $\mathcal{I} \subseteq 2^E$) + f $\rightarrow$ MSIS

Analysis of MWIS Algorithm (One Pass)

Let $I^* = \operatorname{argmax}_{I \in \mathcal{I}} f(I)$, $I^1 =$ output at end of pass

Let $K = \{e \in E : e$ was added to $I\} \setminus I^1$

Lemma 1 $w(I^1) \leq f(I^1)$

Lemma 2 $w(K) \leq w(I^1)/\gamma$

Lemma 3 $f(I^*) \leq (1/\gamma + 1) f(I^1) + w(I^*)$

Conclusion $\mathcal{A}$ is C_γ-approx $\implies$ $\mathcal{A}^f$ is $(C_\gamma + 1 + 1/\gamma)$-approx

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Then $I_e \cup S_e \supseteq \{x \in I^1 : x \prec e\} =: I^1_{\prec e}$ ("$\prec$": precedes in stream)

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$= w(e)$ (definition of w)

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Sum this over $x \in I^1$ in stream order, telescope *QED*

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Proof of Lemma 2: Let $(A_e, J_e) =$ augmenting pair chosen on reading e

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$\implies w(I^1)/\gamma \geq \sum_e w(J_e)$ (sum up)

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Each element in K was removed at some point

So, $K \subseteq \bigcup_e J_e \implies w(K) \leq \sum_e w(J_e) \leq w(I^1)/\gamma$ *QED*

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Proof of Lemma 3: Similar in spirit: submodularity, telescoping sums...

But a more involved argument

Uses Lemma 1 and Lemma 2

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Lemma 2 $w(K) \leq w(I^1)/\gamma$

Lemma 3 $f(I^*) \leq (1/\gamma + 1) f(I^1) + w(I^*)$

Conclusion $\mathcal{A}$ is C_γ-approx $\implies$ $\mathcal{A}^f$ is $(C_\gamma + 1 + 1/\gamma)$-approx

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Proof of Conclusion: $\mathcal{A}$ gives C_γ-approx for MWIS, so $w(I^*) \leq C_\gamma w(I^1)$

Analysis of MWIS Algorithm (One Pass)

Let $I^* = \text{argmax}_{I \in \mathcal{I}} f(I)$, $I^1 =$ output at end of pass

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Proof of Conclusion: $\mathcal{A}$ gives C_γ-approx for MWIS, so $w(I^*) \leq C_\gamma w(I^1)$

So, $f(I^*) \leq (1/\gamma + 1)f(I^1) + C_\gamma w(I^1)$ (by Lemma 3)

$\leq (1/\gamma + 1 + C_\gamma)f(I^1)$ (by Lemma 1)

QED

Applications of the Paradigm

1. Zelke's compliant algorithm for MWM has $C_\gamma = 3 + 2\gamma + \frac{1}{\gamma} - \frac{\gamma}{(1+\gamma)^2}$
   Take f-extension, set $\gamma = 1$ (optimal), get 7.75-approx to MSM

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   Take f-extension, set $\gamma = 1$ for first pass, $\gamma = \varepsilon/3$ for other passes

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- Recently: more sophisticated local search

  Gives $(2+\varepsilon)$-approx for MSM [Feldman-Naor-Schwartz-Ward'11]

Further Applications: Hypermatchings

Stream of hyperedges $e_1, e_2, \ldots, e_m \subseteq [n]$, each $|e_i| \leq p$

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Hypermatching $=$ subset of pairwise disjoint edges

Multi-pass MSM algorithm (compliant)

- Augment using only current edge e
- Use $\gamma = 1$ for first pass, $\gamma = \varepsilon/(p+1)$ subsequently
- Make passes until solution doesn't improve much

Results

- $4p$-approx in one pass
- $(p + 1 + \varepsilon)$-approx in $O(\varepsilon^{-3})$ passes

Further Applications: Maximization Over Matroids

Stream of elements $e_1, e_2, \ldots, e_m$ from ground set E

Matroids $(E, \mathcal{I}_1), \ldots, (E, \mathcal{I}_p)$, given by circuit oracles:

$$\text{Given } A \subseteq E, \text{ returns } \begin{cases} \smile, & \text{if } A \in \mathcal{I}_i \\ \text{a circuit in } A, & \text{otherwise} \end{cases}$$

Independent sets, $\mathcal{I} = \bigcap_i \mathcal{I}_i$; size parameter $n = \max_{I \in \mathcal{I}} |I|$

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Recent MWIS algorithm (compliant) [Varadaraja'11]

- Augment using only current element e
- Remove $J = \{x_1, \ldots, x_p\}$,
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Follow paradigm: use f-extension of above algorithm

Results, using $O(n)$ storage

- $4p$-approx in one pass
- $(p + 1 + \varepsilon)$-approx in $O(\varepsilon^{-3})$ passes *

---

* Multi-pass analysis only works for partition matroids

Conclusions

- Identified framework (compliant algorithms) capturing several semi-streaming algorithms for constrained maximization
- Using framework, extended algs from linear to submodular maximization
- Applied to (hyper)matchings, (intersection of) matroids
- Can smoothly interpolate approx factor between linear f and general submodular f via curvature of f

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Open Problems

- Extend matroid multi-pass result beyond partition matroids
- Capture recent MWM algorithms that beat Zelke [Crouch-Stubbs'14]
- Lower bounds??? Is MSM harder to approximate than MWM?