A Near-Optimal Algorithm for Testing Isomorphism of Two Unknown Graphs

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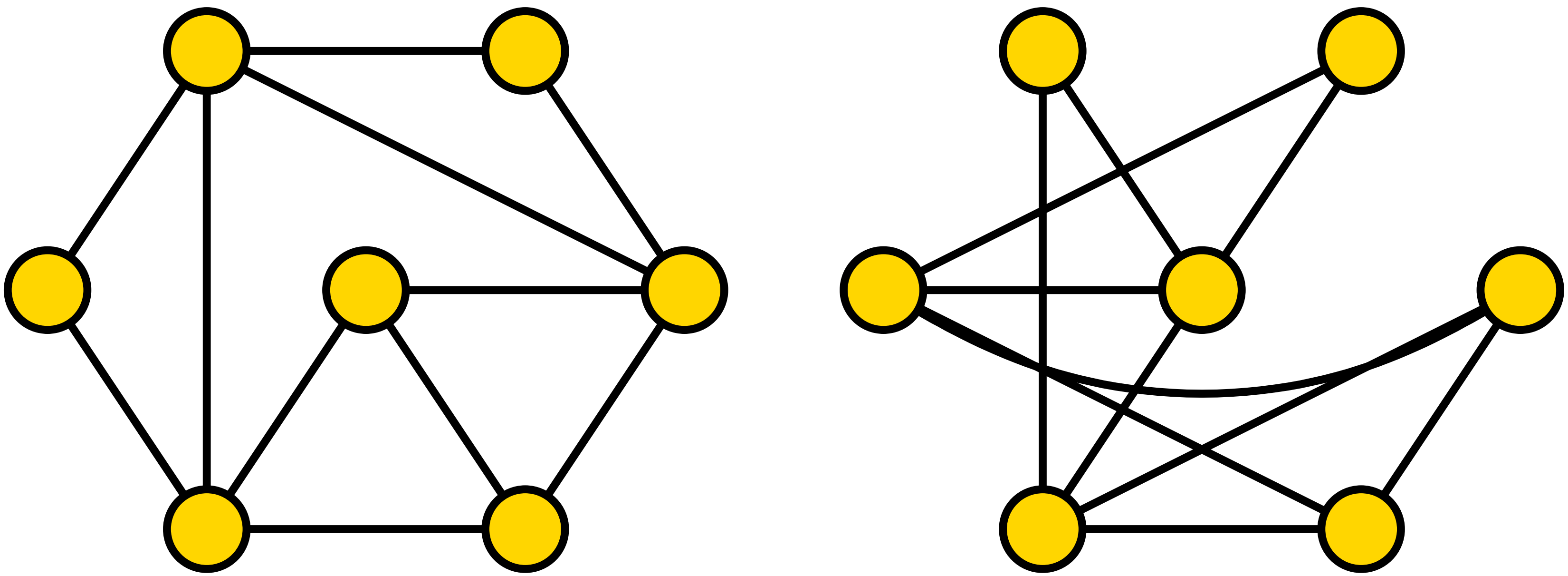
Joint work with **Xiaorui Sun** (Columbia)

Graph Isomorphism

- Problem: Are two graphs identical?



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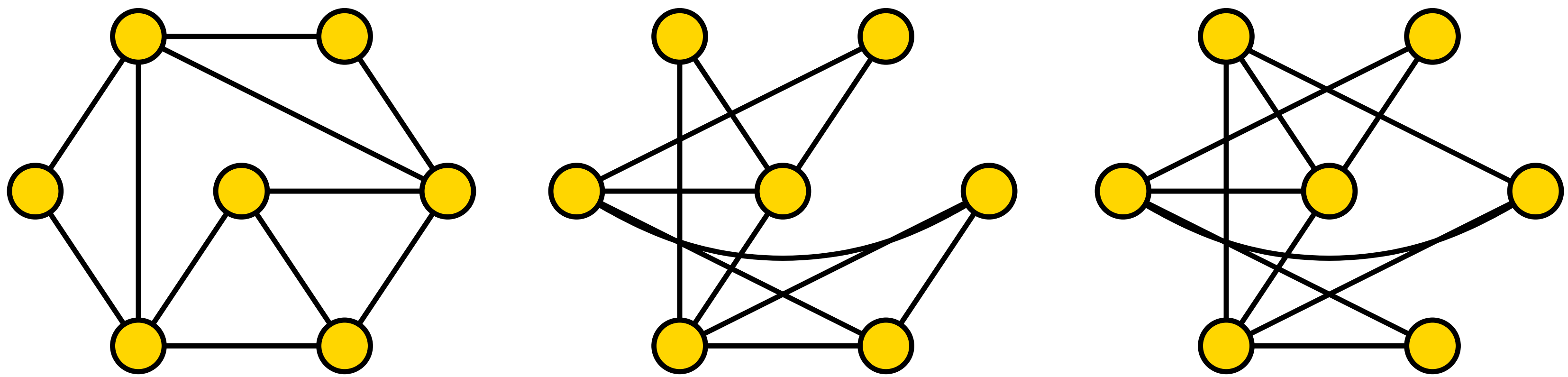

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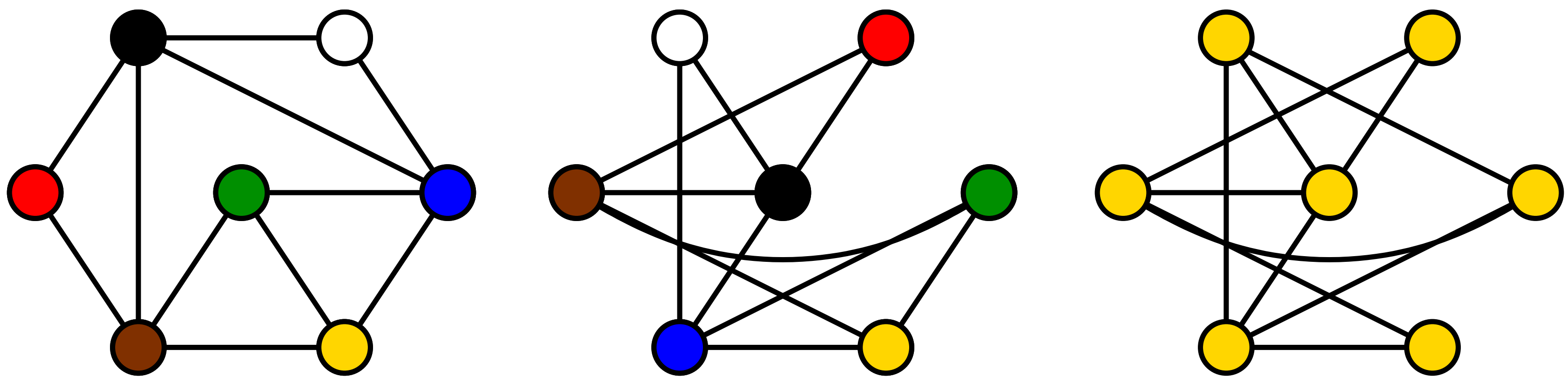

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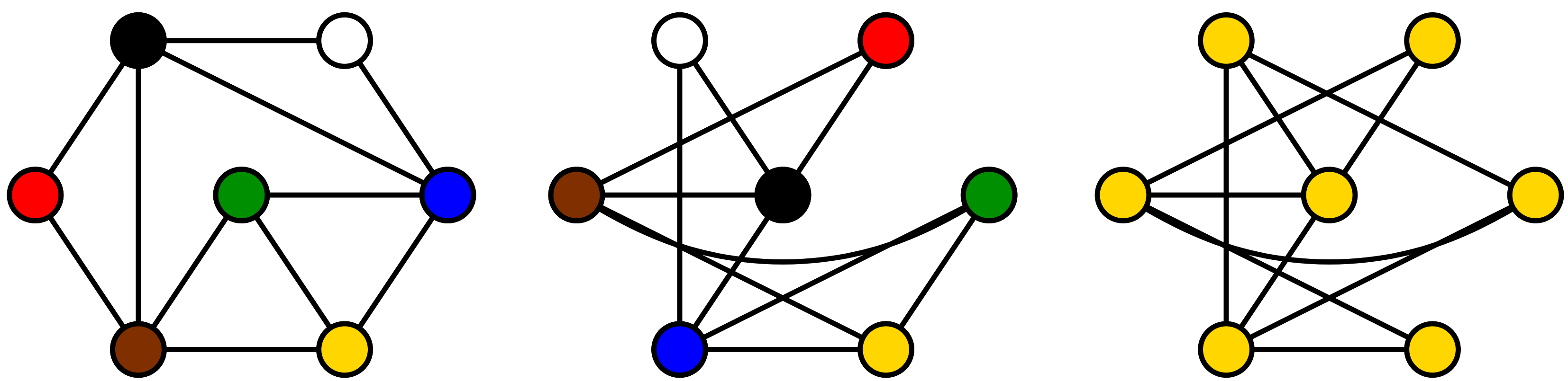
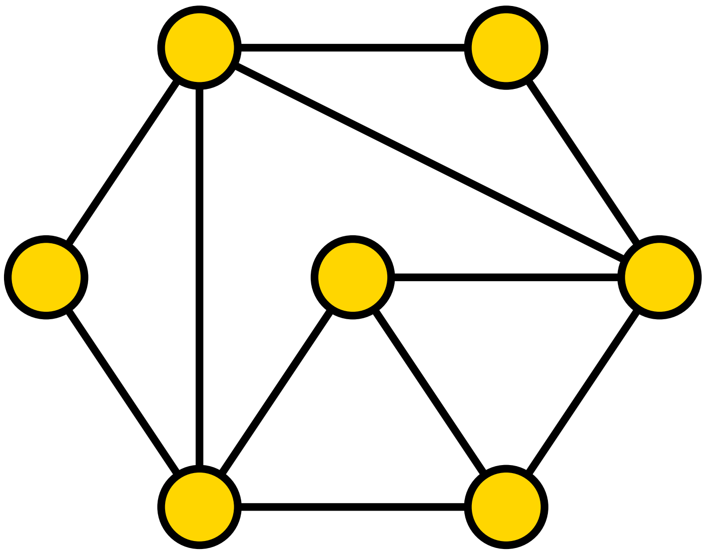


Status:

- Not known to be in P or NP-hard
- Best known algorithm: $2^{\tilde{O}(\sqrt{n})}$ time (early 1980's)

Property Testing Version

- This Talk: Dense graph model



	1	1	0	1	1	0
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1	0		0	0	1	0
0	0	0		1	1	1
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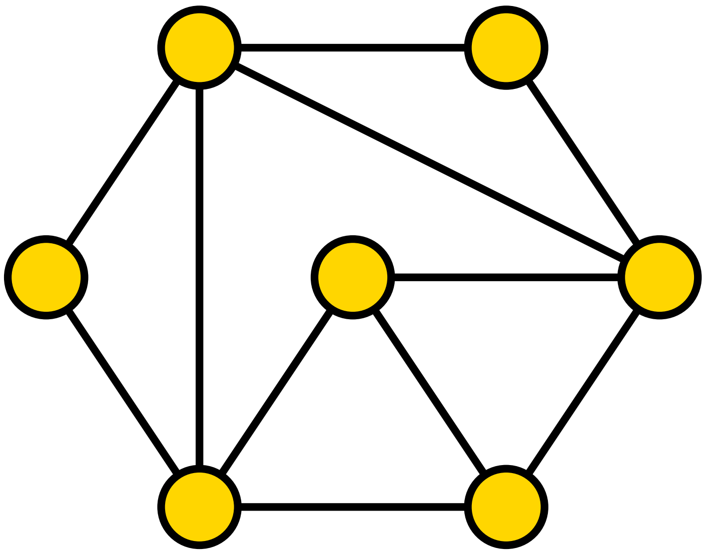

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- Requirements:
  - accept with probability 9/10 if graphs isomorphic
  - reject with probability 9/10 if for any matching of vertices at least ϵn^2 edges disagree

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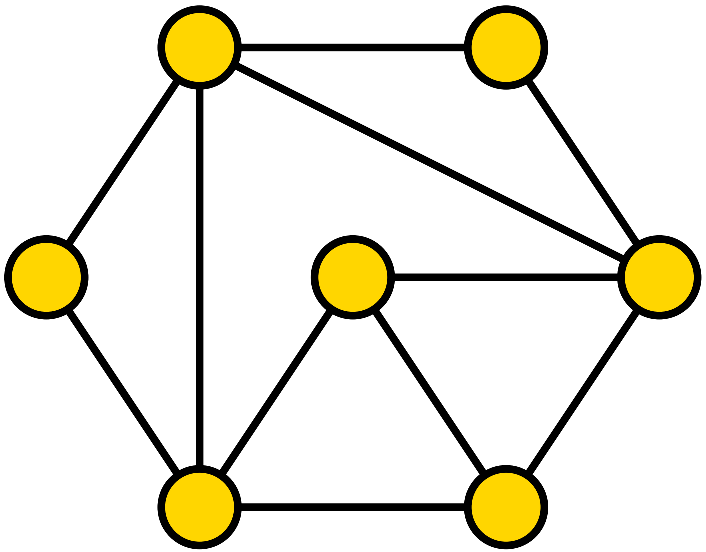

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- Requirements:
  - accept with probability 9/10 if graphs isomorphic
  - reject with probability 9/10 if for any matching of vertices at least ϵn^2 edges disagree
- This talk: focus on the complexity as a function of n, assume that ϵ is a small constant (say, $\epsilon = 10^{-9}$)

Query Complexity

- Fischer, Matsliah (2006):

	Upper bound	Lower bound
One sided error, one graph known	$\widetilde{O}(n)$	$\Omega(n)$
One sided error, both graphs unknown	$\widetilde{O}(n^{3/2})$	$\Omega(n^{3/2})$
Two sided error, one graph known	$\widetilde{O}(n^{1/2})$	$\Omega(n^{1/2})$
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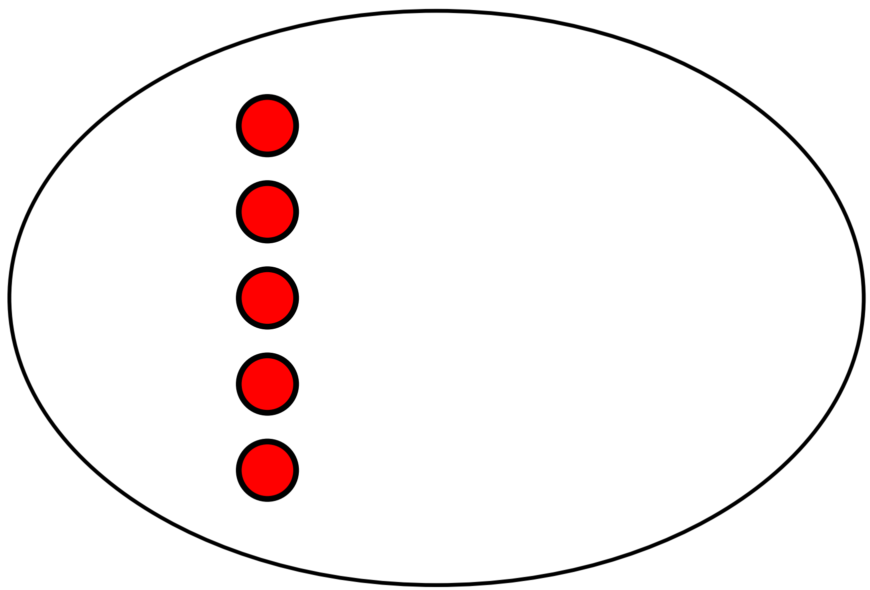
- Our result:

  Algorithm that makes $n \cdot 2^{O(\sqrt{\log n})}$ queries

Review of Fischer-Matsliah Techniques

Core sets

- Core set = list of $\mathrm{polylog}\, n$ vertices $(v_1, v_2, \ldots, v_k)$



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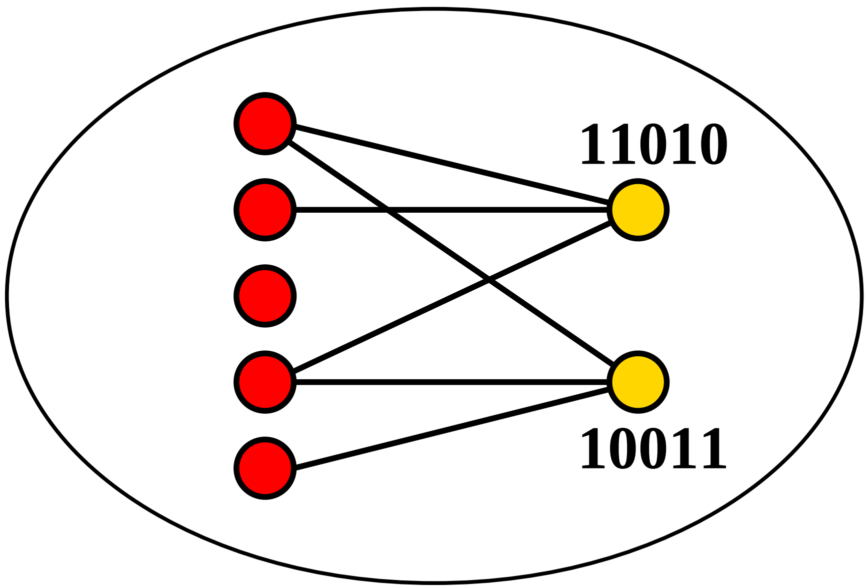
- Every vertex u has a label $l \in \{0,1\}^k$ with respect to core set:

$$l_i = \begin{cases} 1 & \text{if } v_u \text{ and } u \text{ connected} \\ 0 & \text{otherwise} \end{cases}$$



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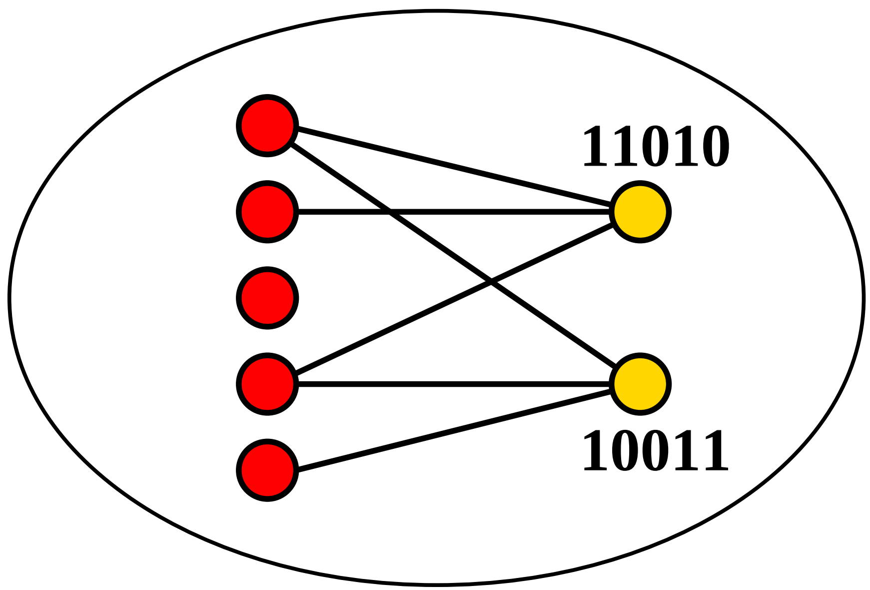
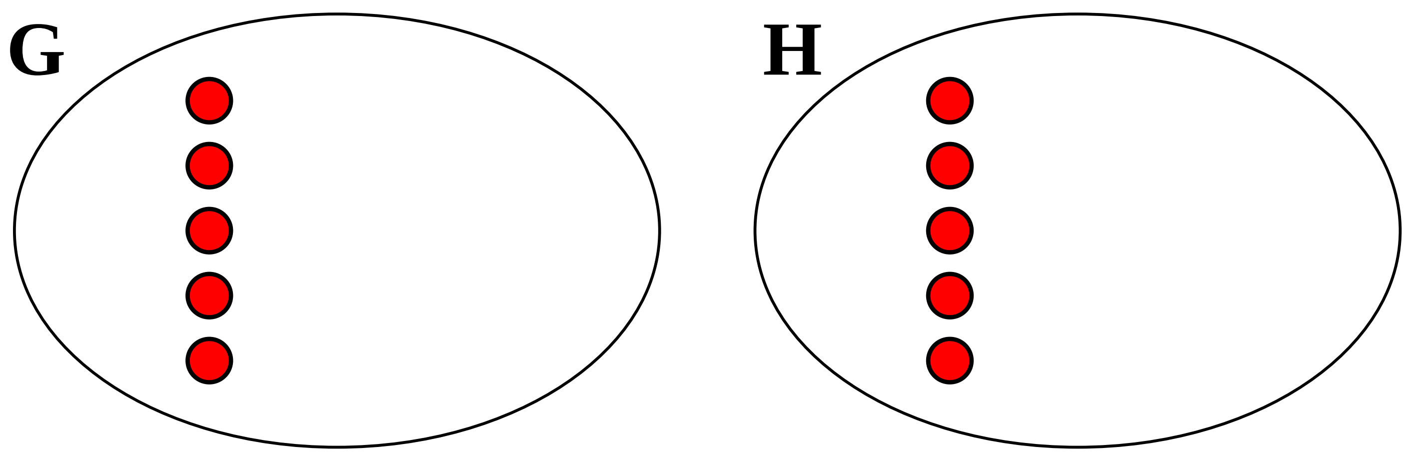


- Intuition: a large random core set partitions vertices into sets of similar vertices

Important Theorem

If G and H have core sets $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$, respectively, such that:



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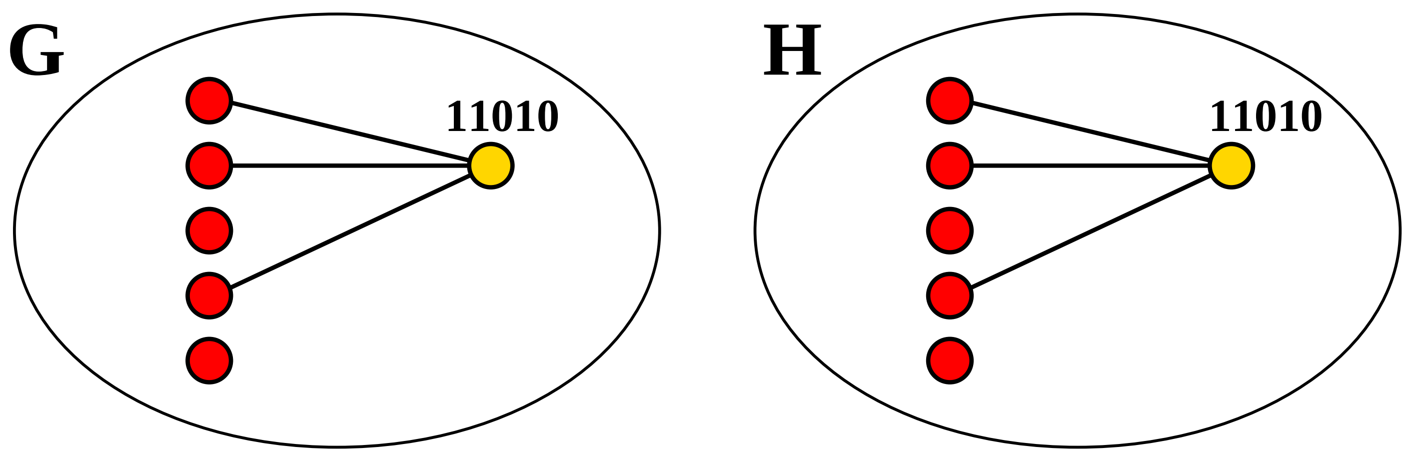

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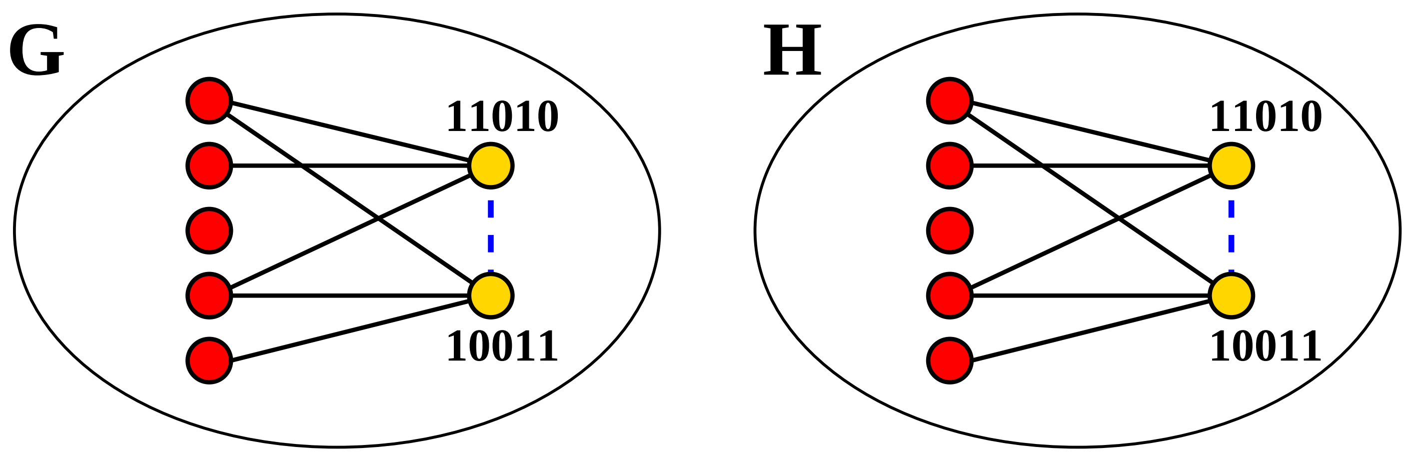

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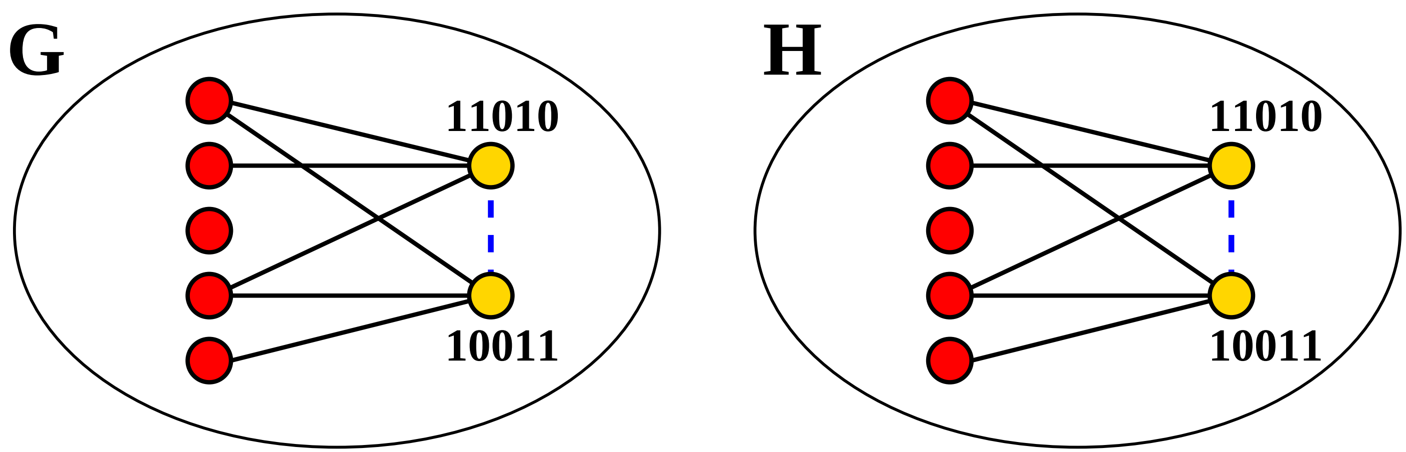

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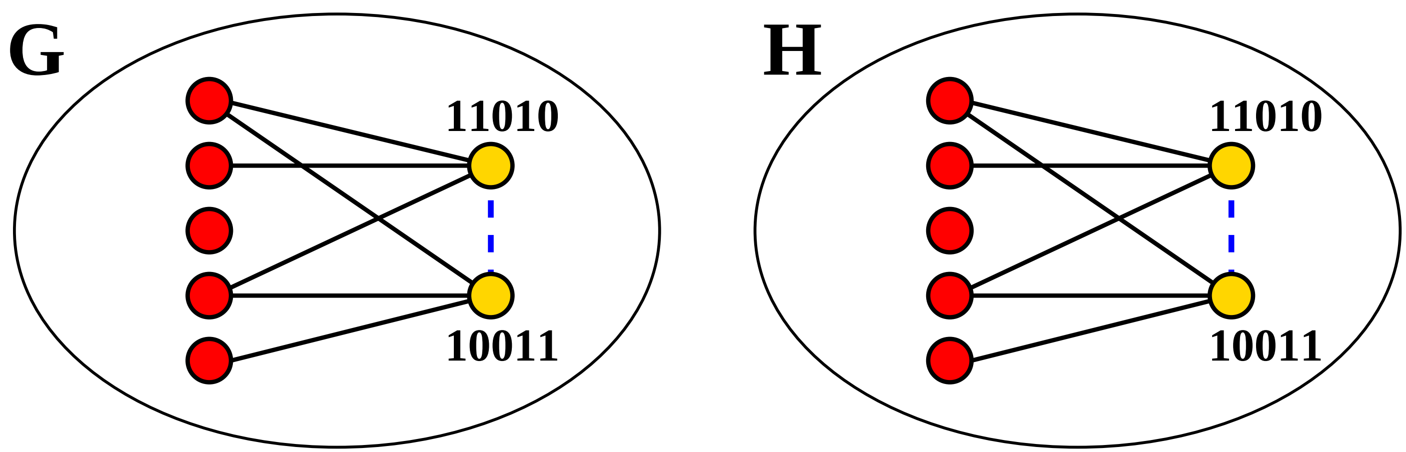
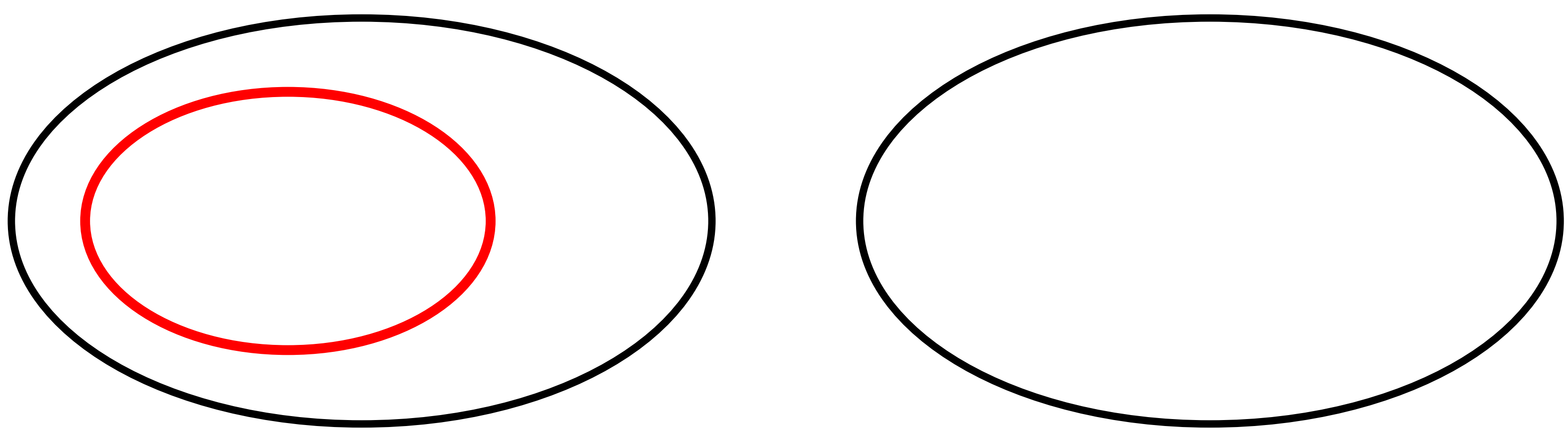


Generic tester:

Search for such a pair of core sets and accept if found

The Fischer-Matsliah Algorithm

- Select $\tilde{O}(n^{3/4})$ random vertices $V_G^\star$ from G



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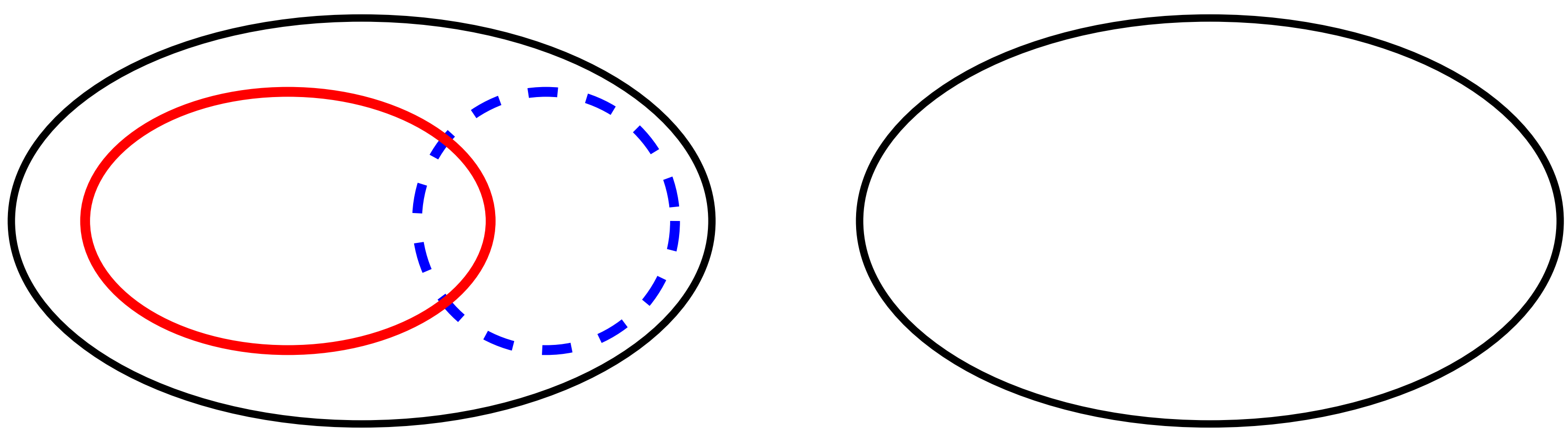

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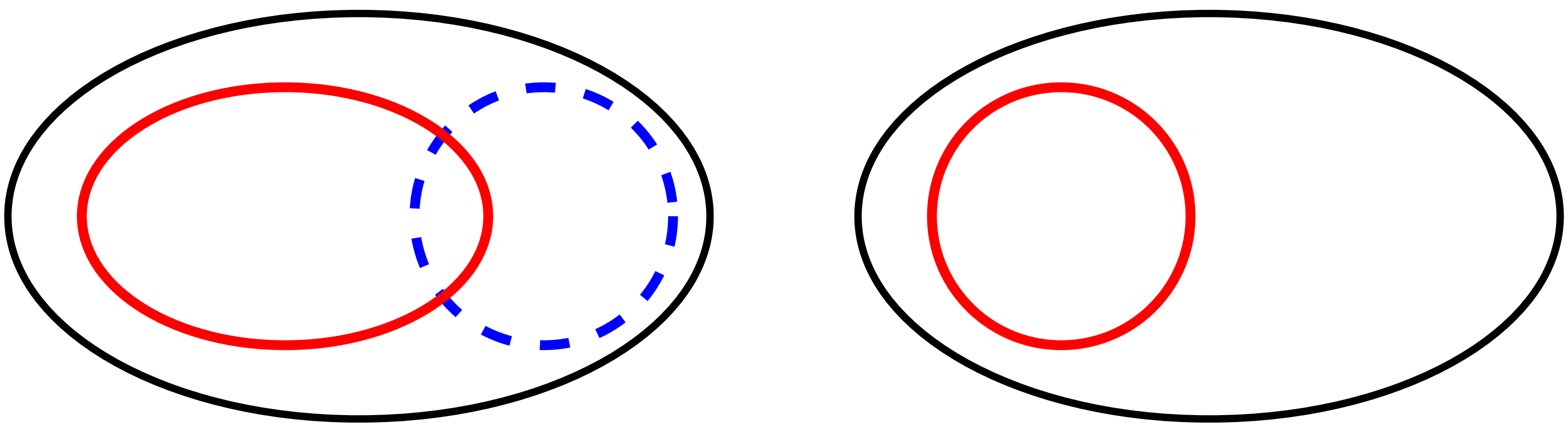

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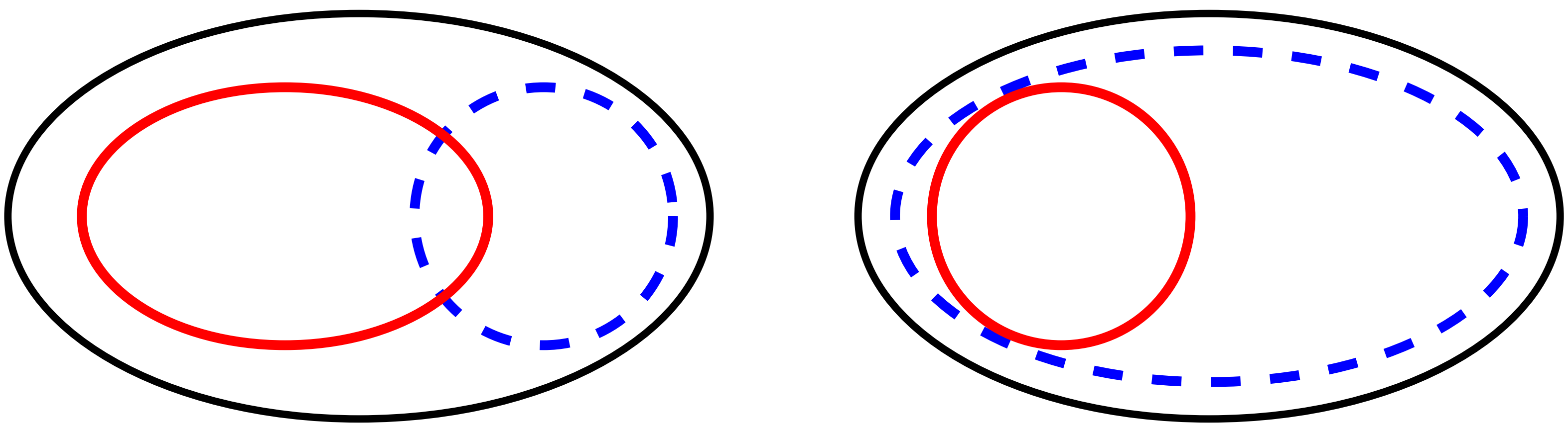

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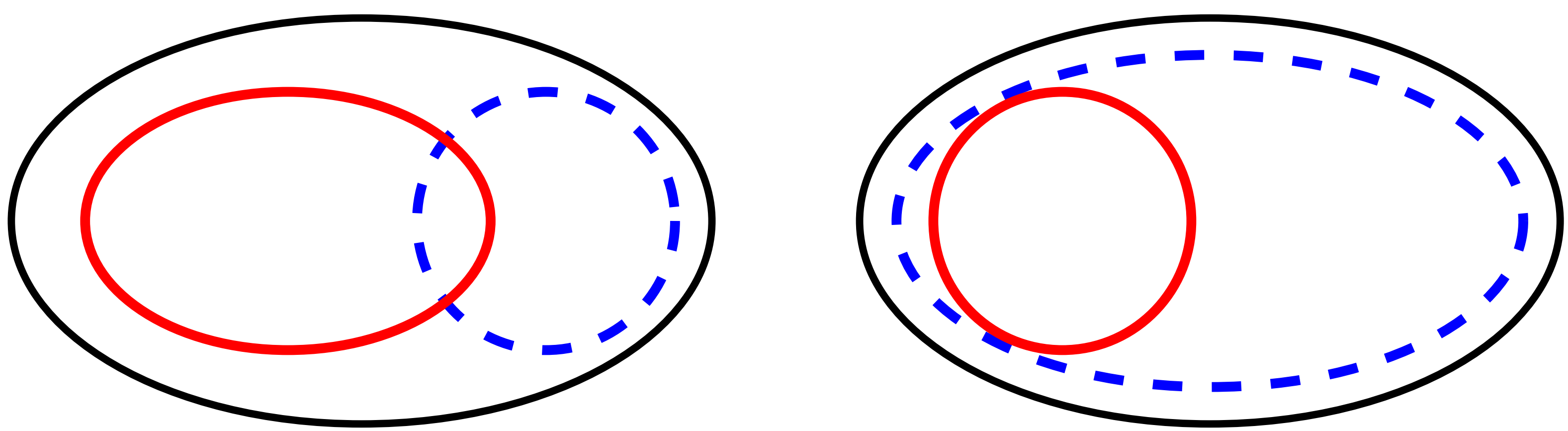

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  - Step 2: Select $\mathrm{polylog}\, n$ pairs of random vertices in G and see if the connectivity of their labels almost the same in H

First Attempts

Better Label Distribution Testing?

- Idea: Use a different distribution tester
  - Select $\tilde{O}(\sqrt{n})$ vertices from G and H
  - Query connections to random subset of size $\tilde{O}(n^{2/3})$
  - Use the distribution tester of Batu et al. (2000)
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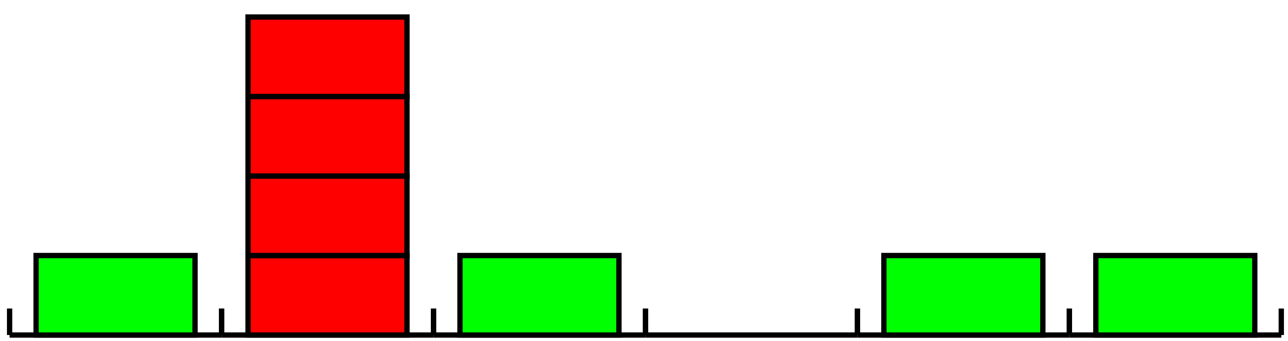

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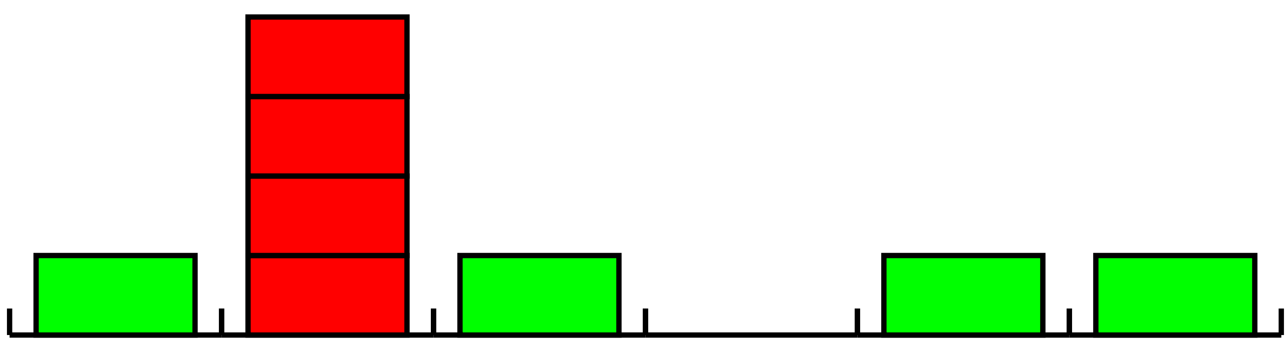

- Ignore this issue for now

Exponent Less than $7/6$?

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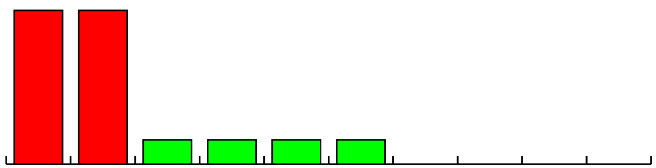

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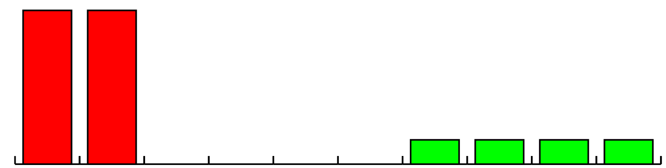

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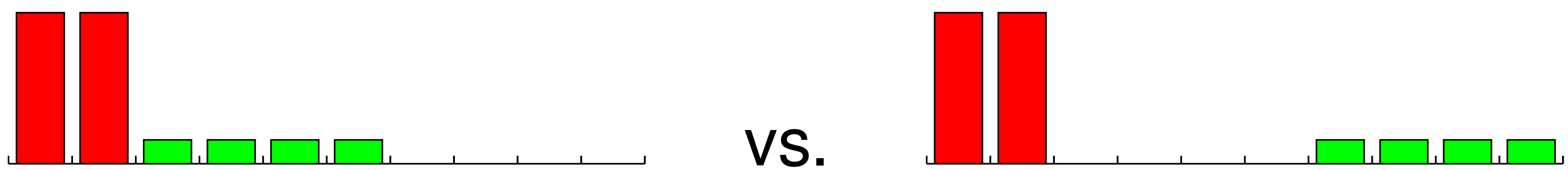

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  - We should be able to distinguish heavy and light elements
- Nice consistent clustering of slightly different adjacency vectors still difficult

Our Algorithm

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Assume labels preserve all distances

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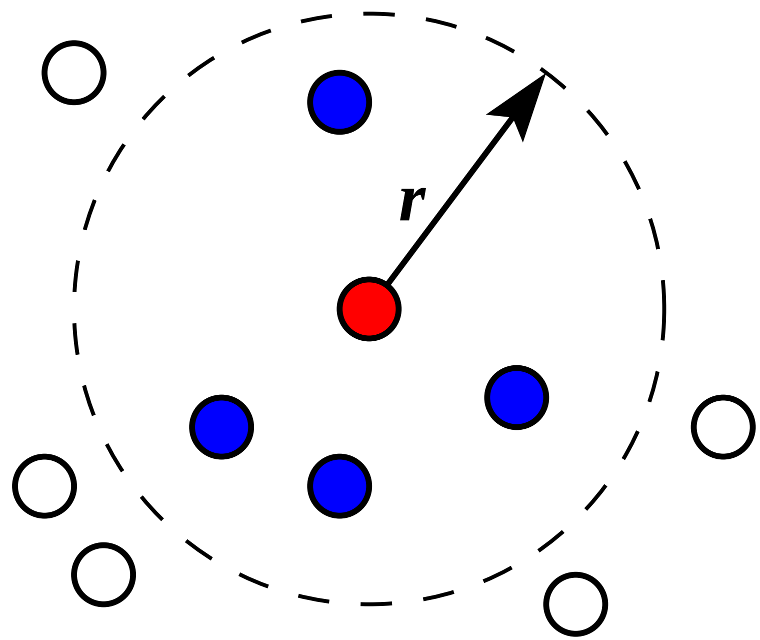

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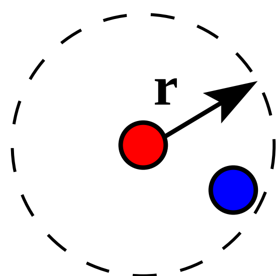
  - Hard to partition them into classes of similar degrees
- Our solution:
  - Randomize the threshold r
  - Design a tool for estimating the number of collisions
  - Reject only if Earth-Mover Distance large

Estimating Number of Collisions

- Found colliding labels for vertices $v \in V[G]$ and $w \in V[H]$



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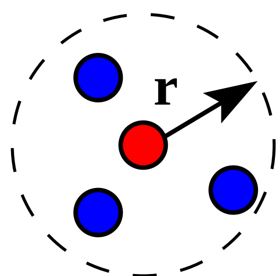

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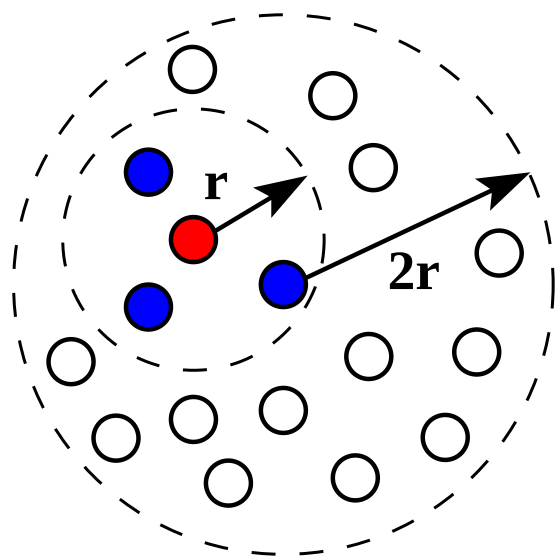

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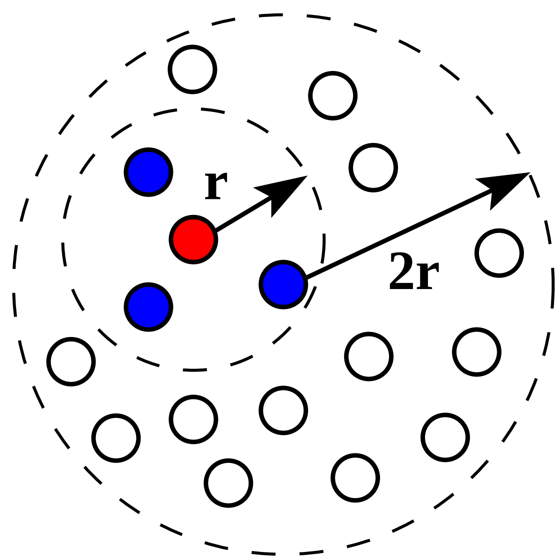
- Requirement: good multiplicative approximation



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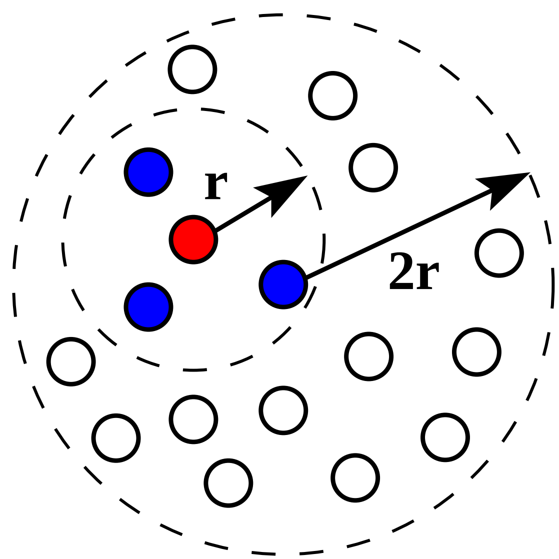
- Problem: What if few of them do?



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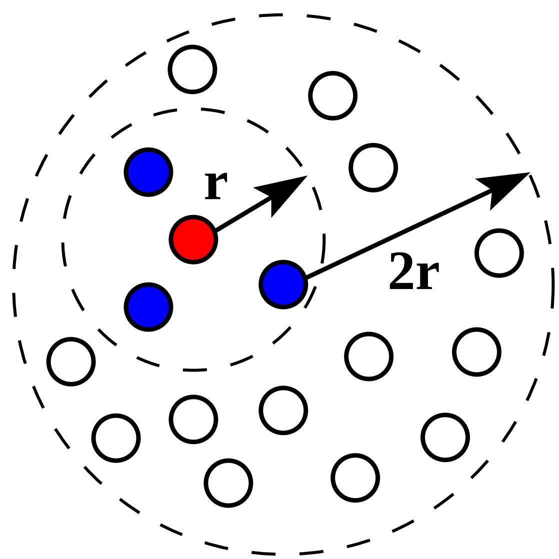

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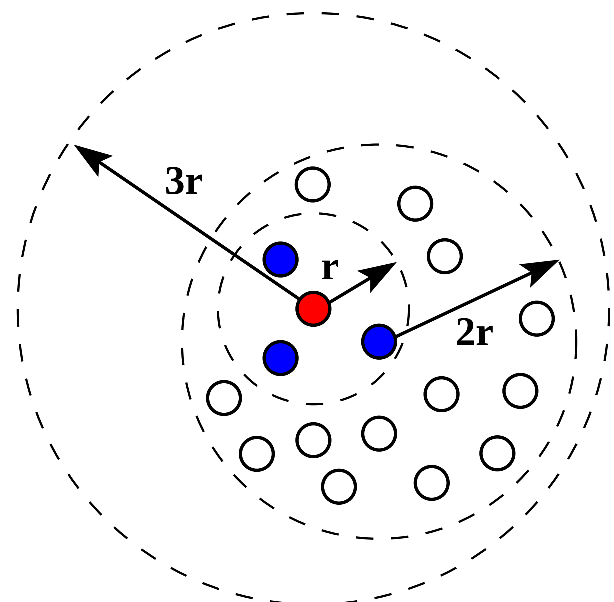

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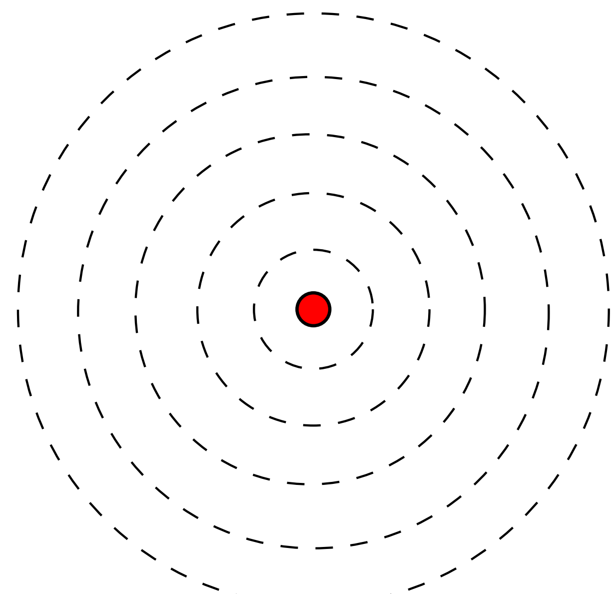

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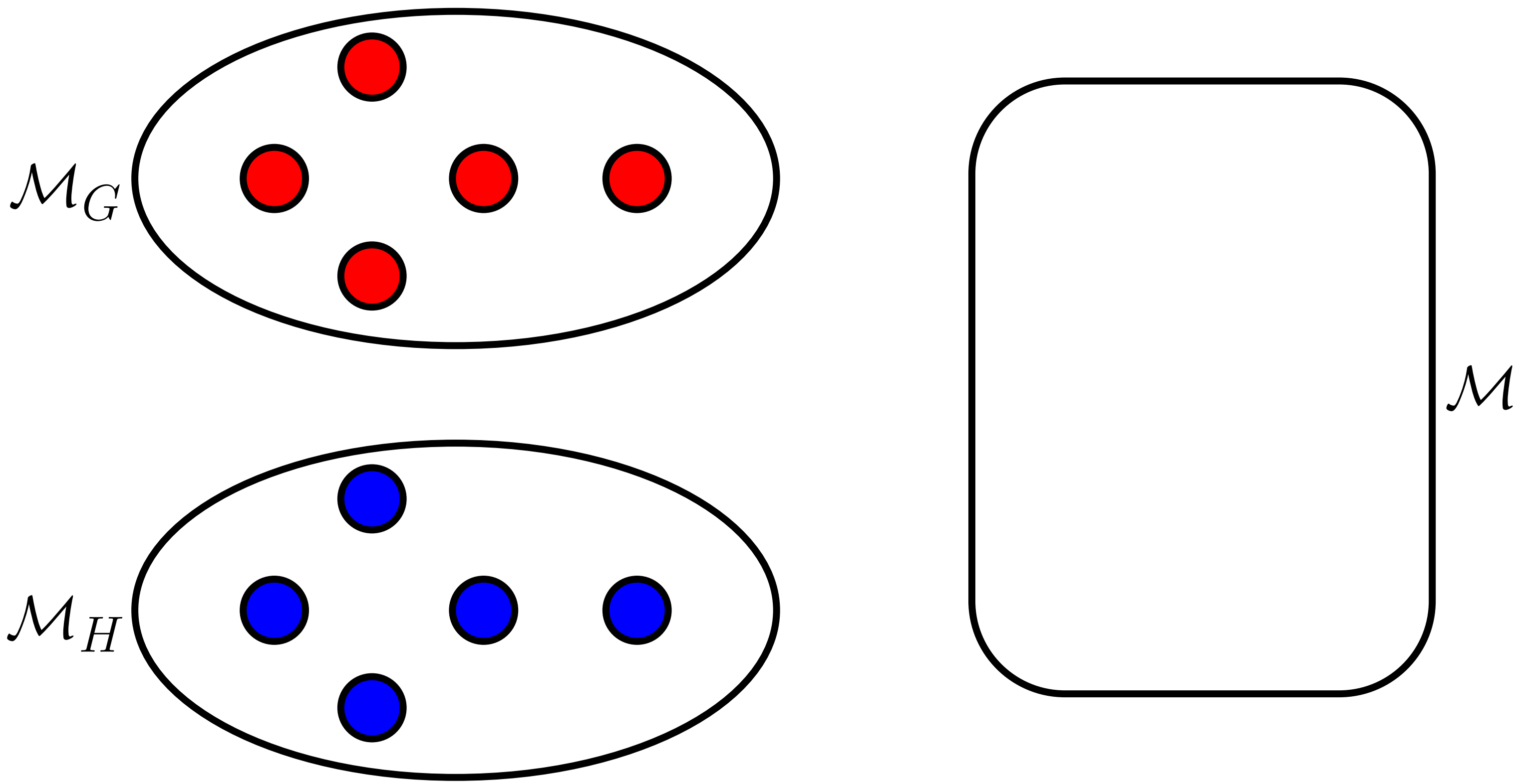

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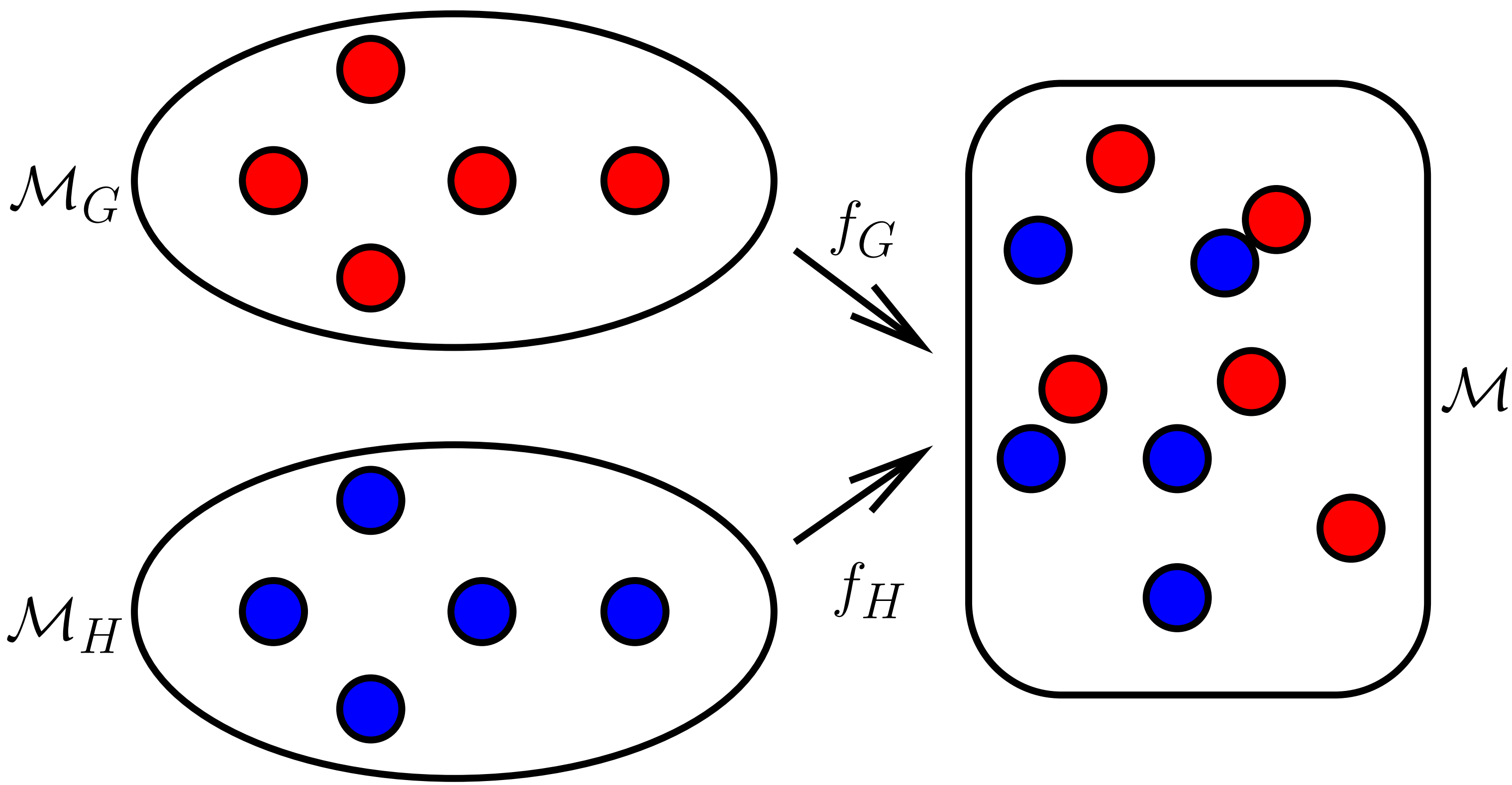

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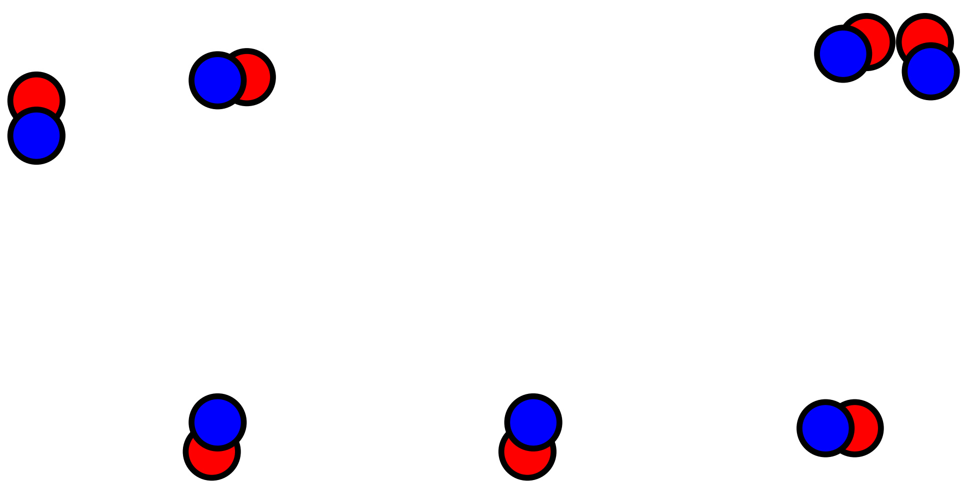

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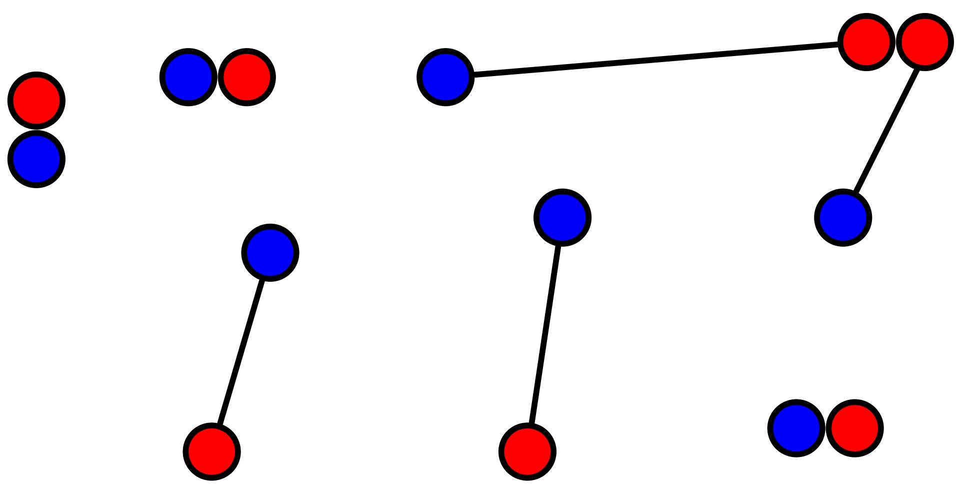

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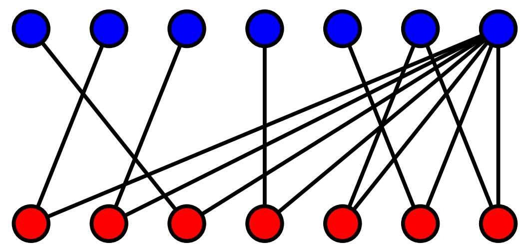

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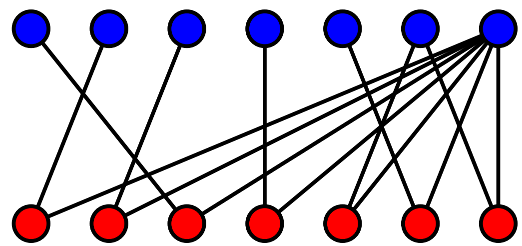

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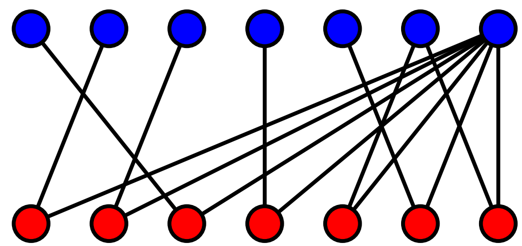

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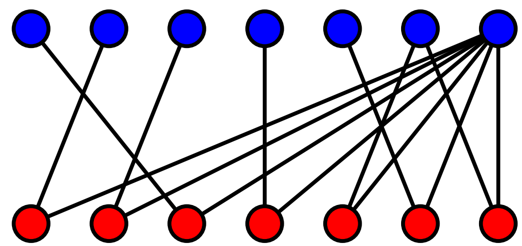

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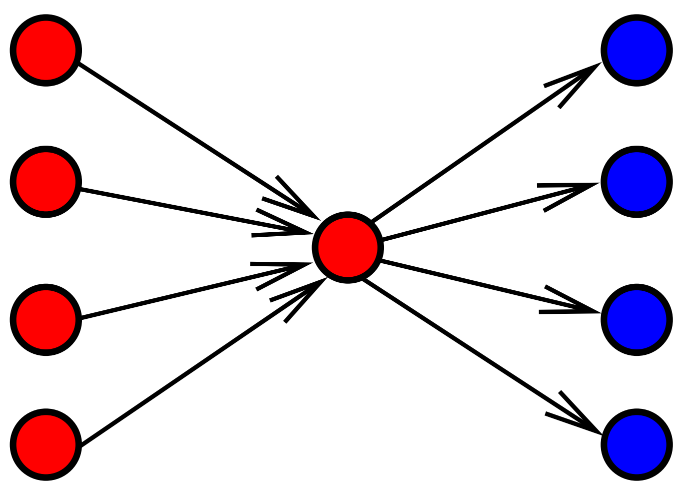

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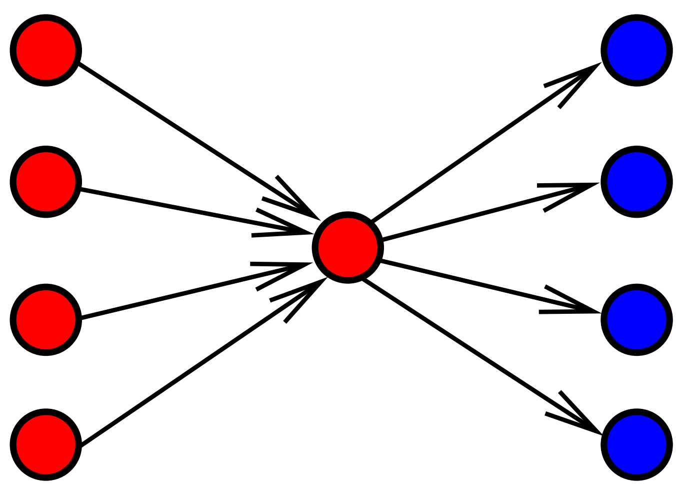

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Final Remarks

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