Reed-Muller testing and approximating small set expansion & hypergraph coloring

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(Spring'14 @ Microsoft Research New England)

2014 Bertinoro Workshop on Sublinear Algorithms

May 27, 2014

Linear codes and testing

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- Dual space $C^\perp = \{y \in \mathbb{F}_2^n \mid \langle y, c \rangle = 0 \ \forall c \in C\}$.

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- Focus on restricted/structured set of dual codewords for test:
  - q query tests: $T \subseteq C^\perp_{\leqslant q}$ (low-weight dual codewords)
  - Hope to have low soundness error when x is **far from** C.

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Goal: Construct codes C of good rate with a *low-query* test

- Always accept codewords, and reject strings far from the code with good prob.

Most work has focused on $q = O(1)$ case.
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Recently, due to connections to approximability, there has been interest in the regime:

- $q \approx \varepsilon n$ ("small linear locality"), and
- codes of large $(n - o(n))$ dimension.

Binary Reed-Muller codes

Let $P(m,u)$ be the $\mathbb{F}_2$-linear space of all multilinear polynomials in $X_1, X_2, \ldots, X_m$ of degree u (coefficients in $\mathbb{F}_2$)

Reed-Muller code

$\mathrm{RM}(m,u) = \{\langle f(\mathbf{a})\rangle_{\mathbf{a}\in\mathbb{F}_2^m} \mid f \in P(m,u)\}.$

- Code length $= 2^m$.
- Dimension $= \sum_{j=0}^{u} \binom{m}{j}$ (number of monomials of degree $\leqslant u$)

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- Dimension $= \sum_{j=0}^{u} \binom{m}{j}$ (number of monomials of degree $\leqslant u$)
- Distance $= 2^{m-u}$ (A min. wt. codeword: $f(\mathbf{X}) = X_1 X_2 \cdots X_u$)

Large rate Reed-Muller codes

Our focus: Large degree $u = m - r - 1$ (think r fixed, $m \to \infty$)

- Code distance $= 2^{r+1}$. (Poly $f(\mathbf{X}) = X_1 X_2 \cdots X_{m-r-1}$)
- Dual space is $\mathrm{RM}(m, r)$
  - $f \in P(m, m-r-1)$ and $g \in P(m, r) \Longrightarrow$
    $f \cdot g \in P(m, m-1) \Longrightarrow \sum_x f(x) g(x) = 0$
  - Dual codewords of minimum weight ($= 2^{m-r}$):
    $L_1 L_2 \ldots L_r$, product of r degree 1 polys (affine forms).

Reed-Muller testing

Canonical test for proximity of $f : \mathbb{F}_2^m \to \mathbb{F}_2$ to deg. $m - r - 1$ polys:

1. Pick linear independent affine forms $L_1, L_2, \ldots, L_r$ u.a.r, and set $h = \prod_{j=1}^{r} L_j$ (random min. wt. dual codeword)
2. Check $\langle f, h \rangle = \sum_x f(x)h(x) = 0$ ($\equiv \deg(f \cdot h) < m$)

queries $= 2^{m-r} = \varepsilon n$ for $\varepsilon = 2^{-r}$.

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Theorem ([Bhattacharyya, Kopparty, Schoenebeck, Sudan, Zuckerman'10])

If f is 2^r-far from $P(m, m-r-1)$, then error of above test is bounded away from 1; i.e., for some absolute constant $\rho < 1$

$$\left| \mathbb{E}[(-1)^{\langle f, L_1 L_2 \cdots L_r \rangle}] \right| \leqslant \rho$$

A beautiful connection: LTCs and SSEs

[Barak, Gopalan, Håstad, Meka, Raghavendra, Steurer'12] made a beautiful connection between locally testable codes (LTCs) and small set expanders (SSEs).

Instantiating with Reed-Muller codes, they constructed SSEs with currently largest known count of bad eigenvalues.

Small set expansion problem

SSE(μ, ε) problem

Given graph $G = (V, E)$ on n vertices, distinguish between:

- YES instance: $\exists$ small non-expanding set,
  i.e., $\exists S \subset V$, $|S| = \mu n$, $\mathrm{EdgeExp}(S) \leqslant \varepsilon$
- NO instance: All small sets expand,
  $\forall S$, $|S| = \mu n$, $\mathrm{EdgeExp}(S) \geqslant 1/2$.

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SSE intractability hypothesis [Raghavendra, Steurer'10]

$\forall \varepsilon > 0$, $\exists \mu$ such that SSE(μ, ε) is hard.

(Implies many other intractability results, including Unique Games conjecture.)

A spectral necessity

A subset S with $\mathrm{EdgeExp}(S) \leqslant \varepsilon$ can be "found" in the eigenspace of eigenvalues $\geqslant 1 - \varepsilon$ (of graph's random walk matrix).

- [Arora, Barak, Steurer'10]: this eigenspace has dimension $\lesssim n^{\varepsilon}$ for No instances (when the graph is a small set expander)

  $\Rightarrow$ $\exp(n^{\varepsilon})$ time algo for SSE problem

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Necessary requirement for SSE intractability hypothesis

Existence of small set expanders (SSEs) with $n^{\Omega_{\varepsilon}(1)}$ "bad" eigenvalues $\gtrsim 1 - \varepsilon$.

SSEs with many bad eigenvalues [BGHMRS'12]

Noisy hypercube

Vertex set $V = \{0,1\}^t$. Edge $x \sim y$ if $\mathrm{HamDist}(x,y) = \varepsilon t$.
Has $\geqslant t = \log|V|$ eigenvalues $\approx 1 - \varepsilon$.

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Derandomization via Reed-Muller code

Take subgraph induced by $V' = \mathrm{RM}(m,r)$ ($t = 2^m$, $\varepsilon = 2^{-r}$).

- Vertices $P(m,r)$, degree r polynomials
- Edges $f \sim g$ if $f - g = L_1 L_2 \cdots L_r$.

Easy: Graph retains $\Omega(t)$ eigenvalues $\approx 1 - \varepsilon$.

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Easy: Graph retains $\Omega(t)$ eigenvalues $\approx 1 - \varepsilon$.

- But now $|V'| \approx 2^{m^r} = 2^{(\log t)^r}$, so we have $2^{(\log|V'|)^{\Omega_\varepsilon(1)}}$ bad eignevalues.

SSE property of Reed-Muller graph

Fourier analysis over $P(m, r)$

Express function $A : P(m, r) \to \mathbb{R}$ as $A(f) = \sum_\beta \widehat{A}(\beta)(-1)^{\langle \beta, f \rangle}$.

- "frequencies" β range over cosets of $P(m, m-r-1)$ (dual group of $P(m, r)$).
- Weight of frequency β = Hamming dist. of β to $P(m, m-r-1)$

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SSE proof has two ingredients

Take A = indicator of a *small* set

1. A has very little Fourier mass on low frequencies *(Hypercontractivity of low-degree polynomials)*
2. Contribution of high frequency $\widehat{A}(\beta)$ killed by edges of graph *(testing of* RM$(m, m-r-1)$*)*, leading to expansion.

LTCs of constant absolute distance

Consider $C \subseteq \mathbb{F}_2^n$ of minimum distance d.

Think d fixed, and $n \to \infty$.

Largest possible dimension (sphere packing bound): $\approx n - \frac{d}{2} \log n$.

- Achieved by BCH codes!
- However, BCH code is not testable even with $0.49n$ queries.

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Reed-Muller code $\mathrm{RM}(m, u)$ of length $n = 2^m$ and $u \approx m - \log d$

- Dimension $\approx n - (\log n)^{\log d}$,
- Testable with $2n/d$ queries (rejecting $d/3$-far strings with $\Omega(1)$ prob.)

Price of local testability?

Where in the spectrum between Reed-Muller and BCH does the best dimension of distance d code testable with $O(n/d)$ queries lie?

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[Guo, Kopparty,Sudan'13] Lifted codes, with dimension $\gtrsim n - \left(\frac{\log n}{\log d}\right)^{\log d}$ slightly improving Reed-Muller codes.

[G.,Sudan,Velingker,Wang'14] For a class of affine-invariant codes containing Reed-Muller, dimension $\lesssim n - \left(\frac{\log n}{\log^2 d}\right)^{\log d}$.

RM testing application II: Hypergraph Coloring

Best known algorithms to color 3-colorable graphs use $n^{\Omega(1)}$ colors

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Recent result based a *"structured"* Reed-Muller testing result:

Theorem ([Dinur, G.'13], [G., Harsha, Håstad, Srinivasan, Varma'14])

Coloring a 2-colorable 8-uniform hypergraph with $\exp(2^{\sqrt{\log\log n}})$ *colors is quasi NP-hard.*

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Previous hardness only ruled out $(\log n)^{O(1)}$ coloring.

Very recently, [Khot, Saket'14] improved bound to $\exp((\log n)^{\Omega(1)})$ via different use of the [Dinur, G.'13] RM testing result.

Won't be able to describe the underlying PCP in any detail,

but will try to give a glimpse of where Reed-Muller tesing fits in.

The Long Code

PCPs *encode* assignments $\{0,1\}^m$ to enable efficient testing.

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Definition (Long Code encoding $a \in \{0,1\}^m$)

$$\mathsf{LONG}(a) := \langle f(a) \rangle_{f:\{0,1\}^m \to \{0,1\}} \ .$$

Gives value of *every* Boolean function on a: *the most redundant encoding.*

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The improvements in hypergraph coloring
(and also earlier integrality gaps in [BGHMRS'12], [Kane-Meka'13])
due to a *"shorter" Reed-Muller based substitute of the long code.*

The low-degree long code

Definition (Degree-r long code)

The degree-r long code encoding of $a \in \{0,1\}^m$ is

$$\langle f(a) \rangle_{f \in P(m,r)} \ .$$

Puncturing of long code to locations indexed by degree $\leqslant r$ fns.
$\Longleftrightarrow$ derandomization of hypercube to Reed-Muller codewords.

Encoding length $\approx 2^{m^r}$ instead of 2^{2^m} for the long code.

- For $r \approx \log m$, almost exponential savings.

Hypergraph gadget on low-degree long code

Underlying hypergh coloring hardness is a "low-degree long code test"

Query patterns give hypergraph on vertex set $P(m, r)$[1] such that:

1. (Completeness) If $A : P(m, r) \to \{0, 1\}$ is a codeword of the degree-r long code, i.e., $\exists \mathbf{a} \in \mathbb{F}_2^m$ such that $\forall f$, $A(f) = f(\mathbf{a})$, then A is a 2-coloring without any monochromatic hyperedge.

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2. (Soundness) If $I : P(m, r) \to \{0, 1\}$ is the indicator function of an independent set of measure μ, then $\exists$ a "sizeable" Fourier coefficient $|\widehat{I}(\beta)|$ for some β of "low" weight ($=$ distance to $P(m, m - r - 1)$)

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8-uniform hypergraph gadget

- Vertex set $= P(m, r)$.
- Hyperedges on 8-tuples:

$$\begin{matrix} e_1 & e_1 + f_1 \\ e_2 & e_2 + f_1 + g \cdot h + 1 \\ e_3 & e_3 + f_2 \\ e_4 & e_4 + f_2 + \overline{g} \cdot h' + 1 \end{matrix} .$$

$$\forall e_i, f_i \in P(m, r),\ g, h, h' \in P(m, r/2).$$

Completeness: Ensured by $(g \cdot h)(\mathbf{a}) = 0$ or $(\overline{g} \cdot h')(\mathbf{a}) = 0$ for every $\mathbf{a} \in \mathbb{F}_2^m$.

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Soundness: Orthogonality to $g \cdot h$ is a good Reed-Muller test that kills high frequencies.

Structured Reed-Muller testing

Recap: To test proximity to $P(m, m-r-1)$, check orthogonality to some degree r polys (the dual space).

Theorem (Dinur, G.'13)

If $\beta : \mathbb{F}_2^m \to \mathbb{F}_2$ *is* 2^r*-far from* $P(m, m-r-1)$*, then*

$$\mathbb{E}_{g,h}\left[(-1)^{\langle \beta, g\cdot h\rangle}\right] \leqslant 2^{-2^{\Omega(r)}} \ ,$$

where $g, h \in_R P(m, r/2)$.

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where $g, h \in_R P(m, r/2)$.

- Compare with [BKSSZ]: Test function $L_1 L_2 \cdots L_r$, constant soundness error (and $2^m/2^r$ queries)
- Here, test function $g \cdot h$, soundness error doubly exponentially small in r (and typically $2^m/4$ queries)

Proof idea of RM testing result

Need to understand when $\langle \beta, gh \rangle = 0 \iff \langle \beta g, h \rangle = 0$,
given β is far from $P(m, m-r-1)$.

$$\mathbb{E}_h[(-1)^{\langle \beta g, h \rangle}] = \begin{cases} 1 & \text{if } \deg(\beta g) \leqslant m - r/2 - 1 \\ 0 & \text{otherwise} \end{cases}$$

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2. [BKSSZ] $\Rightarrow$ If β is D-far from $P(m, m-r-1)$, then $\exists$ a linear form L s.t. $\beta_{|L=0}$ & $\beta_{|L=1}$ are *both* $\frac{D}{3}$-far from $P(m-1, m-r-1)$.

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3. Use 2. to lower bound co-dimension by sum of two similar co-dimensions (recursively for $\Omega(r)$ inductive steps)

Summary

Ability to test high rate Reed-Muller codes is the basis for:

- Quantitative improvements via the low-degree long code
- Applications to approximability: SSE with many eigenvalues, improved integrality gaps for sparsest cut, hardness of hypergraph coloring, size-efficient PCPs.
- More applications?

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Even better testable codes than Reed-Muller codes?

Limits of testability in the "small linear locality" (εn queries) regime?

- Is BCH or RM closer to the largest possible dimension?