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Flow Metrics on Graphs

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# Flow Metrics on Graphs

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## Abstract

Given a graph with non negative edge weights, there are various ways to interpret the edge weights and induce a metric on the vertices of the graph. A few examples are shortest-path, when interpreting the weights as length, and resistance distance, when thinking of the graph as an electrical network and the weights are the electrical resistances of the edges. Each of these metrics has its own properties and applications, for example for studying the structure of the underlying graph, or for clustering the vertices. Another key tool to investigate the underlying graph is flows, for example for finding minimum- $st$ -cuts via the well known min-cut/max-flow Theorem, which can be viewed as a metric by considering the inverse of mincut.

It is known that the 3 abovementioned metrics can all be derived from flows, when formalizing them as convex optimization problems. This key observation led us to studying a family of metrics that are derived from flows, which we call *flow metrics*, that gives a natural interpolation between the above metrics using a parameter  $p$ .

We make the first steps in studying the flow metrics, and mainly focus on two aspects: (a) understanding basic properties of the flow metrics, either as an optimization problem (e.g. finding relations between the flow problem and the dual potential problem) and as a metric function (e.g. understanding their structure and geometry); and (b) considering methods for reducing the size of graphs, either by removing vertices or edges while approximating the flow metrics, and thus attaining a smaller instance that can be used to accelerate running time of algorithms and reduce their storage requirements.

Our main result is a lower bound for the number of edges required for a resistance sparsifier in the worst case. Furthermore, we present a method for reducing the number of edges in a graph while approximating the flow metrics, by utilizing a method of [CP15] for reducing the size of matrices. In addition, we show that the flow metrics satisfy a stronger version of the triangle inequality, which gives some information about their structure and geometry.

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# Chapter 1

## Introduction

Given a graph with non-negative edge weights, there are various ways to interpret the weights and derive a metric on the vertices. Two famous examples are to interpret the weights as lengths or as capacities, and the derived metric on the graph would be the shortest-path metric or the inverse of minimum cut (known to be an ultrametric), respectively. Another example is to think of the graph as an electrical network of resistors, and interpret the weights as conductance (inverse resistance), which yields the effective resistance (also called resistance distance).

It turns out that one can express all the above scenarios via flows. A flow on an edge-weighted graph  $G = (V, E, w)$  is a real function  $f$  over the edges, such that at each vertex, the incoming flow equals the outgoing flow, except for some set of boundary vertices, usually referred to as sources and targets. We fix an arbitrary orientation for the edges, and then the sign of  $f(e)$  determines the direction of the flow on the edge  $e$  (so informally  $f(-e) = -f(e)$  by convention). Let us examine how the abovementioned metrics are derived using flows.

**Shortest Path.** This is attained by considering a flow that ships one unit from a single source  $s \in V$  to a single target  $t \in V$ , while using the edges with the minimum total length. This is formulated as

$$d(s, t) = \min \left\{ \sum_{e \in E} |f(e)| \cdot w(e) : f \text{ ships 1 unit of flow from } s \text{ to } t \right\}. \quad (1.1)$$

We remark that this is a generalization of the shortest path, as shipping flow through multiple shortest paths is allowed.

**Minimum  $st$ -Cuts.** Using the well known mincut-maxflow theorem, the inverse of minimum cut can be viewed as a problem of minimizing congestion (the maximum “load” on an edge), formulated by

$$\frac{1}{\text{mincut}(s, t)} = \min \left\{ \max_{e \in E} \frac{|f(e)|}{w(e)} : f \text{ ships 1 unit of flow from } s \text{ to } t \right\}. \quad (1.2)$$

**Effective Resistance.** There are a couple of equivalent ways to define effective resistance, denoted  $R_{\text{eff}}$ . One way to define it is as a flow that obeys some physical rules

(called an *electrical flow*). Another way is via energy minimization, as

$$R_{\text{eff}}(s, t) = \min \left\{ \sum_{e \in E} \frac{|f(e)|^2}{w(e)} : f \text{ ships 1 unit of flow from } s \text{ to } t \right\}. \quad (1.3)$$

Essentially, each term  $\frac{|f(e)|^2}{w(e)}$  is the energy (heat dissipation) of an edge  $e$  with electrical resistance  $\frac{1}{w(e)}$ , and we look for a flow that minimizes the total energy. The effective resistance is known to capture a lot of properties of the underlying graph, such as commute time and random spanning trees; it also has a strong connection to the Laplacian of the graph.

**Flow Metrics.** A natural question that arises is how to generalize all the metrics seen above. Can we define a metric derived from flows that captures all three cases, and perhaps find more metrics in this family?

The definition we study, which we call the family of *flow metrics*, is the following. Given a weighted graph  $G = (V, E, w)$ , and a parameter  $1 \leq p < \infty$ , define the  $d_p$ -distance between  $s, t \in V$  to be

$$d_p(s, t) = \min \left\{ \left( \sum_{e \in E} \left| \frac{f(e)}{w(e)} \right|^p \right)^{1/p} : f \text{ ships 1 unit of flow from } s \text{ to } t \right\}. \quad (1.4)$$

To define  $d_\infty$ , we take the limit as  $p \rightarrow \infty$ , or equivalently change the objective  $\left( \sum_{e \in E} \left| \frac{f(e)}{w(e)} \right|^p \right)^{1/p}$  in (1.4) to  $\max_{e \in E} \left| \frac{f(e)}{w(e)} \right|$ . Various works give almost linear time algorithms for computing the  $d_p$ -distance, both in the weighted and unweighted case, see e.g. [ABKS21; AS20] where they refer to it as  $p$ -norm flows. It is immediate that this definition yields a metric on  $V$ . Moreover, it is easy to see that the case  $p = 1$  is in fact the ordinary shortest path metric (on a graph with edge lengths  $\frac{1}{w(e)}$ ), and  $\lim_{p \rightarrow \infty} d_p(s, t) = \frac{1}{\text{mincut}(s, t)}$ . Moreover, in the case  $p = 2$ ,  $d_2(s, t)^2$  is just the effective resistance between  $s$  and  $t$  in a graph  $G'$  with squared edge weights.

Thus, the family of  $d_p$ -metrics captures the shortest-path metric ( $p = 1$ ), the effective resistance ( $p = 2$ ), and minimum cuts ( $p = \infty$ ). Our goal is to better understand this family and its properties, and we make the first steps in this direction.

**Connection between different values of  $p$ .** The two extreme cases  $d_1$  (shortest path) and  $d_\infty$  (minimum cuts), are informally, not so well behaved compared to the resistance distance ( $p = 2$ ), and one might be able to interpolate naturally between them by using other values of  $p$  (e.g. see [LN04; CMSV17] for such applications).

**Capturing properties of the underlying graphs.** As mentioned earlier, the effective resistance ( $p = 2$ ) captures key properties of the underlying graph, and a natural direction is to extend this characterizations to other values of  $p$ , or to find other properties of the underlying graphs captured by them.

**Understanding the geometry of the flow metrics.** An important tool for understanding the structure of the  $d_p$ -metrics is embeddings, i.e. mapping  $G$  into a normed

space while preserving the  $d_p$  metric - in which case the mapping is called an isometry, or up to some error - in which case we say that the mapping has distortion  $> 1$ , see e.g. [Mat02; Mat97; Mat13]. Some results are known regarding the three special cases, for example shortest-path ( $d_1$ ) embeds isometrically into  $\ell_\infty$ ; effective resistance ( $d_2^2$ ) embeds isometrically into  $\ell_2^2$ ; and  $d_\infty$  embeds isometrically into  $\ell_1$ . We would like to find and compute such embeddings for other values of  $p$ , and furthermore, we would like to find the best trade-off between dimension and approximation of such embeddings (e.g. the Johnson-Lindenstrauss Lemma [JL84]).

**Small sketches.** Once we understand which metric spaces the flow metrics embed into, and reduce the dimension, we can easily design small sketches and exploit them to improve running time and storage requirements of algorithms. Other examples for graph problems that are naturally solved by such embedding techniques are multicommodity flows problems, cut sparsifiers, as well as spectral sparsifiers.

**Computing all-pairs distances.** Another important line of research is to compute the distance between all pairs of vertices simultaneously, or to construct a data structure that given a query of a pair of vertices, returns the exact  $d_p$ -distance (or an approximation to it) between them. Such constructions are known for the three special cases, e.g. Gomory-Hu tree [GH61] for  $p = \infty$ ; distance oracles [ABCP93; TZ05; Che15], All-Pairs Shortest-Path [Cha10; Sei95], and spanners [PS89; ADDJS93] for  $p = 1$ ; and [SS11; JS18] for  $p = 2$ , and it is an interesting direction to extend these approaches for other values of  $p$ .

**Reducing the size of the graph.** Techniques for reducing the size of the graph while preserving exactly or approximately a given metric, are an important tool that could improve the running time and memory usage of algorithms. One such technique is the well known Delta-Wye transform [Ken99], which removes a vertex of degree 3 from the graph and forms a triangle from its neighbors. It is known that for each of the special cases  $p = 1, 2, \infty$  (shortest path, effective resistance, and minimum cuts) there exists such a transform that preserve  $d_p$  and depends only on the 3 edges incident to the vertex being removed (i.e. oblivious to the rest of the graph). Thus, it is interesting to examine whether this could hold in general for other values of  $p$ .

Another method for reducing the size of the graph is via edge sparsification. This topic is very well studied in the literature and has various applications and results. The most noticeable ones are spanners [PS89; ADDJS93] ( $d_1$ -sparsifiers), resistance sparsifiers [DKW15; JS18; CGPSSW18] ( $d_2$ -sparsifiers), and cut sparsifiers [Kar93; BK96] ( $d_\infty$ -sparsifiers). There is also a stronger notion of spectral sparsifiers [ST04; SS11; BSS12], which preserve the quadratic form of the Laplacian of the graph, and in particular preserve both effective resistance and cuts. Hence, it is natural to try to achieve such results for other values of  $p$ , as well as give lower bounds to this problem.

## 1.1 Results for Graph-Size Reduction

**Lower Bound on Resistance Sparsifiers.** For  $p = 1$  and  $p = \infty$  there are known upper bounds and matching lower bounds, but for  $p = 2$  there is only an upper bound and no known lower bound. Our main result is the first lower bound for resistance sparsifiers

( $p = 2$ ). Formally, an  $\varepsilon$ -resistance sparsifier of a graph  $G$ , is a graph  $G'$  on the same vertex set  $V$  such that

$$\forall s, t \in V, \quad R_{\text{eff},G'}(s, t) \in (1 \pm \varepsilon) R_{\text{eff},G}(s, t). \quad (1.5)$$

We remark that  $G'$  does not have to be a subgraph of  $G$ , and moreover it can have edge weights. Chu et al [CGPSSW18] show that every graph  $G$  with  $n$  vertices admits an  $\varepsilon$ -resistance sparsifier with  $\tilde{O}(n/\varepsilon)$  edges. We conjecture that this is tight (up to the polylog factors), and prove a weaker lower bound.

**Conjecture 1.1.** *For every  $n \geq 2$  and every  $\varepsilon > \frac{1}{n}$ , there exists a graph  $G$  with  $n$  vertices, such that every  $\varepsilon$ -resistance sparsifier of  $G$  has  $\Omega(n/\varepsilon)$  edges.*

**Theorem 1.2.** *For every  $n \geq 2$  and every  $\varepsilon > \frac{1}{n}$ , there exists a graph  $G$  with  $n$  vertices, such that every  $\varepsilon$ -resistance sparsifier of  $G$  has  $\Omega(n/\sqrt{\varepsilon})$  edges.*

In fact, Theorem 1.2 is an easy consequence of the following bound regarding resistance sparsifiers of the clique.

**Lemma 1.3.** *Let  $G = (V, E, w)$  be a graph with  $|V| = n$  and  $|E| < \binom{n}{2}$ . Then,*

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} \geq 1 + \frac{1}{O(n^2)}. \quad (1.6)$$

Moreover, improving the bound in (1.6) to  $1 + \frac{1}{O(n)}$ , would immediately prove Conjecture 1.1. Thus, we focus on studying sparsifiers for the clique. In some cases, we can prove the stronger  $1 + \frac{1}{O(n)}$  bound. One very interesting case is of regular graphs, even when allowing arbitrary edge weights, including 0.

**Theorem 1.4.** *Let  $G = (V, E, w)$  be a  $k$ -regular graph with  $|V| = n$  and  $k < n - 1$ . Then,*

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} \geq 1 + \frac{1}{O(n)}. \quad (1.7)$$

Intuitively, the graphs that seem the best fit for sparsifying the clique are regular expanders, and these graphs are captured by Theorem 1.4. Thus, we believe that our proof can be generalized to any non-complete graph (i.e. with at least one missing edge), which would prove Conjecture 1.1. We discuss all these results in Section 3.1.

**Flow-Metric Sparsifiers.** Cohen and Peng [CP15] show that by sampling rows of a matrix  $A$  via an importance sampling approach, one can preserve up to some error the term  $\|Ax\|_p$  for every vector  $x$ , with high probability. We use this result to sparsify dense graphs while preserving the flow metrics up to some error.

**Theorem 1.5.** *Let  $G = (V, E, w)$  be a graph, fix  $p \in (\frac{4}{3}, \infty]$  with Hölder conjugate  $q$  (i.e.  $\frac{1}{p} + \frac{1}{q} = 1$ ), and let  $\varepsilon > 0$ . Then there exists a graph  $G' = (V, E', w')$  that is a  $d_p$ -sparsifier of  $G$ , i.e.*

$$\forall s, t \in V, \quad d_{p,G'}(s, t) \in (1 \pm \varepsilon) d_{p,G}(s, t), \quad (1.8)$$

and has  $|E'| = f(n, \varepsilon, p)$  edges, where

$$f(n, \varepsilon, p) = \begin{cases} n - 1 & \text{if } p = \Omega(\varepsilon^{-1} \log n), \\ \tilde{O}(n\varepsilon^{-2}) & \text{if } 2 < p < \infty, \\ \tilde{O}(n^{q/2}\varepsilon^{-5}) & \text{if } \frac{4}{3} < p < 2. \end{cases} \quad (1.9)$$



We discuss this result in section 3.2. Note that the case of  $p = 2$  is in fact the case of resistance sparsifiers, for which [CGPSSW18] show a better upper bound of  $\tilde{O}(n\varepsilon^{-1})$  edges. We remark that it is an open question to give lower bounds for this problem for  $p \neq 2$ . In particular, our proof of Theorem 1.2 does not extend to other values of  $p$ .

**Delta-Wye Transform for Flow Metrics.** The well known  $Y - \Delta$  transform [Ken99] (or in general - Schur Complement [Hay68]) is an example for a transform that removes a vertex of degree 3 from the graph in a way that preserves the effective resistance among all other vertices. An important aspect of the  $Y$ - $\Delta$  transform is that it is “local”, namely, its change to the graph depends only on the weights of the edges incident to the vertex being removed, and is oblivious to the rest of the graph. It is known that such transforms exist also for shortest-path and for minimum cuts. We show that such transforms do not exist for the family of flow metric for other values of  $p$ , stated informally as follows. The formal definitions are given in Section 3.3

**Theorem 1.6.** *Let  $p \in [1, \infty]$ . There exists a local transform that removes a vertex of degree 3 and forms a triangle from its neighbors that preserves the  $d_p$  metric among all other vertices, if and only if  $p = 1, 2, \infty$ .*

In addition, we study the interesting case of  $p = \infty$  (minimum cuts), and show that such a transform does not exist for removing a vertex of degree strictly larger than 3, even though there does exist one for shortest-path and effective resistance.

**Theorem 1.7.** *For every  $k > 3$ , there does not exist a local transform that removes a vertex of degree  $k$  and forms a  $k$ -clique from its neighbors that preserves  $d_\infty$  among all other vertices.*

Similarly to the  $Y$ - $\Delta$  transform for effective resistance, there also exist local transforms that remove a vertex of degree 2 or parallel edges, and preserve the effective resistance. We show that this transforms extend naturally to other values of  $p$ . We discuss these results in Section 3.3.

## 1.2 Additional Properties of the Flow Metrics

**The Flow Metrics are  $p$ -strong.** Towards understanding the geometry of the flow metrics, we show that they are  $p$ -strong, i.e. satisfy a stronger version of the triangle inequality.

**Theorem 1.8.** *Let  $G$  be a graph, and let  $p \in [1, \infty)$ . Then,*

$$\forall s, t, v \in V, \quad d_p(s, t)^p \leq d_p(s, v)^p + d_p(v, t)^p. \quad (1.10)$$

This is known to hold for the special cases of  $p = 1, 2$ , and also for  $p = \infty$ , where (1.10) becomes (in the limit after raising both sides to power  $1/p$ ) the ultrametric inequality. We further show that (1.10) is tight, namely for all  $p' > p$ , the metric  $d_p$  is not always  $p'$ -strong. We present it in Section 2.3.

**An Extension of Foster’s Theorem.** Foster’s Theorem [Fos49; Fos61] states that for every connected graph  $G = (V, E, w)$ ,

$$\sum_{xy \in E} w(xy) R_{\text{eff}}(x, y) = |V| - 1. \quad (1.11)$$

Since  $d_2^2$  is the effective resistance in a graph with squared weights, it is immediate that  $\sum_{xy \in E} w(xy)^2 d_2(x, y)^2 = |V| - 1$ . We extend this bound to all  $p > 1$ .

**Proposition 1.9.** *Let  $G = (V, E, w)$  be a connected graph, and let  $p > 1$ . Then,*

$$\text{if } p \geq 2, \quad \frac{|V|}{2} \leq \sum_{xy \in E} (w(xy) d_p(x, y))^{\frac{p}{p-1}} \leq |V| - 1, \quad (1.12)$$

$$\text{if } p \leq 2, \quad |V| - 1 \leq \sum_{xy \in E} (w(xy) d_p(x, y))^{\frac{p}{p-1}} \leq |E|. \quad (1.13)$$

We remark that on trees, the above sum equals  $|V| - 1 = |E|$  for every  $p \in (1, \infty]$ , including in the limit  $p \rightarrow 1$ . Furthermore, on unweighted cycles it holds that

$$\lim_{p \rightarrow \infty} \sum_{xy \in E} d_p(x, y)^{\frac{p}{p-1}} = \frac{|V|}{2}. \quad (1.14)$$

Thus, all four bounds are existentially tight, and cannot be strengthened. We discuss this in Section 2.2.

### 1.3 Related Work

The literature contains several variations of the flow metrics that have found applications. One example is  $p$ -resistance, formulated as

$$R_p(s, t) = \min \left\{ \sum_{e \in E} \frac{1}{w(e)} \cdot |f(e)|^p : f \text{ ships 1 unit of flow from } s \text{ to } t \right\}. \quad (1.15)$$

For fixed  $p$ , one can express  $d_p^p$  as  $R_p$  on a related graph  $G'$  with  $p$ -powered weights. However, as  $p \rightarrow \infty$ , the effect of the weights on  $R_p$  becomes negligible; a phenomenon that  $d_p$  does not suffer from. We utilize the connection between  $R_p$  and  $d_p$  to establish a connection between  $d_p$  and a dual problem of vertex potentials, as done in [AvL11] where such a connection (between flow and potentials) is shown. [AvL11] also shows a transition in the “behavior” of the  $p$ -resistance when moving from small values of  $p$  to large ones. The same  $p$ -resistance is shown in [Her10] to satisfy a variation of the triangle inequality, and we use their technique to show that the flow metrics satisfy a stronger version of the triangle inequality as well. The  $p$ -resistance was later extended in [NM16] by adding some penalty terms to the objective function, which help understanding the structure of the underlying graph, both local and global properties of it, mainly for clustering purposes.

Another variant, called  $p$ -norm flow, was studied in [HJPW21; FWY20; AKPS19; AS20; ABKS21]. Given a graph  $G = (V, E, w)$  with signed edge-vertex incidence matrix  $B$ , and a demands vector  $d \in \mathbb{R}^V$ , the goal is to find a flow  $f \in \mathbb{R}^E$  that satisfies the demands and minimizes some cost function. This is formulated as

$$\min_{B^T f = d} \text{cost}(f) \quad (1.16)$$

The formal definitions are given in Section 1.4. They studied different options for the cost function and for the constraints, such as  $\|f\|_p^p$  (unweighted),  $g^T f + \|W_1 f\|_2^2 + \|W_2 f\|_p^p$  (where  $g, W_1$  and  $W_2$  are some more variables of the problem that represent a gradient, 2-norm weights, and  $p$ -norm weights respectively), or even allowing the constraint  $B^T f \leq d$  or considering cases where  $B$  is some general matrix (i.e. regression problem). Their focus is on fast computation of the  $p$ -norm flows with respect to the proposed cost function and the constraints, either exactly or approximately, and use it to design fast algorithms for approximating maximum flow (see e.g. [AS20]), as well as an applications for graph clustering (see e.g. [FWY20]). Our focus is on objective  $\text{cost}(f) = \|W^{-1} f\|_p$  and demand vector  $d$  corresponding to a single source and a single target, i.e. has 1 in some entry (the source vertex),  $-1$  in another entry (the target vertex) and 0 everywhere else, the formal definitions are given in Section 1.4.

## 1.4 Notations and Problem Definition

Unless stated otherwise, we assume that graphs are connected, and have non-negative edge weights. For a weighted graph  $G = (V, E, w)$ , we fix an arbitrary orientation to the edges, i.e. for every edge, one of its endpoints is said to be the “head” of the edge and the other is said to be its “tail”. Let  $B_{m \times n}$  be a signed edge-vertex incidence matrix of

$G$  with respect to the given orientation, namely  $B_{e,x} = \begin{cases} 1 & \text{if } x \text{ is the head of } e, \\ -1 & \text{if } x \text{ is the tail of } e, \\ 0 & \text{o.w.;} \end{cases}$ . Let

$W_{m \times m}$  be a diagonal matrix where  $W_{e,e} = w(e)$ . For every vertex  $x \in V$  we denote by  $N(x) \subseteq V$  the set of neighbors of  $x$ . We also denote by  $\chi_x \in \mathbb{R}^V$  the unit basis vector with 1 in the entry corresponding to  $x$  and 0 everywhere else. In addition, we denote by  $\mathbf{1} \in \mathbb{R}^V$  the all ones vector. A flow that ships one unit from a source vertex  $s \in V$  to a target vertex  $t \in V$  is a function  $f : E \rightarrow \mathbb{R}$  such that  $B^T f = \chi_s - \chi_t$ , where the sign of  $f(e)$  represents the direction of the flow over the edge  $e$  (plus sign meaning the flow goes from the head to the tail, and minus sign represent flow that goes from the tail to the head). It is easy to verify that the condition  $B^T f = \chi_s - \chi_t$  implies that for every vertex other than  $s$  and  $t$ , the incoming flow equals the outgoing flow.

For fixed  $p \in [1, \infty]$ , the  $\ell_p$  cost of a flow  $f \in \mathbb{R}^E$  is defined to be

$$\|W^{-1} f\|_p = \left( \sum_{e \in E} \left| \frac{f(e)}{w(e)} \right|^p \right)^{1/p}. \quad (1.17)$$

Using this notation we can rewrite (1.4) as

$$d_{p,G}(s, t) = \min \left\{ \|W^{-1} f\|_p : B^T f = \chi_s - \chi_t \right\}. \quad (1.18)$$

We will omit the subscript  $G$  when it is clear from the context. We say that  $q \in [1, \infty]$  is the Hölder conjugate of  $p \in [1, \infty]$  if it is the (unique) parameter such that  $1/p + 1/q = 1$ , namely  $q = \frac{p}{p-1}$ , and by convention, 1 and  $\infty$  are Hölder conjugates of each other.

# Chapter 2

## Properties of the Flow Metrics

In this chapter we discuss some properties of the flow metrics. We begin by presenting in Section 2.1 the dual problem of (1.18), and using it to derive equivalent definitions of  $d_p$  that will be useful in subsequent sections. Moreover, we present a closed-form solution for  $d_p$ , which can be viewed as a generalization of Ohm's law and electrical flows.

In Section 2.2 we discuss some monotonicity properties of the flow metrics. Specifically, we show that for fixed  $p$ , the  $d_p$  distance is monotone in the edge weights, and for fixed edge weights, it is monotone in  $p$ . Moreover, we show a stronger monotonicity property of  $d_p$ , which leads to a generalization of Foster's Theorem (proving Proposition 1.9).

Finally, in Section 2.3 we show that for fixed  $p \in [1, \infty)$ ,  $d_p$  satisfies a stronger version of the triangle inequality (proving Theorem 1.8), which gives some information about the geometry of the flow metrics.

### 2.1 Basic Properties

In this section we show the dual problem of (1.18), and use it to provide characterizations of the flow metrics, which will be useful in different contexts.

**Dual Problem.** It is well known that (1.18) has a dual problem, that asks to optimize vertex potentials. Let  $G = (V, E, w)$  be a graph, and fix  $p \in [1, \infty]$ , having Hölder conjugate  $q$ . For a vector  $\varphi \in \mathbb{R}^V$ , viewed as vertex potentials, define its  $\ell_q$  cost to be

$$\|WB\varphi\|_q = \left( \sum_{xy \in E} w(xy)^q |\varphi_x - \varphi_y|^q \right)^{1/q}. \quad (2.1)$$

Similarly to [AvL11], for every  $s \neq t \in V$  we can consider the optimization problem,

$$\bar{d}_p(s, t) = \min \left\{ \|WB\varphi\|_q : (\chi_s - \chi_t)^T \varphi = 1 \right\}. \quad (2.2)$$

Proposition 4 in [AvL11] implies that the optimization problems (1.18) and (2.2) are equivalent in the following sense.

**Claim 2.1.** *Let  $p \in (1, \infty)$ . Let  $G = (V, E, w)$  be a weighted graph. Then,*

$$\forall s \neq t \in V, \quad d_p(s, t) = (\bar{d}_p(s, t))^{-1}. \quad (2.3)$$

We show how it is derived formally in Appendix A.1. We remark that  $\bar{d}_p$  is strongly related to  $\text{mincut}(s, t)$  because in the limit  $p \rightarrow \infty$ ,

$$\begin{aligned} \lim_{p \rightarrow \infty} \bar{d}_p(s, t) &= \lim_{p \rightarrow \infty} \min_{\varphi_s - \varphi_t = 1} \left( \sum_{xy \in E} |\varphi_x - \varphi_y|^{1 + \frac{1}{p-1}} w(xy)^{1 + \frac{1}{p-1}} \right)^{1 - \frac{1}{p}} \\ &= \min_{\varphi_s - \varphi_t = 1} \sum_{xy \in E} |\varphi_x - \varphi_y| \cdot w(xy) \\ &= \text{mincut}(s, t). \end{aligned} \quad (2.4)$$

Moreover, this shows that Claim 2.1 extends to  $p = \infty$ , because  $d_\infty(s, t) = (\text{maxflow}(s, t))^{-1}$ .

Furthermore, by swapping the constraints and the objectives of problem (2.2), we get the optimization problem,

$$\tilde{d}_p(s, t) = \max \left\{ (\chi_s - \chi_t)^T \varphi : \|WB\varphi\|_q = 1 \right\}, \quad (2.5)$$

and establish another connection.

**Claim 2.2.** *Let  $G = (V, E, w)$  be a graph, let  $p \in [1, \infty]$ . Then,*

$$\forall s \neq t \in V, \quad \tilde{d}_p(s, t) = (\bar{d}_p(s, t))^{-1}. \quad (2.6)$$

*Proof.* Indeed, let  $\varphi^*$  be a solution to (2.2), and define  $\tilde{\varphi} = \frac{\varphi^*}{\|WB\varphi^*\|_q}$ . Thus,

$$\tilde{d}_p(s, t) \geq (\chi_s - \chi_t)^T \tilde{\varphi} = (\chi_s - \chi_t)^T \frac{\varphi^*}{\|WB\varphi^*\|_q} = \frac{1}{\|WB\varphi^*\|_q} = (\bar{d}_p(s, t))^{-1}.$$

We can similarly show that  $\tilde{d}_p(s, t) \leq (\bar{d}_p(s, t))^{-1}$ , and the claim follows.  $\square$

Combining Claims 2.1 and 2.2, we conclude that  $d_p = \tilde{d}_p$ . Next, we use this connection to show another characterization of the flow metrics, which will be very useful in later sections.

**Claim 2.3.** *Let  $G = (V, E, w)$  be a graph, let  $p \in [1, \infty]$ , with Hölder conjugate  $q$ . Then,*

$$\forall s, t \in V, \quad d_p(s, t) = \max \left\{ \frac{(\chi_s - \chi_t)^T \varphi}{\|WB\varphi\|_q} : \varphi \in \mathbb{R}^V, \varphi \notin \text{span}\{\bar{\mathbf{1}}\} \right\}. \quad (2.7)$$

*Proof.* Since  $d_p = \tilde{d}_p$ , it suffices to show it for  $\tilde{d}_p$ . Fix some  $s \neq t \in V$ . First, let  $\phi^* \in \underset{\|WB\varphi\|_q=1}{\text{argmax}} (\chi_s - \chi_t)^T \varphi$ , and since  $\|WB\phi^*\|_q = 1$ , it holds that  $\phi^* \notin \text{span}\{\bar{\mathbf{1}}\}$  (otherwise

the norm would have been 0 since  $\ker B = \text{span}\{\bar{\mathbf{1}}\}$ ) and  $(\chi_s - \chi_t) \phi^* = \frac{(\chi_s - \chi_t) \phi^*}{\|WB\phi^*\|_q}$ . Hence, it is immediate that

$$\tilde{d}_p(s, t) = (\chi_s - \chi_t)^T \phi^* \leq \max_{\varphi \in \mathbb{R}^V: \varphi \notin \text{span}\{\bar{\mathbf{1}}\}} \frac{(\chi_s - \chi_t)^T \varphi}{\|WB\varphi\|_q}. \quad (2.8)$$

In the other direction, let  $\varphi^* \in \underset{\varphi \in \mathbb{R}^V: \varphi \notin \text{span}\{\bar{\mathbf{1}}\}}{\text{argmax}} \frac{(\chi_s - \chi_t)^T \varphi}{\|WB\varphi\|_q}$ , and define  $\tilde{\varphi} = \frac{\varphi^*}{\|WB\varphi^*\|_q}$ . It is

easy to see that  $\tilde{\varphi} \notin \text{span}\{\bar{\mathbf{1}}\}$ ,  $\|WB\tilde{\varphi}\|_q = 1$ , and moreover,

$$\tilde{d}_p(s, t) \geq (\chi_s - \chi_t)^T \tilde{\varphi} = \frac{(\chi_s - \chi_t)^T \varphi^*}{\|WB\varphi^*\|_q} = \max_{\varphi \in \mathbb{R}^V: \varphi \notin \text{span}\{\bar{\mathbf{1}}\}} \frac{(\chi_s - \chi_t)^T \varphi}{\|WB\varphi\|_q}. \quad (2.9)$$

$\square$

**Closed-Form Solution.** We remark that via the KKT-conditions we can get a closed-form solution for the flow metrics, which can be viewed as a generalization of Ohm's law and electrical flows, similarly to [HJPW21].

**Fact 2.4** (KKT conditions for  $d_p$ ). *Let  $G = (V, E, w)$  be a graph, let  $p \in (1, \infty)$  with Hölder conjugate  $q$ , let  $s, t \in V$ , and let  $f \in \mathbb{R}^E$  such that  $B^T f = \chi_s - \chi_t$ . Then,  $f$  is an optimal flow for  $d_p(s, t)$ , if and only if there exists a potentials vector  $\varphi \in \mathbb{R}^V$  such that for every edge  $xy \in E$ ,  $\varphi_x - \varphi_y = \frac{f(xy)|f(xy)|^{p-2}}{w(xy)^p}$ , or equivalently,  $f(xy) = w(xy)^q (\varphi_x - \varphi_y) |\varphi_x - \varphi_y|^{q-2}$ .*

We remark that using Fact 2.4, we can deduce a connection between the flow metrics and the  $p$ -Laplacian of the graph, as well as its second smallest eigenvalue (see [BH09] for details about the graph  $p$ -Laplacian and its second smallest eigenvalue). We present it in appendix A.2.

## 2.2 Monotonicity Properties

In this section we show that the flow metrics are “monotone” in two manners - in the edge weights and in the parameter  $p$ . We begin by showing that for fixed  $p$ , as the weights increase, the  $d_p$ -distance decrease. We then show that as  $p$  increases, the  $d_p$ -distance decreases. Moreover, we show even stronger monotonicity in  $p$ , that as  $p$  increases even  $d_p^q$  decreases, where  $q$  is the Hölder conjugate of  $p$  (and changes with  $p$ ). We then use these properties to show a generalization of Foster's Theorem.

### Monotonicity in the edge weights.

**Claim 2.5.** *Let  $G = (V, E)$  be a graph and let  $w, w' : E \rightarrow \mathbb{R}_+$  be two weight functions on  $E$ , such that for every edge  $e \in E$ ,  $w(e) \leq w'(e)$ . Fix  $p \in [1, \infty]$ , and let  $d_p, d'_p$  be the corresponding flow metrics on  $G$  with weight functions  $w$  and  $w'$  respectively. Then,*

$$\forall s, t \in V, \quad d_p(s, t) \geq d'_p(s, t). \quad (2.10)$$

*Proof.* Fix some  $s \neq t \in V$ , and let  $f^*$  be some minimizing  $st$  flow with respect to the weight function  $w$ , i.e.  $f^* \in \operatorname{argmin}_{B^T f = \chi_s - \chi_t} \|W^{-1} f\|_p$ . Thus, we can see that

$$\begin{aligned} d_p(s, t) &= \left( \sum_{e \in E} \left| \frac{f^*(e)}{w(e)} \right|^p \right)^{1/p} && \text{(by definition of } f^*) \\ &\geq \left( \sum_{e \in E} \left| \frac{f^*(e)}{w'(e)} \right|^p \right)^{1/p} && \text{(by } w' \geq w) \\ &\geq d'_p(s, t). \end{aligned}$$

□

**Monotonicity in  $p$ .** We recall that for all  $1 \leq p \leq p' \leq \infty$ , we have

$$\forall x \in \mathbb{R}^n, \quad \|x\|_{p'} \leq \|x\|_p \leq n^{1/p-1/p'} \|x\|_{p'}. \quad (2.11)$$

Using this bound, we get the following.

**Claim 2.6.** Let  $G = (V, E, w)$  be a graph. Let  $1 \leq p \leq p' \leq \infty$ , and let  $s, t \in V$ . Then

$$d_{p'}(s, t) \leq d_p(s, t) \leq |E|^{1/p-1/p'} d_{p'}(s, t). \quad (2.12)$$

*Proof.* Let  $f_p^* \in \operatorname{argmin}_{B^T f = \chi_s - \chi_t} \|W^{-1} f\|_p$  and  $f_{p'}^* \in \operatorname{argmin}_{B^T f = \chi_s - \chi_t} \|W^{-1} f\|_{p'}$ . Then,

$$\begin{aligned} d_p(s, t) &\leq \|W^{-1} f_p^*\|_p && \text{(by definition of } d_p) \\ &\leq |E|^{1/p-1/p'} \|W^{-1} f_{p'}^*\|_{p'} && \text{(by (2.11))} \\ &= |E|^{1/p-1/p'} d_{p'}(s, t) && \text{(by definition of } f_{p'}^*). \end{aligned}$$

By similar calculations we conclude that  $d_{p'}(s, t) \leq d_p(s, t)$ .  $\square$

The following corollary will come handy in Section 3.2, where we discuss flow metric sparsifiers.

**Corollary 2.7.** Let  $G = (V, E, w)$  be a graph on  $n$  vertices, let  $0 < \varepsilon < 1$ , and let  $p \geq 4\varepsilon^{-1} \log n$ . Then,

$$\forall s, t \in V, \quad d_\infty(s, t) \leq d_p(s, t) \leq (1 + \varepsilon) d_\infty(s, t). \quad (2.13)$$

*Proof.* By Claim 2.6, we have

$$d_\infty \leq d_p \leq |E|^{1/p} d_\infty. \quad (2.14)$$

Hence, for  $p \geq 4\varepsilon^{-1} \log n$  we get,

$$|E|^{1/p} = e^{\frac{1}{p} \log |E|} \leq e^{\varepsilon/2} \leq 1 + \varepsilon. \quad (2.15)$$

Thus, plugging this into (2.14) gives the desired result.  $\square$

## 2.2.1 A Generalization of Foster's Theorem

Next, we present another monotonicity property for the flow metrics, and use it to conclude lower and upper bounds on the sum  $\sum_{xy \in E} w(xy)^q d_p(x, y)^q$  (proving Proposition 1.9). These bounds can be viewed as a generalization of Foster's Theorem. We restate it here for clarity.

**Proposition 2.8.** Let  $G = (V, E, w)$  be a connected graph, and fix  $p > 1$  with Hölder conjugate  $q$ . Then,

$$\text{if } p \geq 2, \quad \frac{|V|}{2} \leq \sum_{xy \in E} w(xy)^q d_p(x, y)^q \leq |V| - 1, \quad (2.16)$$

$$\text{if } p \leq 2, \quad |V| - 1 \leq \sum_{xy \in E} w(xy)^q d_p(x, y)^q \leq |E|. \quad (2.17)$$

We remark that for defining the sum for  $p = 1$ , we take power  $1/q$  on both sides and take the limit  $p \rightarrow 1$ . Thus, (2.17) turns into  $\max_{xy \in E} \{w(xy) d_1(x, y)\} = 1$ . Proposition 2.8 is a consequence of the following lemma.

**Lemma 2.9.** Let  $G = (V, E, w)$  be a graph (possibly with parallel edges), fix  $p, p' \in (1, \infty]$  with Hölder conjugates  $q, q'$  respectively. Define  $w' = w^{q/q'}$ , and denote  $G' = (V, E, w')$ . If  $p \leq p'$  then

$$\forall s, t \in V, \quad d_{p,G}(s, t)^q \geq d_{p',G'}(s, t)^{q'}. \quad (2.18)$$

Otherwise, the inequality is reversed (by symmetry).

*Proof.* Let  $s \neq t \in V$ , and recall that by Claim 2.1

$$d_{p,G}(s, t) = (\bar{d}_{p,G}(s, t))^{-1} = \left( \min_{\varphi_s - \varphi_t = 1} \left( \sum_{xy \in E} w(xy)^q |\varphi_x - \varphi_y|^q \right)^{1/q} \right)^{-1}.$$

Denote a minimizer  $\varphi$  by

$$\varphi^* \in \operatorname{argmin}_{\varphi_s - \varphi_t = 1} \left( \sum_{xy \in E} w'(xy)^{q'} |\varphi_x - \varphi_y|^{q'} \right)^{1/q'}.$$

Then,

$$\begin{aligned} d_{p',G'}(s, t)^{-q'} &= \sum_{xy \in E} w'(xy)^{q'} |\varphi_x^* - \varphi_y^*|^{q'} && \text{(by definition of } \varphi^*) \\ &\geq \sum_{xy \in E} w'(xy)^{q'} |\varphi_x^* - \varphi_y^*|^q && \text{(by } q' \leq q \text{ and } |\varphi_x^* - \varphi_y^*| \leq 1) \\ &= \sum_{xy \in E} w(xy)^q |\varphi_x^* - \varphi_y^*|^q && \text{(by } w^q = (w')^{q'}) \\ &\geq d_{p,G}(s, t)^{-q}. \end{aligned}$$

□

Lemma 2.9 implies two of the bounds in Proposition 2.8 as easy corollaries.

**Corollary 2.10.** Let  $G = (V, E, w)$  be a connected graph (possibly with parallel edges) with  $|V| = n$  vertices. Fix  $p \in (1, \infty]$  with Hölder conjugate  $q$ . Then,

$$\sum_{xy \in E} w(xy)^q d_p(x, y)^q \leq n - 1 \iff p \geq 2. \quad (2.19)$$

*Proof.* For  $p \geq 2$  with Hölder conjugate  $q$ , denote  $w' = w^{q/2}$  and let  $d'_2$  is the corresponding flow metric on  $G' = (V, E, w')$ . By Lemma 2.9,

$$\sum_{xy \in E} w(xy)^q d_p(x, y)^q \leq \sum_{xy \in E} w'(xy)^2 d'_2(x, y)^2 = n - 1. \quad (2.20)$$

where the last equality is exactly Foster's Theorem about effective resistance. For  $p \in (1, 2]$  we use the symmetric case of Lemma 2.9 and Foster's Theorem again. □

We remark that the bound in (2.19) is tight (for every  $p > 1$ ), since on trees the sum always equals to  $n - 1$ .

Next, we give tight upper and lower bounds for the remaining cases. We begin by showing the following claim.



**Claim 2.11.** For every graph  $G = (V, E, w)$ , and every  $p > 1$  with Hölder conjugate  $q$ ,

$$\forall uv \in E, \quad w(uv)^q d_p(u, v)^q \leq 1, \quad (2.21)$$

which implies the upper bound  $\sum_{xy \in E} w(xy)^q d_p(x, y)^q \leq |E|$ .

We remark that these bounds hold also when taking the limit  $p \rightarrow 1$ .

*Proof.* Consider first the case  $p > 1$ . Fix an edge  $uv \in E$ , and fix  $p > 1$  with Hölder conjugate  $q$ . Then,

$$\begin{aligned} w(uv)^{-q} d_p(u, v)^{-q} &= w(uv)^{-q} \cdot \min_{\varphi_u - \varphi_v = 1} \left( \sum_{xy \in E} w(xy)^q |\varphi_x - \varphi_y|^q \right) \\ &= \min_{\varphi_u - \varphi_v = 1} \left( \sum_{xy \in E} \left( \frac{w(xy)}{w(uv)} \right)^q |\varphi_x - \varphi_y|^q \right) \\ &= \min_{\varphi_u - \varphi_v = 1} \left( 1 + \sum_{xy \in E \setminus \{uv\}} \left( \frac{w(xy)}{w(uv)} \right)^q |\varphi_x - \varphi_y|^q \right) \\ &\geq 1. \end{aligned} \quad (2.22)$$

The case  $p = 1$  follows by taking the limit.  $\square$

In fact, we have something stronger for several graphs.

**Claim 2.12.** For every unweighted graph  $G = (V, E)$  with no parallel edges,

$$\forall uv \in E, \quad \lim_{p \rightarrow 1} d_p(u, v)^q = 1. \quad (2.23)$$

and thus the upper bound  $\sum_{xy \in E} w(xy)^q d_p(x, y)^q \leq |E|$  is existentially tight in the limit  $p \rightarrow 1$ .

*Proof.* Similarly to the proof of Claim 2.11, we have

$$d_p(u, v)^{-q} = \min_{\varphi_u - \varphi_v = 1} \left( 1 + \sum_{xy \in E \setminus \{uv\}} |\varphi_x - \varphi_y|^q \right). \quad (2.24)$$

Define a potential function  $\varphi$  by  $\varphi_x = \begin{cases} 1 & \text{if } x = u, \\ 0 & \text{if } x = v, \\ \frac{1}{2} & \text{o.w.;} \end{cases}$ . Plugging this into (2.24) we get

$$\begin{aligned} d_p(u, v)^{-q} &\leq 1 + \sum_{xy \in E \setminus \{uv\}} |\varphi_x - \varphi_y|^q \\ &= 1 + \sum_{x \in N(u) \setminus \{v\}} \left( \frac{1}{2} \right)^q + \sum_{y \in N(v) \setminus \{u\}} \left( \frac{1}{2} \right)^q \\ &\leq 1 + \frac{2\Delta}{2^q}. \end{aligned} \quad (2.25)$$

where  $\Delta = \max_{v \in V} \deg(v)$ . Together with the bound from Claim 2.11, we have

$$1 \leq d_p(u, v)^{-q} \leq 1 + \frac{2\Delta}{2^q} \xrightarrow{q \rightarrow \infty} 1.$$

and the claim follows.  $\square$

We conclude the case  $p \leq 2$  in Proposition 2.8, by using Corollary 2.10 and Claim 2.11. Moreover, these bounds are tight - the lower bound is tight for  $p = 2$ , and the upper bound is tight for  $p \rightarrow 1$  (Claim 2.12). In particular, on trees the sum always equals  $|V| - 1 = |E|$ .

Finally, we give a lower bound for the sum  $\sum_{xy \in E} w(xy)^q d_p^q(x, y)$  when  $p > 2$ , which will conclude Proposition 2.8. Due to Lemma 2.9, it suffices to give a lower bound for the case  $p = \infty$ . Indeed, we have the following bound.

**Claim 2.13.** *Let  $G = (V, E, w)$  be a graph with  $|V| = n$  vertices. Then*

$$\sum_{xy \in E} w(xy) d_\infty(x, y) \geq \frac{n}{2}. \quad (2.26)$$

*Proof.* For every vertex  $x \in V$ ,

$$\begin{aligned} \sum_{y \in N(x)} w(xy) d_\infty(x, y) &= \sum_{y \in N(x)} \frac{w(xy)}{\text{mincut}(x, y)} \\ &\geq \sum_{y \in N(x)} \frac{w(xy)}{\text{weighted-deg}(x)} \\ &= 1. \end{aligned}$$

By rearranging the sum over the edges into a sum over the vertices, we obtain

$$\begin{aligned} \sum_{xy \in E} w(xy) d_\infty(x, y) &= \frac{1}{2} \sum_{x \in V} \sum_{y \in N(x)} w(xy) d_\infty(x, y) \\ &\geq \frac{n}{2}. \end{aligned}$$

□

We remark that the above lower bound is tight too, for example for an  $n$ -cycle the sum is indeed  $n/2$ .

Using Claim 2.13, we conclude the case of  $p \geq 2$  in Proposition 2.8.

## 2.3 $p$ -strong Triangle Inequality for Flow Metrics

In this section we show that the flow-metrics satisfy a stronger version of the triangle inequality, and discuss some of its properties.

**Definition 2.14** ( $p$ -strong metric). *Let  $(X, d)$  be a metric space, and fix  $p \geq 1$ . We say that  $d$  is  $p$ -strong if it satisfies the  $p$ -strong triangle inequality,*

$$\forall x, y, z \in X, \quad d(x, y)^p \leq d(x, z)^p + d(z, y)^p. \quad (2.27)$$

To extend the definition to  $p = \infty$ , we take power  $1/p$  of both side of the inequality and let  $p \rightarrow \infty$ . Inequality (2.27) then turns to  $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ .

**Theorem 2.15.** *Let  $G = (V, E, w)$  be a graph, and fix  $p \in [1, \infty)$ . Then  $d_p$  is  $p$ -strong.*

For  $p = 1, 2, \infty$  this was known to be true: For  $p = 1$  it is trivial, as (2.27) is just the regular triangle inequality. For  $p = \infty$  this is known to be true since  $d_\infty$  is an ultrametric. For  $p = 2$ , since  $d_2(s, t)^2$  is the effective resistance between  $s$  and  $t$ , it is related to the commute time. Namely, if we denote  $w_2(E) = \sum_{e \in E} w(e)^2$ , then  $\text{commute}(s, t) = 2w_2(E)d_2(s, t)^2$ . It is easy to see that for every  $s, t, v$  it holds that  $\text{commute}(s, t) \leq \text{commute}(s, v) + \text{commute}(v, t)$ , which leads to  $d_2(s, t)^2 \leq d_2(s, v)^2 + d_2(v, t)^2$ .

The proof of Theorem 2.15 uses the same technique as in [Her10], who showed a variation of the triangle inequality for the family of  $p$ -resistance.

*Proof.* (of Theorem 2.15) Let  $s, t, v \in V$ , and define a new graph  $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{w})$  that consists of two copies of  $G$ , and a single copy of  $v$ . Namely, a copy  $G_1 = (V_1, E_1, w)$  and another copy  $G_2 = (V_2, E_2, w)$ , where the two copies of  $v$  from  $V_1$  and  $V_2$  are identified with the same vertex  $v$ . An illustration is given in figure 2.1.

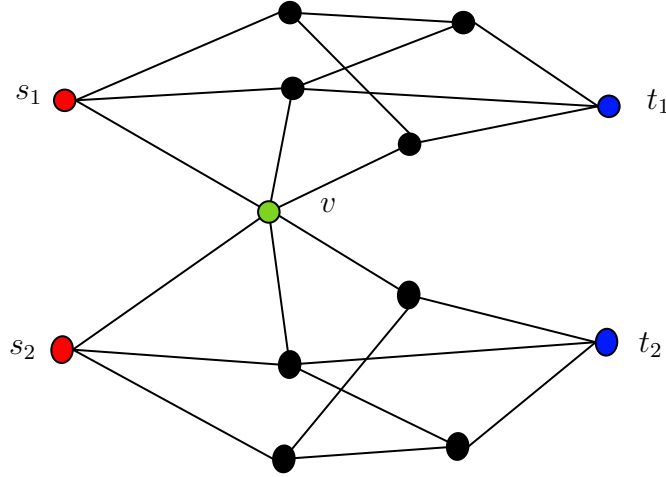


Figure 2.1: Illustration of the construction of the new graph

Given a flow  $f_1$  that ships 1 unit from  $s$  to  $v$  in the original graph  $G$ , and another flow  $f_2$  that ships 1 unit from  $v$  to  $s$  (also in  $G$ ), we can define a new flow  $\tilde{f}$  from  $s_1$  to  $t_2$  in  $\tilde{G}$  that will be the same on the first copy of  $G$  as  $f_1$ , and on the second copy of  $G$  it will be the same as  $f_2$ . This is a feasible flow that ships 1 unit from  $s_1$  to  $t_2$ , and it is easy to see that by taking  $f_1$  and  $f_2$  to be the minimizing flows that attain  $d_{p,G}(s, v)^p$  and  $d_{p,G}(v, t)^p$  respectively, we get,

$$d_{p,\tilde{G}}(s_1, t_2)^p \leq \sum_{e \in E_1} \left| \frac{f_1(e)}{w(e)} \right|^p + \sum_{e \in E_2} \left| \frac{f_2(e)}{w(e)} \right|^p = d_{p,G}(s, v)^p + d_{p,G}(v, t)^p. \quad (2.28)$$

Thus, it suffices to prove that  $d_{p,G}(s, t) \leq d_{p,\tilde{G}}(s_1, t_2)$ . By Claim 2.1 it is enough to prove that

$$\bar{d}_{p,G}(s, t) \geq \bar{d}_{p,\tilde{G}}(s_1, t_2). \quad (2.29)$$

Let  $q$  be the Hölder conjugate of  $p$ . Denote by  $W_{m \times m}, B_{m \times n}$  the diagonal weight matrix and the signed edge-vertex incident matrix of  $G$ , and denote by  $\tilde{W}, \tilde{B}$  the diagonal weight matrix and the signed edge-vertex incident matrix of  $\tilde{G}$ .

To prove (2.29), let  $\varphi^* \in \underset{\varphi \in \mathbb{R}^V: \varphi_s - \varphi_t = 1}{\operatorname{argmin}} \|WB\varphi\|_q$ , be a minimizing potential function on the old graph  $G$ , and we will use it to define a new potential function  $\tilde{\varphi}$  on  $\tilde{G}$  such

that  $\|WB\varphi^*\|_q \geq \left\| \widetilde{W}\widetilde{B}\widetilde{\varphi} \right\|_q$ . This will give,

$$\bar{d}_{p,G}(s,t) = \|WB\varphi^*\|_q \geq \left\| \widetilde{W}\widetilde{B}\widetilde{\varphi} \right\|_q \geq \bar{d}_{p,\widetilde{G}}(s_1,t_2), \quad (2.30)$$

which will conclude the proof.

For every  $u_1 \in V_1$ , define  $\widetilde{\varphi}(u_1) = \max\{\varphi^*(u), \varphi^*(v)\}$  and for every  $u_2 \in V_2$ , define  $\widetilde{\varphi}(u_2) = \min\{\varphi^*(u), \varphi^*(v)\}$  (note that  $\widetilde{\varphi}(v) = \varphi^*(v)$ ). This potential function is at least  $\varphi^*(v)$  on the first copy of  $G$ , and at most  $\varphi^*(v)$  on the second copy of  $G$ . Note that in particular,  $\widetilde{\varphi}(s_1) - \widetilde{\varphi}(t_2) = \varphi^*(s) - \varphi^*(t) = 1$ .

By translation of the potential, we may assume that  $\varphi^*(v) = 0$ . Fix some edge  $uu' \in E$ , and note that its contribution to  $\|WB\varphi^*\|_q^q$  is  $w(e)^q \cdot |\varphi^*(u) - \varphi^*(u')|^q$ , and the contribution of its corresponding two edges in  $\widetilde{G}$  to  $\left\| \widetilde{W}\widetilde{B}\widetilde{\varphi} \right\|_q^q$  is

$$w(u_1u'_1)^q \cdot |\widetilde{\varphi}(u_1) - \widetilde{\varphi}(u'_1)|^q + w(u_2u'_2)^q \cdot |\widetilde{\varphi}(u_2) - \widetilde{\varphi}(u'_2)|^q. \quad (2.31)$$

Next, let us examine (2.31) more carefully by separating into four cases.

- Case 1:  $\varphi^*(u), \varphi^*(u') \geq 0$ . In this case it holds that:

$$\widetilde{\varphi}(u_1) = \varphi^*(u), \quad \widetilde{\varphi}(u'_1) = \varphi^*(u'),$$

$$\widetilde{\varphi}(u_2) = \widetilde{\varphi}(u'_2) = 0.$$

Thus, the contribution of the corresponding edges in (2.31) is the same as in  $\|WB\varphi^*\|_q^q$  (i.e.  $w(uu')^q \cdot |\varphi^*(u) - \varphi^*(u')|^q$ ).

- Case 2:  $\varphi^*(u), \varphi^*(u') \leq 0$ . This case is very similar to the previous case since now:

$$\widetilde{\varphi}(u_1) = \widetilde{\varphi}(u'_1) = 0,$$

$$\widetilde{\varphi}(u_2) = \varphi^*(u), \quad \widetilde{\varphi}(u'_2) = \varphi^*(u').$$

Thus, again, the contribution of the edges is  $w(uu')^q \cdot |\varphi^*(u) - \varphi^*(u')|^q$ .

- Case 3:  $\varphi^*(u) \geq 0 \geq \varphi^*(u')$ . In this case we have:

$$\widetilde{\varphi}(u_1) = \varphi^*(u), \quad \widetilde{\varphi}(u'_1) = 0,$$

$$\widetilde{\varphi}(u_2) = 0, \quad \widetilde{\varphi}(u'_2) = \varphi^*(u').$$

Hence, the contribution is:

$$w(uu')^q \cdot (|\varphi^*(u) - 0|^q + |0 - \varphi^*(u')|^q) \leq w(uu')^q \cdot |\varphi^*(u) - \varphi^*(u')|^q.$$

where we used the fact that  $\varphi^*(u)$  and  $-\varphi^*(u')$  are non-negative, which implies that  $|\varphi^*(u) - \varphi^*(u')| = |\varphi^*(u)| + |-\varphi^*(u')|$ , and thus we could apply the inequality

$$\forall \alpha \geq 1, a, b \geq 0, \quad (a + b)^\alpha \geq a^\alpha + b^\alpha.$$

This inequality is proved below as Claim 2.17.

To conclude, we got that the contribution of the edges in this case is at most the contribution of the corresponding edge in  $G$ .

- **Case 4:**  $\varphi^*(u) \leq 0 \leq \varphi^*(u')$ . This case is analogous to the previous case, where we use the fact that  $|a - b| = |b - a|$  for any  $a, b \in \mathbb{R}$ , and then repeat the same arguments as in the previous case in order to reach the same conclusion.

To summarize, we got that for every edge  $e = uu' \in E$ , its contribution to  $\|WB\varphi^*\|_q^q$  is larger than the sum of the contributions of the corresponding edges  $e_1 = u_1u'_1 \in E_1$  and  $e_2 = u_2u'_2 \in E_2$  to  $\|\widetilde{W}\widetilde{B}\widetilde{\varphi}\|_q^q$ . Thus, it indeed holds that  $\|WB\varphi^*\|_q \geq \|\widetilde{W}\widetilde{B}\widetilde{\varphi}\|_q$ , which concludes the proof.  $\square$

Next, we present two simple properties of the  $p$ -strong triangle inequality.

**The  $p$ -strong Triangle Inequality is Monotone in  $p$ .** We show that if a metric is  $p$ -strong for some  $p \geq 1$ , then it is also  $p'$ -strong for all  $p' \in [1, p]$ .

**Proposition 2.16.** *Let  $(X, d)$  be a metric space that is  $p$ -strong for some  $p \geq 1$ . Then, for every  $p' \in [1, p]$ ,  $d$  is  $p'$ -strong as well.*

In particular, this proves that for every  $p \in [1, \infty)$ ,  $d_p$  is a metric. In order to prove Proposition 2.16, we will need the following simple claim, which we prove here for completeness.

**Claim 2.17.** *Let  $a_1, \dots, a_n \geq 0$  and let  $p > 0$ , then:*

$$\text{if } p \leq 1, \quad \sum_{i=1}^n a_i^p \geq \left( \sum_{i=1}^n a_i \right)^p. \quad (2.32)$$

$$\text{if } p \geq 1, \quad \sum_{i=1}^n a_i^p \leq \left( \sum_{i=1}^n a_i \right)^p. \quad (2.33)$$

*Proof.* (of Claim 2.17) Suppose  $p \geq 1$  (the other case is similar), let  $A = \sum_{i=1}^n a_i$ , and note that if  $A = 0$  then the statement clearly holds, hence we may assume that  $A > 0$ . By direct calculations,

$$\begin{aligned} \sum_{i=1}^n a_i^p &= A^p \cdot \sum_{i=1}^n \left( \frac{a_i}{A} \right)^p \\ &\leq A^p \cdot \sum_{i=1}^n \frac{a_i}{A} && \text{(by } a_i/A \leq 1 \text{ and } p \geq 1) \\ &= A^p. \end{aligned}$$

$\square$

*Proof.* (of Proposition 2.16) Let  $x, y, z \in X$ . Then,

$$\begin{aligned} d(x, y)^{p'} &\leq (d(x, z)^p + d(z, y)^p)^{\frac{p'}{p}} && \text{(by } p\text{-strong)} \\ &\leq d(x, z)^{p'} + d(z, y)^{p'} && \text{(by } p' \leq p \text{ and Claim 2.17)}. \end{aligned}$$

$\square$

**The  $p$ -strong Triangle Inequality is Tight for  $d_p$ .** We have shown that  $d_p$  is  $p$ -strong, and thus in particular it is also  $p'$ -strong for  $1 \leq p' \leq p$ . Next, we present a simple example that shows that in the general case, the power  $p$  cannot be strengthened.

**Claim 2.18.** *There exists a graph  $G = (V, E)$  such that for every  $p \geq 1$  and every  $\varepsilon > 0$ , the metric  $d_p$  is not  $(p + \varepsilon)$ -strong.*

*Proof.* Let  $G = (V, E)$  be a 2-path, i.e.  $V = \{s, t, v\}$  and  $E = \{\{s, v\}, \{v, t\}\}$ . It is easy to see that for every  $p \geq 1$ ,  $d_p(s, v) = d_p(v, t) = 1$ . Moreover, it holds that  $d_p(s, t) = (1^p + 1^p)^{1/p} = 2^{1/p}$ . It is easy to verify that indeed the  $p$ -strong triangle inequality holds, but for any  $\varepsilon > 0$ , we can see that

$$d_p(s, t)^{p+\varepsilon} = (2^{1/p})^{p+\varepsilon} = 2^{1+\varepsilon/p} > 2 = 1^{p+\varepsilon} + 1^{p+\varepsilon} = d_p(s, v)^{p+\varepsilon} + d_p(v, t)^{p+\varepsilon}, \quad (2.34)$$

where we used the fact that  $2^a$  for  $a > 0$  is greater than 1. □

# Chapter 3

## Graph-Size Reduction

In this chapter we discuss techniques for reducing the size of a graph while preserving its flow metric up to some error. We begin by examining the method of edge sparsification.

**Definition 3.1** ( $d_p$ -sparsifier). *Let  $G = (V, E, w)$  be a graph, let  $p \in [1, \infty]$ , and let  $\varepsilon > 0$ . A  $d_p$ -sparsifier of  $G$  is a graph  $G' = (V, E', w')$ , such that*

$$\forall s, t \in V, \quad d_{p,G'}(s, t) \in (1 \pm \varepsilon)d_{p,G}(s, t). \quad (3.1)$$

We remark that for the special cases of  $p = 1, 2, \infty$  there are known upper bounds: for  $p = 1$ , the definition coincides with multiplicative spanners; for  $p = 2$ , it coincides with resistance-sparsifiers; for  $p = \infty$ , there is the Gomory-Hu tree. We wish to generalize these constructions to other values of  $p$ .

Moreover, there are matching lower bounds for  $p = 1, \infty$ , but for  $p = 2$  no non-trivial lower bound is known. In section 3.1 we give the first lower bound for  $p = 2$  (Theorem 1.2). It is essentially a lower bound for resistance sparsifiers of the clique (Lemma 1.3), and we also discuss some special graph families (for the sparsifier) in which the lower bound can be strengthened, including regular graphs (Theorem 1.4), which intuitively should be the best fit for sparsifying the clique.

In Section 3.2 we present constructions of  $d_p$ -sparsifiers for other values of  $p$  (Theorem 1.5), that follow easily from a Theorem by Cohen and Peng [CP15]. Furthermore, we discuss the relation between the size of the sparsifier and the parameter  $p$ , as well as the gaps between the known constructions for  $p = 1, 2, \infty$  and our construction for other values of  $p$ .

In Section 3.3 we discuss a different method to reduce the size of graphs while preserving exactly the  $d_p$ -metric, known as the Delta-Wye transform, and its generalization the  $k$ -star-mesh transform for effective resistance; the formal definitions are given in Section 3.3. We examine for which values of  $p$  and  $k$  the  $d_p$  metrics admit such transforms. Specifically we show in Theorem 1.6 that for  $k = 3$  there exists an analogue of the Delta-Wye transform if and only if  $p = 1, 2, \infty$ , and in Theorem 1.7 that for  $p = \infty$  there exists an analogue of the  $k$ -star-mesh transform if and only if  $k \leq 3$ .

### 3.1 Lower Bound on Resistance Sparsifiers

In this section we make partial progress towards proving Conjecture 1.1, which asserts that in the worst case, an  $\varepsilon$ -resistance-sparsifier of a graph with  $n$  vertices requires at least  $\Omega(n/\varepsilon)$  edges. We begin by showing a weaker lower bound of  $\Omega(n/\sqrt{\varepsilon})$  edges, and then discuss some special cases in which we achieve the stronger lower bound.

**Definition 3.2** ( $\varepsilon$ -resistance sparsifier [DKW15]). Let  $G = (V, E, w)$  be a graph. An  $\varepsilon$ -resistance sparsifier of  $G$  is a graph  $G' = (V, E', w')$  such that

$$\forall x, y \in V, \quad R_{\text{eff}, G'}(x, y) \in (1 \pm \varepsilon) R_{\text{eff}, G}(x, y). \quad (3.2)$$

Conjecture 1.1 is inspired by the following open question.

**Open Question 3.3.** *Chu et al [CGPSSW18] show that every graph with  $n$  vertices admits an  $\varepsilon$ -resistance sparsifiers with  $\tilde{O}(n/\varepsilon)$  edges. Is this tight?*

It is known that a clique over  $n$  vertices admits an  $\varepsilon$ -resistance sparsifier with  $O(n/\varepsilon)$  edges. We present it formally in 3.1.2 for completeness. Thus, the best lower bound for sparsifying the clique is of  $\Omega(n/\varepsilon)$  edges (compared to  $\tilde{O}(n/\varepsilon)$  as stated in the question), which leads the following question, where we think of  $G$  as a possible resistance sparsifier of the clique.

**Open Question 3.4.** *Let  $G = (V, E, w)$  be a graph with  $|V| = n$  and  $|E| < \binom{n}{2}$ . Is it true that*

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} > 1 + \frac{1}{10n}?$$

We show a weaker bound than Question 3.4, that gives a lower bound of  $\Omega(n/\sqrt{\varepsilon})$  edges for an  $\varepsilon$ -resistance sparsifier in the worst case (Theorem 1.2). This is in fact Lemma 1.3, which we restate here for clarity.

**Lemma 3.5.** *For any graph  $G = (V, E, w)$  with  $|V| = n$  and  $|E| < \binom{n}{2}$ , it holds that*

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} > 1 + \frac{1}{O(n^2)}.$$

Before presenting the proof of Lemma 3.5, we show how this proves Theorem 1.2, which we restate here for clarity.

**Theorem 3.6.** *For every  $n \geq 2$  and every  $\varepsilon > \frac{1}{n}$ , there exists a graph  $G$  with  $n$  vertices, such that every  $\varepsilon$ -resistance sparsifier of  $G$  has  $\Omega(n/\sqrt{\varepsilon})$  edges.*

*Proof.* (of Theorem 3.6) To see this, take  $\Theta(n\sqrt{\varepsilon})$  distinct cliques, each of size  $\Theta(\varepsilon^{-1/2})$ ; removing even one edge will fail to achieve  $1 + \varepsilon$  approximation. This graph has  $\Theta(\varepsilon^{-1}) \cdot \Theta(n\sqrt{\varepsilon}) = \Theta(n/\sqrt{\varepsilon})$  edges, which concludes the lower bound.  $\square$

We remark that if Question 3.4 is true, it implies that every  $\frac{1}{10n}$ -resistance sparsifier of the clique must have  $\Omega(n^2)$  edges, and by following a similar proof as in Theorem 3.6 we conclude that in the worst case, an  $\varepsilon$ -resistance sparsifier requires  $\Omega(n/\varepsilon)$  edges (Conjecture 1.1).

We proceed to present the proof of Lemma 3.5. Throughout the proof, as well as in the next sections, we use the following equivalent definition of effective resistance for a graph  $G = (V, E, w)$ .

$$\forall s \neq t \in V, \quad R_{\text{eff}, G}(s, t) = \left( \min_{\varphi \in \mathbb{R}^V: \varphi_s - \varphi_t = 1} \sum_{xy \in E} w(xy) (\varphi_x - \varphi_y)^2 \right)^{-1}. \quad (3.3)$$

Moreover, we introduce the following notations. For a subset of edges  $F \subseteq E$  we denote  $w(F) = \sum_{e \in F} w(e)$ , and for every vertex  $x$  we denote its weighted degree by  $\deg_w(x) = \sum_{y \in V} w(xy)$ .



*Proof.* (of Lemma 3.5) Without loss of generality, we may assume that  $|E| = \binom{n}{2} - 1$  since a missing edge is an edge of weight 0. Let  $s, t \in V$  be the pair with a missing edge between them, denote by  $A$  the set of all the edges that touch  $s$  or  $t$ , and by  $B$  denote the set of the edges that do not touch them. Formally,

$$\begin{aligned} A &= \{\{x, y\} \in E : |\{x, y\} \cap \{s, t\}| = 1\}, \\ B &= \{\{x, y\} \in E : |\{x, y\} \cap \{s, t\}| = 0\}. \end{aligned}$$

Denote the average edge weight in each set by

$$\begin{aligned} \bar{\alpha} &= \mathbb{E}_{xy \in A} [w(xy)] = \frac{1}{|A|} w(A), \\ \bar{\beta} &= \mathbb{E}_{xy \in B} [w(xy)] = \frac{1}{|B|} w(B). \end{aligned}$$

Note that  $|A| = 2(n-2)$  and  $|B| = \binom{n-2}{2}$ . Now, by considering a potential function

$$\varphi \in \mathbb{R}^V \text{ given by } \varphi_x = \begin{cases} 1 & x = s, \\ 0 & x = t, \\ 1/2 & x \neq s, t; \end{cases} \text{ , we see that}$$

$$\begin{aligned} R_{\text{eff}}(s, t)^{-1} &\leq \sum_{xy \in A} w(xy) \frac{1}{4} \\ &= \frac{|A| \cdot \bar{\alpha}}{4} \\ &= \frac{n-2}{2} \cdot \bar{\alpha}. \end{aligned} \tag{3.4}$$

Similarly, for every  $x, y \in V \setminus \{s, t\}$ , by considering a potential function with values  $0, \frac{1}{2}, 1$  (similarly to the previous case), we get

$$R_{\text{eff}}(x, y)^{-1} \leq \frac{\deg_w(x) + \deg_w(y) + 2w(xy)}{4}. \tag{3.5}$$

Let us compute  $\mathbb{E}_{xy \in B} [\deg_w(x) + \deg_w(y) + 2w(xy)]$ . We will first compute the sum, and then divide by  $|B|$ .

$$\begin{aligned} \sum_{xy \in B} (\deg_w(x) + \deg_w(y) + 2w(xy)) &= 2w(B) + \sum_{xy \in B} (\deg_w(x) + \deg_w(y)) \\ &= 2w(B) + \sum_{x \in V \setminus \{s, t\}} (n-3) \deg_w(x) \\ &= 2w(B) + (n-3) \cdot (2w(E) - \deg_w(s) - \deg_w(t)) \\ &= 2w(B) + (n-3) \cdot (2w(B) + w(A)) \\ &= 2(n-2)w(B) + (n-3)w(A). \end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}
\mathbb{E}_{xy \in B} [\deg_w(x) + \deg_w(y) + 2w(xy)] &= \frac{1}{|B|} (2(n-2)w(B) + (n-3)w(A)) \\
&= 2(n-2)\bar{\beta} + \frac{n-3}{\binom{n-2}{2}}w(A) \\
&= 2(n-2)\bar{\beta} + \frac{n-3}{\frac{(n-2)(n-3)}{2}}w(A) \\
&= 2(n-2)\bar{\beta} + \frac{2}{n-2}w(A) \\
&= 2(n-2)\bar{\beta} + 4\bar{\alpha}.
\end{aligned} \tag{3.7}$$

We get that there exists  $uv \in B$  such that

$$R_{\text{eff}}(u, v)^{-1} \leq \mathbb{E}_{xy \in B} [R_{\text{eff}}(x, y)^{-1}] \leq \frac{2(n-2)\bar{\beta} + 4\bar{\alpha}}{4} = \frac{n-2}{2}\bar{\beta} + \bar{\alpha}. \tag{3.8}$$

where we used (3.5) and (3.7) in the second transition.

Moreover, note that for all  $\tau \in [0, 1]$ ,

$$\min \{R_{\text{eff}}(s, t)^{-1}, R_{\text{eff}}(u, v)^{-1}\} \leq \tau \cdot R_{\text{eff}}(s, t)^{-1} + (1 - \tau) \cdot R_{\text{eff}}(u, v)^{-1}. \tag{3.9}$$

We set  $\tau = \frac{2}{n-2}$ , thus  $1 - \tau = \frac{n-4}{n-2}$ , which combined with the bounds from (3.4) and (3.8), yields

$$\begin{aligned}
\min_{x \neq y \in V} R_{\text{eff}}(x, y)^{-1} &\leq \frac{2}{n-2} \cdot \frac{n-2}{2}\bar{\alpha} + \frac{n-4}{n-2} \cdot \left(\bar{\alpha} + \frac{n-2}{2}\bar{\beta}\right) \\
&= \left(2 - \frac{2}{n-2}\right)\bar{\alpha} + \frac{n-4}{2}\bar{\beta} \\
&= \left(\frac{n-3}{(n-2)^2}\right) \cdot 2(n-2)\bar{\alpha} + \frac{n-4}{2}\bar{\beta} \\
&= \left(\frac{n-3}{(n-2)^2}\right) \cdot w(E) - \left(\frac{n-3}{(n-2)^2}\right) \cdot \binom{n-2}{2}\bar{\beta} + \frac{n-4}{2}\bar{\beta} \\
&= \left(\frac{n-3}{(n-2)^2}\right) \cdot w(E) + \left(-\frac{n^2-6n+9}{2(n-2)} + \frac{n^2-6n+8}{2(n-2)}\right)\bar{\beta} \\
&< \left(\frac{n-3}{(n-2)^2}\right) \cdot w(E).
\end{aligned} \tag{3.10}$$

This gives us a lower bound on the maximum effective resistance in the graph.

In order to get an upper bound on the minimum effective resistance in the graph, let us compute the expectation of the effective resistance of a random edge  $e \in E$  sampled with probability  $\frac{w(e)}{w(E)}$ . By Foster's Theorem,

$$\begin{aligned}
\mathbb{E}_{xy \in E} [R_{\text{eff}}(x, y)] &= \frac{\sum_{xy \in E} w(xy) R_{\text{eff}}(x, y)}{w(E)} \\
&= \frac{n-1}{w(E)}.
\end{aligned} \tag{3.11}$$

There exists an edge  $u'v' \in E$  whose effective resistance is at most the expectation, i.e.

$$R_{\text{eff}}(u', v') \leq \frac{n-1}{w(E)}. \quad (3.12)$$

Altogether, we see that

$$\begin{aligned} \frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)^{-1}}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)^{-1}} &> \frac{\frac{w(E)}{n-1}}{\left(\frac{n-3}{(n-2)^2}\right) \cdot w(E)} \\ &= \frac{(n-2)^2}{(n-1)(n-3)} \\ &= \frac{n^2 - 4n + 4}{n^2 - 4n + 3} \\ &= 1 + \frac{1}{n^2 - 4n + 3}. \end{aligned} \quad (3.13)$$

□

### 3.1.1 Stronger Bound for Special Cases

In this section we present some special cases in which we can prove the stronger bound from Question 3.4. We begin by showing a technical lemma stating that if some condition holds, then the graph cannot guarantee better than  $(1 + 1/O(n))$ -approximation of the clique. Later, we show that as a matter of fact, this condition holds in some interesting special cases. One very interesting case is when the graph is regular and arbitrary edge weights are allowed, including zero (Theorem 1.4).

**Lemma 3.7.** *Let  $G = (V, E, w)$  be a graph with  $|V| = n$  vertices. Denote by  $\overline{D}_w$  the average weighted degree of the vertices, i.e.  $\overline{D}_w = \frac{1}{n} \sum_{x \in V} \deg_w(x) = \frac{2w(E)}{n}$ . Suppose that one of the following conditions holds.*

1. *There exists  $v \in V$  such that  $\deg_w(v) \leq \frac{\overline{D}_w}{2} \cdot \left(1 + \frac{1}{2n}\right)$ .*
2. *There exist  $s, t \in V$  such that  $\deg_w(s) + \deg_w(t) + 2w(st) \leq 2\overline{D}_w \cdot \left(1 + \frac{1}{2n}\right)$ .*

Then,

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} \geq 1 + \frac{1}{O(n)}. \quad (3.14)$$

We remark that the factor 2 in the denominator (in  $1 + \frac{1}{2n}$ ) is arbitrary, and in fact this can be proved for any constant  $c > 1$ .

*Proof.* Observe that by Foster's Theorem, we can calculate the expected effective resistance of an edge when sampling an edge  $e$  with probability proportional to its weight  $w(e)$ , as follows.

$$\begin{aligned} \mathbb{E}_{xy \in E} [R_{\text{eff}}(x, y)] &= \frac{\sum_{xy \in E} w(xy) R_{\text{eff}}(x, y)}{w(E)} \\ &= \frac{n-1}{\frac{n\overline{D}_w}{2}} \\ &= \frac{2}{\overline{D}_w} \left(1 - \frac{1}{n}\right). \end{aligned} \quad (3.15)$$

In particular, there exists an edge  $uv \in E$  such that  $R_{\text{eff}}(u, v) \leq \frac{2}{\overline{D_w}} \left(1 - \frac{1}{n}\right)$ .

Next, if the first condition holds, then by considering a potential function  $\varphi$  such that  $\varphi_x = \begin{cases} 1 & x = v, \\ 0 & \text{o.w.}; \end{cases}$ , we can see that for any vertex  $u \in V \setminus \{v\}$ ,

$$R_{\text{eff}}(v, u)^{-1} \leq \deg_w(v) \leq \frac{\overline{D_w}}{2} \cdot \left(1 + \frac{1}{2n}\right). \quad (3.16)$$

Similarly, if the second condition holds, then by considering a potential function  $\varphi$  such

that  $\varphi_x = \begin{cases} 1 & x = s, \\ 0 & x = t, \\ \frac{1}{2} & \text{o.w.}; \end{cases}$ , we see that

$$R_{\text{eff}}(s, t)^{-1} \leq \frac{\deg_w(s) + \deg_w(t) + 2w(st)}{4} \leq \frac{\overline{D_w}}{2} \cdot \left(1 + \frac{1}{2n}\right). \quad (3.17)$$

Hence, in either case it follows that,

$$\begin{aligned} \frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} &\geq \frac{\frac{2}{\overline{D_w} \cdot \left(1 + \frac{1}{2n}\right)}}{\frac{2}{\overline{D_w}} \left(1 - \frac{1}{n}\right)} \\ &\geq \left(1 + \frac{1}{n-1}\right) \cdot \left(1 - \frac{1}{2n}\right) \\ &= 1 + \frac{1}{n-1} - \frac{1}{2n} - \frac{1}{2n(n-1)} \\ &= 1 + \frac{1}{2(n-1)}. \end{aligned} \quad (3.18)$$

as desired.  $\square$

Observe that Lemma 3.7 gives the following theorem as a corollary.

**Theorem 3.8.** *Let  $G = (V, E, w)$  be a graph with  $|V| = n$  and  $|E| = m < \binom{n}{2}$ . For every vertex  $x \in V$ , denote  $N_x = |N(x)|$  and  $\deg_w(x) = \sum_{y \in N(x)} w(xy)$ . Suppose that one of the following conditions holds.*

1. *The graph is regular, i.e. there is  $a > 0$  such that for every  $x \in V$ ,  $N_x = a$ .*
2. *All the weighted degrees are the same, i.e. there is  $b > 0$  such that for every  $x \in V$ ,  $\deg_w(x) = b$ .*
3. *All the edge weights are the same, i.e. there is  $c > 0$  such that for every  $xy \in E$ ,  $w(xy) = c$ .*

Then,

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} \geq 1 + \frac{1}{O(n)}. \quad (3.19)$$

We remark that condition #1 in the above is in fact Theorem 1.4 presented in the introduction.

*Proof.* We will show that if one of the conditions in the theorem holds, then there exists a non-edge pair  $s, t \in V$  such that  $\deg_w(s) + \deg_w(t) \leq 2\overline{D_w}$  (i.e. condition #2 in Lemma 3.7 holds), which will give the desired result.

First, assume that

$$\sum_{x \in V} \deg_w(x) \cdot N_x \geq \frac{4m \cdot w(E)}{n}. \quad (3.20)$$

We will show later that if one of the conditions in the theorem holds, then (3.20) holds as well. Denote by  $F$  the set of all the missing edges in  $G$ , namely  $F = E(K_n) \setminus E$ . Now, we can see that when sampling a non-edge pair uniformly from  $F$ , the following holds.

$$\begin{aligned} \mathbb{E}_{st \in F} [\deg_w(s) + \deg_w(t)] &= \frac{1}{\binom{n}{2} - m} \left( \sum_{x \in V} \deg_w(x) (n - 1 - N_x) \right) \\ &= \frac{1}{\binom{n}{2} - m} \left( (n - 1) \cdot 2w(E) - \sum_{x \in V} \deg_w(x) \cdot N_x \right) \\ &\leq \frac{1}{\binom{n}{2} - m} \left( \binom{n}{2} \cdot \frac{4w(E)}{n} - \frac{4m \cdot w(E)}{n} \right) \\ &= 2\overline{D_w}. \end{aligned} \quad (3.21)$$

Hence, there exists a pair  $s, t \in V$  such that  $\deg_w(s) + \deg_w(t) \leq 2\overline{D_w}$  (since  $w(st) = 0$ ). Thus, condition #2 in Lemma 3.7 holds as claimed.

All we are left to show is that if one of the three conditions in the theorem holds, then (3.20) holds as well. Indeed, this can be seen as follows.

$$\begin{aligned} \sum_{x \in V} \deg_w(x) \cdot N_x &\stackrel{?}{\geq} \frac{4m \cdot w(E)}{n} \\ \iff \frac{\sum_{x \in V} \deg_w(x) \cdot N_x}{n} &\stackrel{?}{\geq} \frac{\sum_{y \in V} N_y}{n} \cdot \frac{\sum_{z \in V} \deg_w(z)}{n} \\ &\iff \mathbb{E}_{x \in V} [\deg_w(x) \cdot N_x] \stackrel{?}{\geq} \mathbb{E}_{y \in V} [\deg_w(y)] \cdot \mathbb{E}_{z \in V} [N_z] \\ &\iff \text{Cov}(\deg_w(x), N_x) \stackrel{?}{\geq} 0 \\ &\iff \mathbb{E}_{x \in V} [(\deg_w(x) - \overline{D_w})(N_x - \overline{d})] \stackrel{?}{\geq} 0 \end{aligned}$$

where we denote by  $\overline{d}$  the average number of neighbors of the vertices, i.e.

$$\overline{d} = \frac{1}{n} \sum_{x \in V} N_x = \frac{2m}{n}.$$

Observe that in the first two cases, one of the random variables ( $\deg_w(x)$  and  $N_x$ ) is constant, and thus  $(\deg_w(x) - \overline{D_w})(N_x - \overline{d}) = 0$  for every vertex  $x$ . In the third case, the two random variables are equal up to scaling by a constant, and we know  $\mathbb{E}[Z^2] \geq 0$ .

Thus, we arrive at the desired result.  $\square$

We remark that Theorem 3.8 does not say that any graph with maximal degree (number of neighbors)  $\leq (n - 2)$  cannot guarantee better than  $1 + \frac{1}{O(n)}$  approximation, as not every such graph can be “completed” to form an  $(n - 2)$ -regular graph. However, it does immediately give the following corollary.

**Corollary 3.9.** *Let  $G = (V, E, w)$  be a graph with  $|V| = n$  where  $n$  is even and with maximal degree  $\Delta < \frac{n}{2}$ . Then,*

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} \geq 1 + \frac{1}{O(n)} \quad (3.22)$$

*Proof.* Recall that by Dirac's Theorem, every graph with minimal degree  $\geq \frac{n}{2}$ , must be Hamiltonian. Thus, we may apply it on the complement of  $G$ , which leads to the conclusion that there exists a complete matching (complete since  $n$  is even) that does not belong to  $G$ . Hence, we can refer to  $G$  as an  $(n - 2)$ -regular graph, and by Theorem 3.8 we are done.  $\square$

In addition, observe that in Theorem 3.8, we essentially showed that if (3.20) holds, then the graph cannot guarantee better than  $\left(1 + \frac{1}{O(n)}\right)$ -approximation of the clique. Similarly, we can show the following.

**Claim 3.10.** *Let  $G = (V, E, w)$  with  $|V| = n$  vertices and  $|E| = m$  edges. For every vertex  $x \in V$ , denote  $N_x = |N(x)|$  and  $\deg_w(x) = \sum_{y \in N(x)} w(xy)$ . If*

$$\sum_{x \in V} \deg_w(x) \cdot N_x \leq \frac{4m \cdot w(E)}{n} - 2w(E), \quad (3.23)$$

Then,

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} \geq 1 + \frac{1}{O(n)} \quad (3.24)$$

*Proof.* Similarly to the proof of Theorem 3.8, we will show that there exists an edge  $xy \in E$  such that  $\deg_w(x) + \deg_w(y) + 2w(xy) \leq 2\overline{D}_w$  (where  $\overline{D}_w = \frac{1}{n} \sum_{x \in V} \deg_w(x) = \frac{2w(E)}{n}$ ), which by Lemma 3.7 will give the desired result.

By similar calculations as in Theorem 3.8 we can see that by sampling an edge uniformly the following holds.

$$\begin{aligned} \mathbb{E}_{xy \in E} [\deg_w(x) + \deg_w(y) + 2w(xy)] &= \frac{1}{m} \left( 2w(E) + \sum_{x \in V} \deg_w(x) \cdot N_x \right) \\ &\leq \frac{1}{m} \cdot \frac{4m \cdot w(E)}{n} \\ &= 2\overline{D}_w \end{aligned} \quad (3.25)$$

Hence, again, by Lemma 3.7 we are done.  $\square$

## The Symmetric Case

If we further add the assumption that the graph is symmetric, in the sense that all the edges that touch  $s$  and  $t$  (where  $st$  is the missing edge) have the same weight  $\alpha$ , and the rest of the edges have the same weight  $\beta$ , then we can prove the stronger bound from Question 3.4. Formally,

**Claim 3.11.** *Let  $G = (V, E, w)$  be a graph with  $|E| = \binom{n}{2} - 1$ . Let  $s, t \in V$  be the pair with a missing edge between them, and suppose that for every edge  $e \in E$ ,  $w(e) = \begin{cases} \alpha & e \text{ touches } s \text{ or } t, \\ \beta & \text{o.w.;} \end{cases}$  where  $\alpha, \beta \in \mathbb{R}^+$ . Then,*

$$\frac{\max_{x \neq y \in V} R_{\text{eff}}(x, y)}{\min_{x \neq y \in V} R_{\text{eff}}(x, y)} > 1 + \frac{1}{10n}.$$

We remark that this case is not necessarily contained within any other case which we have presented so far.

We present here a proof via direct calculations. We give an additional proof via the connection between effective resistance and commute time in Appendix B.1.

*Proof.* Again, denote by  $A$  the set of all the edges that touch  $s$  or  $t$ , and by  $B$  denote the set of the edges that do not touch them. By symmetry, for every  $uv, u'v' \in A$  it holds that  $R_{\text{eff}}(u, v) = R_{\text{eff}}(u', v')$ , and the same holds for every pair of edges in  $B$ . We will denote by  $R_A$  and  $R_B$  the resistances of the edges from  $A$  and  $B$  respectively.

Let us compute the effective resistance for each pair according to the sets.

1.  $R_{\text{eff}}(s, t)$ : On the one hand we can suggest a flow that splits equally to the neighbors of  $s$ , and then each of the neighbors ships the same amount of flow directly to  $t$ .

$$\begin{aligned} R_{\text{eff}}(s, t) &\leq \frac{1}{(n-2)^2} \cdot \left( \sum_{x \in N(s)} \frac{1}{\alpha} + \sum_{y \in N(t)} \frac{1}{\alpha} \right) \\ &= \frac{2}{(n-2)\alpha} \end{aligned} \quad (3.26)$$

On the other hand we can suggest a potential function  $\varphi_x = \begin{cases} 1 & , x = s \\ 0 & , x = t \\ 1/2 & , x \neq s, t \end{cases}$ .

$$\begin{aligned} R_{\text{eff}}(s, t)^{-1} &\leq \sum_{xy \in E} \alpha (\varphi_x - \varphi_y)^2 \\ &\leq 2(n-2)\alpha \cdot \frac{1}{4} \\ &= \frac{(n-2)\alpha}{2} \end{aligned} \quad (3.27)$$

Thus we conclude that

$$R_{\text{eff}}(s, t) = \frac{2}{(n-2)\alpha} \quad (3.28)$$

2.  $R_B$ : Take some vertices  $u, v \in V \setminus \{s, t\}$ . On the one hand we can suggest a flow that ships  $\tau/2$  amount of flow to each of  $s$  and  $t$ , and then ships the same amount from  $s$  and  $t$  to  $v$ . In addition, the flow will send  $\sigma$  on the edge  $uv$ , and then split the rest of the flow equally between every vertex  $x \in V \setminus \{s, t, v\}$ . Thus,

$$\begin{aligned} R_B &\leq \min_{\tau, \sigma \in [0,1]: \tau + \sigma \leq 1} \left( 2 \cdot 2 \cdot \frac{(\frac{\tau}{2})^2}{\alpha} + \frac{\sigma^2}{\beta} + 2(n-4) \frac{(\frac{1-\tau-\sigma}{n-4})^2}{\beta} \right) \\ &= \min_{\tau, \sigma \in [0,1]: \tau + \sigma \leq 1} \left( \frac{\tau^2}{\alpha} + \frac{\sigma^2}{\beta} + \frac{2}{(n-4)\beta} (1-\tau-\sigma)^2 \right) \end{aligned} \quad (3.29)$$

Let us denote the RHS by  $f(\tau, \sigma)$ , differentiate it with respect to  $\sigma$ , and compare

to 0 in order to find a minimum.

$$\begin{aligned}
& \frac{\partial f}{\partial \sigma}(\tau, \sigma) = 0 \\
\iff & \frac{2\sigma}{\beta} - \frac{4}{(n-4)\beta}(1-\tau-\sigma) = 0 \\
& \iff \left(\frac{n-2}{n-4}\right)\sigma = \frac{2}{n-4}(1-\tau) \\
& \iff \sigma = \frac{2}{n-2}(1-\tau)
\end{aligned}$$

Denote  $\sigma_0 = \frac{2}{n-2}(1-\tau)$ , and let  $f_{\sigma_0}(\tau) = f(\tau, \sigma_0)$ , thus,

$$\begin{aligned}
f_{\sigma_0}(\tau) &= \frac{\tau^2}{\alpha} + \frac{\left(\frac{2}{n-2}(1-\tau)\right)^2}{\beta} + \frac{2}{(n-4)\beta} \left(1-\tau - \frac{2}{n-2}(1-\tau)\right)^2 \\
&= \frac{\tau^2}{\alpha} + \frac{4}{(n-2)^2\beta}(1-\tau)^2 + \frac{2}{(n-4)\beta} \left((1-\tau) \left(1 - \frac{2}{n-2}\right)\right)^2 \\
&= \frac{\tau^2}{\alpha} + \frac{4}{(n-2)^2\beta}(1-\tau)^2 + \frac{2(n-4)}{(n-2)^2\beta}(1-\tau)^2 \tag{3.30} \\
&= \frac{\tau^2}{\alpha} + \frac{2}{(n-2)^2\beta}(1-\tau)^2(2+n-4) \\
&= \frac{\tau^2}{\alpha} + \frac{2}{(n-2)\beta}(1-\tau)^2
\end{aligned}$$

Again, let us now differentiate  $f_{\sigma_0}$  and compare to 0 in order to find a minimum.

$$\begin{aligned}
& f'_{\sigma_0}(\tau) = 0 \\
\iff & \frac{2}{\alpha}\tau - \frac{4}{(n-2)\beta}(1-\tau) = 0 \\
& \iff \tau \left(\frac{1}{\alpha} + \frac{2}{(n-2)\beta}\right) = \frac{2}{(n-2)\beta} \tag{3.31} \\
& \iff \tau = \frac{1}{1 + \frac{n-2}{2} \cdot \frac{\beta}{\alpha}}
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
R_B &\leq \frac{1}{\alpha} \cdot \left(\frac{1}{1 + \frac{n-2}{2} \cdot \frac{\beta}{\alpha}}\right)^2 + \frac{2}{(n-2)\beta} \left(\frac{\frac{n-2}{2} \cdot \frac{\beta}{\alpha}}{1 + \frac{n-2}{2} \cdot \frac{\beta}{\alpha}}\right)^2 \\
&= \frac{\frac{1}{\alpha} \left(1 + \frac{n-2}{2} \cdot \frac{\beta}{\alpha}\right)}{\left(1 + \frac{n-2}{2} \cdot \frac{\beta}{\alpha}\right)^2} \tag{3.32} \\
&= \frac{1}{\frac{n-2}{2} \cdot \beta + \alpha}
\end{aligned}$$



On the other hand we can suggest a potential function  $\varphi_x = \begin{cases} 1 & , x = u \\ 0 & , x = v \\ 1/2 & , x \neq u, v \end{cases}$ .

$$\begin{aligned} R_B^{-1} &\leq \sum_{xy \in E} w(xy) (\varphi_x - \varphi_y)^2 \\ &= \beta + 2 \cdot 2 \cdot \alpha \cdot \frac{1}{4} + (2(n-4))\beta \cdot \frac{1}{4} \\ &= \frac{1}{2} \cdot (2\alpha + (n-2)\beta) \end{aligned} \quad (3.33)$$

and thus

$$R_B = \frac{1}{\alpha + \frac{n-2}{2}\beta} \quad (3.34)$$

3.  $R_A$ : Recall that by Foster's Theorem we have

$$\sum_{xy \in E} w(xy) R_{\text{eff}}(x, y) = n - 1 \quad (3.35)$$

Thus,

$$\begin{aligned} n - 1 &= \sum_{xy \in A} \alpha R_A + \sum_{x'y' \in B} \beta R_B \\ &= 2(n-2)\alpha R_A + \binom{n-2}{2} \beta R_B \end{aligned}$$

which leads to the conclusion that

$$\begin{aligned} R_A &= \frac{(n-1) - \binom{n-2}{2} \beta R_B}{2(n-2)\alpha} \\ &= \frac{1}{2\alpha} \cdot \left( 1 + \frac{1}{n-2} - \frac{(n-3)}{2} \beta R_B \right) \end{aligned} \quad (3.36)$$

Hence, by using (3.34) we get that

$$R_A = \frac{1}{2\alpha} \cdot \left( 1 + \frac{1}{n-2} - \frac{(n-3)\beta}{2\alpha + (n-2)\beta} \right) \quad (3.37)$$

Assume towards contradiction that  $\frac{\max_{x \neq y} R_{\text{eff}}(x, y)}{\min_{x \neq y} R_{\text{eff}}(x, y)} \leq 1 + \frac{1}{10n}$ . On the one hand, by comparing  $R_{\text{eff}}(s, t)$  and  $R_B$  we can see that

$$\begin{aligned} \left( 1 + \frac{1}{10n} \right) &\geq \frac{\frac{2}{(n-2)\alpha}}{\frac{2}{2\alpha + (n-2)\beta}} \\ &= \frac{2}{n-2} + \frac{\beta}{\alpha} \\ \implies \frac{\beta}{\alpha} &\leq 1 + \frac{1}{10n} - \frac{2}{n-2} \end{aligned}$$

On the other hand, by comparing  $R_B$  and  $R_A$ , we see that

$$\begin{aligned}
\left(1 + \frac{1}{10n}\right) &\geq \frac{\frac{1}{\alpha + \frac{n-2}{2}\beta}}{\frac{1}{2\alpha} \cdot \left(1 + \frac{1}{n-2} - \frac{(n-3)\beta}{2\alpha + (n-2)\beta}\right)} \\
&= \frac{\frac{4\alpha}{2\alpha + (n-2)\beta}}{1 + \frac{1}{n-2} - \frac{(n-3)\beta}{2\alpha + (n-2)\beta}} \\
&= \frac{4\alpha}{2\alpha + (n-2)\beta + \frac{2\alpha + (n-2)\beta}{n-2} - (n-3)\beta} \\
&= \frac{2}{1 + \frac{1}{n-2} + \frac{\beta}{\alpha}}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\beta}{\alpha} + \frac{1}{n-2} + 1 &\geq 2 \left(1 - \frac{1}{10n+1}\right) \\
\iff \frac{\beta}{\alpha} &\geq 1 - \frac{1}{n-2} - \frac{2}{10n+1}
\end{aligned}$$

Hence, we conclude that

$$1 - \frac{1}{n-2} - \frac{2}{10n+1} \leq \frac{\beta}{\alpha} \leq 1 - \frac{2}{n-2} + \frac{1}{10n} \quad (3.38)$$

Which is a contradiction.  $\square$

### Discussion.

We remark that (3.23) can be viewed as follows.

$$\begin{aligned}
\sum_{x \in V} \deg_w(x) \cdot N_x &\leq \frac{4m \cdot w(E)}{n} - 2w(E) \\
\iff \sum_{x \in V} \frac{\deg_w(x)}{2w(E)} \cdot N_x &\leq \frac{2m}{n} - 1 \\
\iff \mathbb{E}_{x \sim \deg_w(x)} [N_x] &\leq \mathbb{E}_{x \in V} [N_x] - 1
\end{aligned} \quad (3.39)$$

where by  $x \sim \deg_w(x)$  we mean that a vertex  $x \in V$  is sampled with probability proportional to its weighted degree.

Recall that as we mentioned earlier, the proof of Theorem 3.8 essentially shows that if (3.20) holds, then the lower bound of  $1 + \frac{1}{O(n)}$  holds. Note that (3.20) can be viewed as follows.

$$\begin{aligned}
\sum_{x \in V} \deg_w(x) \cdot N_x &\geq \frac{4m \cdot w(E)}{n} \\
\iff \sum_{x \in V} \frac{\deg_w(x)}{2w(E)} \cdot N_x &\geq \frac{2m}{n} \\
\iff \mathbb{E}_{x \sim \deg_w(x)} [N_x] &\geq \mathbb{E}_{x \in V} [N_x]
\end{aligned} \quad (3.40)$$

A summary of our results is presented in Table 3.1.

Condition	Lower Bound on $\frac{\max R_{\text{eff}}}{\min R_{\text{eff}}}$
All graphs except the clique	$1 + \frac{1}{O(n^2)}$
$\min_{x \in V} \deg_w(x) \leq \frac{\overline{D}_w}{2} \left(1 + \frac{1}{(1+O(1))n}\right)$	$1 + \frac{1}{O(n)}$
$\min_{x \neq y \in V} (\deg_w(x) + \deg_w(y) + 2w(xy)) \leq 2\overline{D}_w \left(1 + \frac{1}{(1+O(1))n}\right)$	$1 + \frac{1}{O(n)}$
$\mathbb{E}_{x \sim \deg_w(x)} [N_x] \geq \mathbb{E}_{x \in V} [N_x]$	$1 + \frac{1}{O(n)}$
$\mathbb{E}_{x \sim \deg_w(x)} [N_x] \leq \mathbb{E}_{x \in V} [N_x] - 1$	$1 + \frac{1}{O(n)}$
Symmetric case	$1 + \frac{1}{O(n)}$

Table 3.1: Summary of lower bounds for resistance sparsifiers of the clique.

### 3.1.2 Upper Bound for Resistance Sparsifier of the Clique

In this subsection we complete the discussion about Question 3.3 by showing the upper bound of  $O(n/\varepsilon)$  edges for resistance sparsifier of the clique, as observed in [DKW15].

**Claim 3.12.** *The clique  $K_n$  admits an  $\varepsilon$ -resistance sparsifier with  $O(n/\varepsilon)$  edges.*

Claim 3.12 follows from the fact that  $R_{\text{eff}}(s, t) = \frac{2}{n}$  for all pairs of vertices in the clique (Fact 3.14) and the following theorem presented in [vLRH14].

**Theorem 3.13** (Proposition 5 in [vLRH14]). *Let  $G = (V, E, w)$  be a graph. For every vertex  $x \in V$  denote its weighted degree by  $\deg_w(x) = \sum_{y \in N(x)} w(xy)$ . Denote the minimum weighted degree by  $d_{\min}$  and the maximal edge weight by  $w_{\max}$ . Denote by  $\lambda_2(G)$  the second smallest eigenvalue of the normalized Laplacian of  $G$ . Then*

$$\left| R_{\text{eff}}(s, t) - \left( \frac{1}{\deg_w(s)} + \frac{1}{\deg_w(t)} \right) \right| \leq \frac{w_{\max}}{d_{\min}} \left( \frac{1}{\lambda_2(G)} + 2 \right) \left( \frac{1}{\deg_w(s)} + \frac{1}{\deg_w(t)} \right) \quad (3.41)$$

Since for expanders  $\lambda_2 = \Omega(1)$ , an  $\Theta(\varepsilon^{-1})$ -regular expander with all edges having the same weight  $\Theta(\varepsilon \cdot n)$  is an  $\varepsilon$ -resistance sparsifier for  $K_n$ .

**Fact 3.14.** *Let  $G = K_n$  be a clique over  $n$  vertices. Then*

$$\forall s, t \in V, \quad R_{\text{eff}}(s, t) = \frac{2}{n}.$$

*Proof.* By symmetry and Foster's theorem,

$$R_{\text{eff}}(s, t) = \frac{\sum_{xy \in E} R_{\text{eff}}(x, y)}{\binom{n}{2}} = \frac{n-1}{\binom{n}{2}} = \frac{2}{n}. \quad (3.42)$$

□

## 3.2 Flow Metric Sparsifiers

In this section we show that for some values of  $p$  there exists a  $d_p$ -sparsifier, where the sparsity of the graph depends on  $p$  and some additional parameters of the problem. This is in fact Theorem 1.5, which we restate here for clarity.

**Theorem 3.15.** Let  $G = (V, E, w)$  be a graph, fix  $p \in (\frac{4}{3}, \infty]$  having Hölder conjugate  $q$ , and let  $\varepsilon > 0$ . Then there exists a graph  $G' = (V, E', w')$  that is a  $d_p$ -sparsifier of  $G$ , i.e.

$$\forall s, t \in V, \quad d_{p,G'}(s, t) \in (1 \pm \varepsilon) d_{p,G}(s, t), \quad (3.43)$$

and has  $|E'| = f(n, \varepsilon, p)$  edges, where

$$f(n, \varepsilon, p) = \begin{cases} n - 1 & \text{if } p = \Omega(\varepsilon^{-1} \log n), \\ O(n \log(n/\varepsilon) (\log \log(n/\varepsilon))^2 \varepsilon^{-2}) & \text{if } 2 < p < \infty, \\ O(n^{q/2} \log(n) \log(1/\varepsilon) \varepsilon^{-5}) & \text{if } \frac{4}{3} < p < 2. \end{cases} \quad (3.44)$$

We remark that in general, the last row of the table applies for all  $1 < p < 2$ , but for  $1 < p \leq \frac{4}{3}$  it gives trivial bounds since this implies that  $q > 4$  and thus we have  $n^{q/2} > n^2$ . Moreover, we remark that as  $p$  tends to  $\infty$ , the  $d_p$  metric tends to the inverse of mincut( $s, t$ ) (ultra)metric, for which there exists cut sparsifiers [Kar93; BK96], that preserve *all* of the cuts in the graph, and for which there exists a lower bound [CKST19] of  $\Omega(n/\varepsilon^2)$  edges. However in our case, in order to preserve the  $d_\infty$  metric, it suffices to preserve only the minimum- $st$ -cuts. In addition, note that the case  $p = 2$  is the special case where  $d_2^2$  is in fact the resistance distance, for which [CGPSSW18] showed the existence of a resistance sparsifier with  $\tilde{O}(n\varepsilon^{-1})$  edges.

We now turn to proving Theorem 3.15 which follows easily from a theorem of Cohen and Peng [CP15].

**Theorem 3.16** (Theorem 7.1 in [CP15]). Given a matrix  $A \in \mathbb{R}^{m \times n}$  and parameters  $q \in (1, \infty), \varepsilon > 0$ , there exists a set of scores  $\{\tau_i(A, q)\}_{i=1}^m$  summing up to at most  $n$ , such that for any set of sampling values  $\{\sigma_i\}_{i=1}^m$  satisfying

$$\sigma_i \geq \tau_i(A, q) \cdot g(n, \varepsilon, q)$$

if we generate a matrix  $S$  with  $N = \sum_{i=1}^m \sigma_i$  rows, each chosen independently as  $\frac{1}{\sigma_i^{1/q}} \cdot \vec{e}_i$  with probability  $\frac{\sigma_i}{N}$  ( $\vec{e}_i \in \mathbb{R}^m$  is the  $i^{\text{th}}$  basis vector), then with probability at least  $1 - \frac{1}{n^{\Omega(1)}}$  we have

$$\forall \varphi \in \mathbb{R}^n, \quad \|SA\varphi\|_q \in (1 \pm \varepsilon) \|A\varphi\|_q$$

where

$$g(n, \varepsilon, q) = \begin{cases} \log(n) \varepsilon^{-2} & \text{if } q = 1, \\ \log(n/\varepsilon) (\log(\log(n/\varepsilon)))^2 \varepsilon^{-2} & \text{if } 1 < q < 2, \\ n^{\frac{q}{2}-1} \log(n) \log(1/\varepsilon) \varepsilon^{-5} & \text{if } 2 < q. \end{cases} \quad (3.45)$$

Note that for the proof of existence, we can set  $\sigma_i = \tau_i(A, q) \cdot g(n, \varepsilon, q)$  and thus a sufficient number of rows in the matrix  $S$  will satisfy  $N = \sum_{i=1}^m \tau_i(A, q) g(n, \varepsilon, q) \leq n \cdot g(n, \varepsilon, q)$ . We are ready to present the proof of Theorem 3.15.

*Proof.* (of Theorem 3.15) We will first deal with the last two rows in the table of the theorem ( $\frac{4}{3} < p < \infty$ ). Let  $A = WB$  where  $W_{m \times m}$  is the diagonal weight matrix and  $B_{m \times n}$  is the signed edge-vertex incidence matrix of  $G$ . By Theorem 3.16, there exists a matrix  $S$  with  $N$  rows (same  $N$  as defined in Theorem 3.16) where each row is a reweighted basis vector, such that

$$\forall \varphi \in \mathbb{R}^n, \quad \|SWB\varphi\|_q \in (1 \pm \varepsilon) \|WB\varphi\|_q \quad (3.46)$$

with  $q$  being the Hölder conjugate of  $p$ . Note that  $q = \frac{p}{p-1}$ , and thus the case  $2 < p < \infty$  corresponds to the case  $1 < q < 2$ , and the case  $2 < q$  corresponds to the case  $1 < p < 2$ .

Let  $W' = (S^T S)^{1/2} W$ , and note that it is a diagonal matrix of dimensions  $m \times m$  where the entries on the diagonal are in fact the weights of the edges corresponding to the entries in  $SW$ . Let  $G'$  be the graph defined by the weights  $W'$ , and let  $d'_p$  be the flow metric on  $G'$ . It is important to note that since  $S$  consists of  $N$  rows, then  $W'$  has at most  $N$  non-zero entries, and moreover it is easy to see that for any  $\varphi \in \mathbb{R}^n$  it holds that  $\|W' B \varphi\|_q = \|S W B \varphi\|_q$ .

Next, recall that by Claim 2.1

$$\forall s \neq t \in V, \quad d_p(s, t) = \left( \min_{\varphi_s - \varphi_t = 1} \|W B \varphi\|_q \right)^{-1}. \quad (3.47)$$

and similarly for  $d'_p$  using  $W'$ .

Now, we can see that

$$\begin{aligned} d'_p(s, t)^{-1} &= \min_{\varphi_s - \varphi_t = 1} \|W' B \varphi\|_q \\ &\leq (1 + \varepsilon) \min_{\varphi_s - \varphi_t = 1} \|W B \varphi\|_q \\ &= (1 + \varepsilon) d_p(s, t)^{-1} \end{aligned} \quad (3.48)$$

where the inequalities follow because we can take a minimizer  $\varphi^*$  of  $\|W B \varphi\|_q$ , apply (3.46) on it, and conclude an upper bound on the minimum of  $\|W' B \varphi\|_q$ . The other direction is similar, and thus we can conclude that  $G'$  is a  $d_p$  sparsifier of  $G$  that satisfies the guarantee of (3.43).

Finally, recalling that  $N \leq n \cdot g(n, \varepsilon, q)$ , gives the desired bound on the number of the edges in  $G'$ .

Next, for the first row in the table (the case  $p = \Omega(\varepsilon^{-1} \log n)$ ) we recall that by Corollary 2.7,

$$\forall s, t \in V, \quad d_\infty(s, t) \leq d_p(s, t) \leq (1 + \varepsilon) d_\infty(s, t). \quad (3.49)$$

Thus, we choose  $G'$  to be the Gomory-Hu tree of  $G$ , and clearly Corollary 2.7 and (3.49) can be applied also to  $d'_p$ . It is easy to see that this is indeed a  $d_p$  sparsifier for  $G$ , since we have

$$\begin{aligned} d'_p(s, t) &\leq (1 + \varepsilon) d'_\infty(s, t) && \text{(by (3.49) on } G') \\ &= (1 + \varepsilon) d_\infty(s, t) && \text{by GH guarantee} \\ &\leq (1 + \varepsilon) d_p(s, t) && \text{(by (3.49) on } G) \end{aligned}$$

The other direction is similar, and this completes the proof of Theorem 3.15.  $\square$

### 3.3 Transforms that Preserve the Flow Metrics, and Those that do not Exist

In this section, we present transformations that reduce the number of edges/vertices in the graph in some cases, while preserving the flow metrics on them. These transformations are closely related to known transformations for effective resistance. We begin with reductions of parallel edges and of sequential edges, that are natural extensions of corresponding reductions for effective resistance. We then proceed to discuss the well known  $Y$ - $\Delta$  transform for effective resistance, as well as the more general  $k$ -star-mesh transform, and examine for which values of  $k$  and  $p$  there exists an analogue of it for  $d_p$ .

### 3.3.1 Sequential Edges Reduction

Think of the case presented in figure 3.1, where a vertex  $x$  of degree 2 is connected with an edge  $e_1$  of weight  $\alpha$  to a vertex  $a$ , and with an edge  $e_2$  of weight  $\beta$  to another vertex  $b \neq a$ . We wish to remove the vertex  $x$  and find a weight  $\gamma = \gamma(\alpha, \beta)$  for a new edge that will now connect  $a$  and  $b$ , and will preserve the flow metric of the graph.

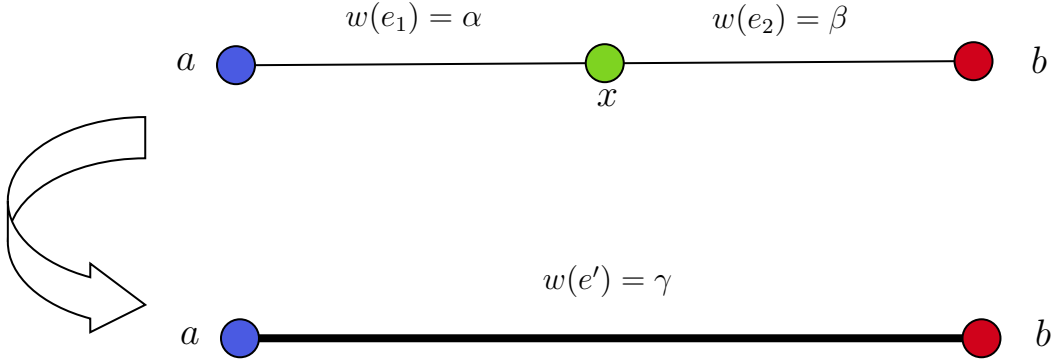


Figure 3.1: Sequential edges to single edge transform.

**Claim 3.17.** *Let  $p \in [1, \infty]$ . Let  $G = (V, E, w)$  be a graph that contains a vertex  $x$  that is incident to exactly 2 edges  $e_1 = \{x, a\}$  and  $e_2 = \{x, b\}$  of weights  $\alpha$  and  $\beta$  respectively. Let  $G' = (V', E', w')$  be a graph with  $V' = V \setminus \{x\}$ ,  $E' = E \setminus \{e_1, e_2\} \cup \{\{a, b\}\}$  and*

$$w'(e) = \begin{cases} \frac{1}{\left(\frac{1}{\alpha^p} + \frac{1}{\beta^p}\right)^{1/p}} & \text{if } e = \{a, b\}, \\ w(e) & \text{o.w.;} \end{cases}$$

*Then, for every  $s, t \in V'$ ,  $d_{p,G}(s, t) = d_{p,G'}(s, t)$ .*

Before we begin the proof, note that for  $p = 1$ , we have  $\frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta}$ , which is the desired behavior since  $d_1$  coincides with the shortest-path metric in the graph with inverse edge weights. Moreover, note that as  $p \rightarrow \infty$ , it holds that  $\gamma \rightarrow \min\{\alpha, \beta\}$ , which is the desired behavior for the minimum cut in the graph.

*Proof.* Note that in the graph  $G$ , for every amount of flow  $\tau$  that flows from  $a$  to  $x$ , the contribution of the flow (without the  $1/p$  power) over the edges that connect  $a$  to  $b$  in this case is exactly  $\left|\frac{\tau}{\alpha}\right|^p + \left|\frac{\tau}{\beta}\right|^p = |\tau|^p \cdot \left(\frac{1}{\alpha^p} + \frac{1}{\beta^p}\right)$ , where in  $G'$  it will be  $\left|\frac{\tau}{\gamma}\right|^p$ . So setting

$$\gamma = \frac{1}{\left(\frac{1}{\alpha^p} + \frac{1}{\beta^p}\right)^{1/p}}$$

indeed satisfies our demands, since it holds that  $\frac{1}{\gamma^p} = \frac{1}{\alpha^p} + \frac{1}{\beta^p}$ . □

### 3.3.2 Parallel Edges Reduction

Think of the case presented in figure 3.2, where two vertices  $a$  and  $b$  are connected with two edges  $e_1$  and  $e_2$  of some weight  $\alpha$  and  $\beta$  respectively. Again, we wish to find a weight  $\gamma = \gamma(\alpha, \beta)$  that will replace the two edges and preserve the flow metric in the graph.

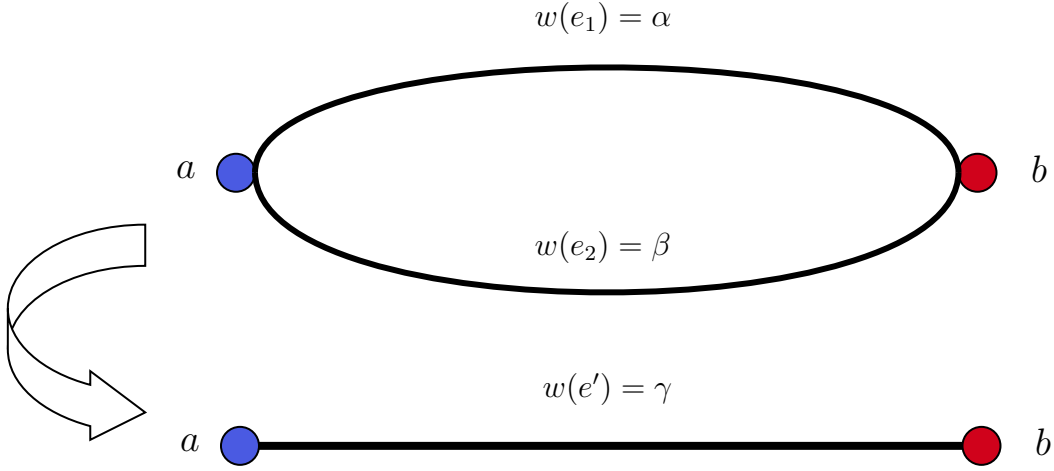


Figure 3.2: Parallel edges to single edge transform.

**Claim 3.18.** *Let  $p \in [1, \infty]$ , and let  $q$  be the Hölder conjugate of  $p$ . Let  $G = (V, E, w)$  be a graph that contains two parallel edges  $e_1$  and  $e_2$  of weights  $\alpha$  and  $\beta$  respectively between some vertices  $a$  and  $b$ . Let  $G' = (V, E', w')$  be a graph with  $E' = E \setminus \{e_1\}$  and*

$$w'(e) = \begin{cases} (\alpha^q + \beta^q)^{1/q} & \text{if } e = e_2, \\ w(e) & \text{o.w.;} \end{cases}$$

*Then, for every  $s, t \in V'$ ,  $d_{p,G}(s, t) = d_{p,G'}(s, t)$ .*

Let us examine what happens to  $\gamma$  as  $p \rightarrow \infty$ . In this case,  $q \rightarrow 1$ , and thus  $\gamma \xrightarrow{p \rightarrow \infty} \alpha + \beta$ , which is the desired behavior for minimum cut. Moreover, note that as  $p \rightarrow 1$  then  $q \rightarrow \infty$ . Then  $\gamma \xrightarrow{p \rightarrow 1} \max\{\alpha, \beta\}$ , and hence  $\frac{1}{\gamma} \xrightarrow{p \rightarrow 1} \min\left\{\frac{1}{\alpha}, \frac{1}{\beta}\right\}$ , which is again the desired behavior since  $d_1$  coincides with the shortest-path metric in the graph with inverse edge weights.

We present here a proof via the dual problem. We give an additional proof via flows in Appendix B.2.1.

*Proof.* We consider the dual problem. Recall that  $d_p(s, t) = (\bar{d}_p(s, t))^{-1}$  (Claim 2.1), where

$$\bar{d}_p(s, t) = \min_{\varphi_s - \varphi_t = 1} \|WB\varphi\|_q.$$

Thus, we can finish the proof by showing that the  $\bar{d}_p$  metric is preserved in the new graph  $G'$ . Suppose we have potentials  $\varphi_a, \varphi_b$ , and we wish to find  $\gamma$  such that the potential difference between  $a$  and  $b$  is preserved, i.e.

$$\gamma^q \cdot |\varphi_a - \varphi_b|^q = \alpha^q \cdot |\varphi_a - \varphi_b|^q + \beta^q \cdot |\varphi_a - \varphi_b|^q = (\alpha^q + \beta^q) \cdot |\varphi_a - \varphi_b|^q.$$

and thus it is easy to see that setting  $\gamma = (\alpha^q + \beta^q)^{1/q}$  will satisfy our requirement.  $\square$

### 3.3.3 Non-Existence of Y- $\Delta$ Transform

Suppose that a graph has a vertex of degree 3, and we wish to remove it while preserving the  $d_p$  metric between the remaining vertices, thus obtaining a smaller equivalent instance to work with. The way to do this for the resistance distance is via the well known Y- $\Delta$  transform (as shown in figure 3.3), and we wish to generalize it to all flow metrics.

It is known that such a transformation exists for  $p = 1$  (shortest-path metric) and for  $p = \infty$  (specifically for the case where the middle vertex has degree 3 - see Theorem 1 in [CSWZ00], we elaborate on this in subsection 3.3.3). In addition, it is important to note that the Y- $\Delta$  transform for effective resistance, depends solely on the weights of the edges incident to the vertex of degree 3, i.e. it is a local transformation that does not depend on the rest of the graph.

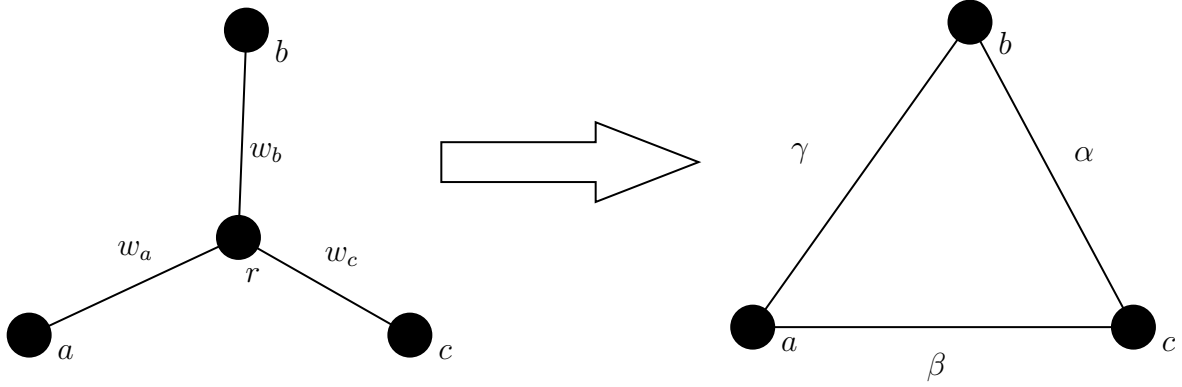


Figure 3.3: Transforming “Y” (or a 3-star) into a “ $\Delta$ ” (a triangle) by deleting the middle vertex  $r$ , and creating new edges while preserving the effective resistance on the graph. In the general case, an  $n$ -star transforms into a clique  $K_n$ .

In the case of effective resistance, the transformation is as follows (the notations of the weights are as presented in figure 3.3).

$$\alpha = \frac{w_b \cdot w_c}{w_a + w_b + w_c}, \quad \beta = \frac{w_a \cdot w_c}{w_a + w_b + w_c}, \quad \gamma = \frac{w_a \cdot w_b}{w_a + w_b + w_c}$$

Unfortunately, such a transformation does not exist when  $p \neq 1, 2, \infty$ , as we show next.

We first define the transform for general  $k$ . Note that for general values of  $k$ , the transform is called a  $k$ -star-mesh transform, where in this section we focus on the special case of  $k = 3$  (and we will continue to call it Y- $\Delta$ -transform).

**Definition 3.19** ( $k$ -star-mesh transform). *A  $k$ -star-mesh transform is an operation that given a graph  $G$  and a vertex  $r \in V(G)$  of degree  $k$ , removes  $r$  from  $G$  and replaces the  $k$ -star formed by  $r$  and its neighbors with a (possibly weighted) clique on the neighbor set  $N(r)$ .*

**Definition 3.20** (local  $k$ -star-mesh transform). *A  $k$ -star-mesh transform is called **local** if the edge weights inside the clique  $N(r)$  in the transformed graph depend only on the edge weights of the  $k$ -star in  $G$  (obviously to the rest of the graph  $G$ ).*

**Definition 3.21** (local  $k$ -star-mesh transform preserving  $d_p$ ). *We say that a local  $k$ -star-mesh transform **preserves**  $d_p$  if for every graph  $G = (V, E, w)$  and a vertex  $r \in V$  of degree  $k$ , applying the transform on  $G$  and  $r$  yields a graph  $G'$  where*

$$\forall s, t \in V \setminus \{r\}, \quad d_{p, G'}(s, t) = d_{p, G}(s, t).$$

We can now state our main theorem of this section.

**Theorem 3.22.** *For every  $p \neq 1, 2, \infty$ , there is no local Y- $\Delta$  transform that preserves the  $d_p$  metric.*



We prove this by showing two graphs that contain the same unweighted 3-star (“Y”) as an induced subgraph, on which such a transformation, if one existed, must have acted differently. Of course, this is not possible, since the transformation should be local and thus the same in the two graphs. Before we present the claim formally, let us add some notation. Denote by  $V_T = \{a, b, c\}$  a set of 3 vertices we will call “terminals”, denote  $V_Y = V_T \cup \{r\}$ , and denote by  $G_Y = (V_Y, E_Y)$  the 3-star over the terminals, i.e.  $E_Y = \{\{a, r\}, \{b, r\}, \{c, r\}\}$ . In addition, denote by  $G_\Delta = (V_T, E_\Delta)$  the triangle over the terminals, i.e.  $E_\Delta = \{\{a, b\}, \{b, c\}, \{c, a\}\}$ .

**Claim 3.23.** *Let  $p \in (1, 2) \cup (2, \infty)$ . There exist graphs  $G_1 = (V_1, E_1, w_1), G_2 = (V_2, E_2, w_2)$ , with  $V_Y \subseteq V_i$ , and  $G_i[V_Y] = G_Y$  for  $i = 1, 2$ , that satisfy the following. Suppose that we have graphs  $G'_1 = (V'_1, E'_1, w'_1), G'_2 = (V'_2, E'_2, w'_2)$  on which the “Y” transformed into a “ $\Delta$ ”, i.e. satisfying for  $i = 1, 2$ ,*

$$V'_i = V_i \setminus \{r\} \quad (3.50)$$

$$E(G'_i[V_T]) = E_\Delta \quad (3.51)$$

$$E'_i \setminus E_\Delta = E_i \setminus E_Y \quad (3.52)$$

$$w'_i|_{E'_i \setminus E_\Delta} = w_i|_{E_i \setminus E_Y} \quad (3.53)$$

If in addition we have that

$$\forall s, t \in V_T, \quad d_{p, G'_i}(s, t) = d_{p, G_i}(s, t) \quad (3.54)$$

for  $i = 1, 2$ , then there exists  $e \in E_\Delta$  such that  $w'_1(e) \neq w'_2(e)$ .

Observe that Claim 3.23 immediately gives Theorem 3.22 as a corollary.

*Proof.* We will take  $G_1 = G_Y$  (with unit weights), and  $G_2$  to be  $G_Y$  with the addition of a new vertex and two edges as presented in figure 3.4 below. Formally, we define

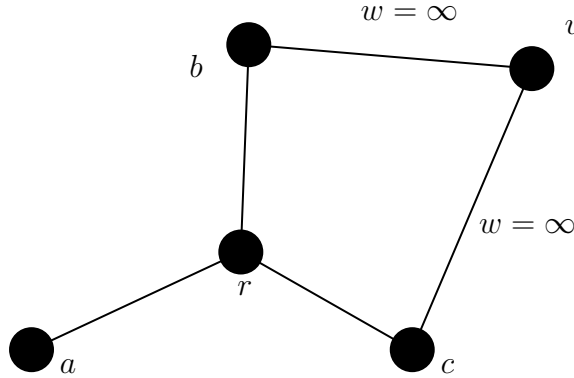


Figure 3.4: Illustration of the graph  $G_2$ .

$V_2 = V_Y \cup \{v\}$ ,  $E_2 = E_Y \cup \{\{v, b\}, \{v, c\}\}$ , and  $w_2(\{v, b\}) = w_2(\{v, c\}) = \infty$ , and the rest of the edges are of unit weight.

Next, suppose that we have transformed the “Y” into “ $\Delta$ ” in the two graphs, and received  $G'_1, G'_2$  that satisfy the conditions in the claim, and assume towards contradiction that for all  $e \in E_\Delta$ ,  $w'_1(e) = w'_2(e)$ . Denote the weights by  $w'_i(\{b, c\}) = \alpha$ ,  $w'_i(\{a, c\}) = \beta$ , and  $w'_i(\{a, b\}) = \gamma$  (as presented in figure 3.3).

We first focus on the case of  $G_1$ , and examine the new weights in  $G'_1$ . Note that in our case,  $G'_1$  is simply the triangle over the terminals with some new weights. In addition,

note that by symmetry, the new weights must satisfy  $\alpha = \beta = \gamma$ . It is easy to see that for any two pairs  $(s, t), (s', t') \in V_T \times V_T$  with  $s \neq t$  and  $s' \neq t'$ , it holds that  $d_{p, G_1}(s, t) = d_{p, G_1}(s', t')$ . Thus, by our assumption, this also holds in  $G'_1$ . In order for this to hold in the triangle, it must hold that  $\alpha = \beta = \gamma$ . Next, let us compute  $\bar{d}_{p, G_1}(a, b)$  (recall that  $d_p = (\bar{d}_p)^{-1}$  and thus preserving  $\bar{d}_p$  implies preserving  $d_p$  and vice versa). Note that by definition,  $\bar{d}_{p, G_1}(a, b)^q = \min_{\varphi \in \mathbb{R}^{V_1}: \varphi_a - \varphi_b = 1} \{|\varphi_a - \varphi_r|^q + |\varphi_b - \varphi_r|^q + |\varphi_c - \varphi_r|^q\}$ . Thus, w.l.o.g we can set  $\varphi_a = 1, \varphi_b = 0, \varphi_r = x, \varphi_c = y$  for some  $x, y \in \mathbb{R}$  and get that,

$$\begin{aligned} \bar{d}_{p, G_1}(a, b)^q &= \min_{x, y \in \mathbb{R}} \{|1 - x|^q + |0 - x|^q + |y - x|^q\} \\ &= \min_{x \in [0, 1]} \{(1 - x)^q + x^q\} \quad (\text{by setting } y = x \text{ and } x \in [0, 1]) \end{aligned}$$

Define  $f(x) = (1 - x)^q + x^q$ , and we wish to find a minimum in the interval  $[0, 1]$ . Hence,

$$\begin{aligned} & f'(x) = 0 \\ \iff & -(1 - x)^{q-1} + x^{q-1} = 0 \\ \iff & x^{q-1} = (1 - x)^{q-1} \\ \iff & x = \frac{1}{2} \end{aligned}$$

Thus, we get that

$$\bar{d}_{p, G_1}(a, b)^q = \left(1 - \frac{1}{2}\right)^q + \left(\frac{1}{2} - 0\right)^q = 2 \cdot \left(\frac{1}{2}\right)^q = 2^{1-q} \quad (3.55)$$

On the other hand, let us compute  $\bar{d}_{p, G'_1}(a, b)$  in terms of  $\alpha$ . Again, w.l.o.g. we can set  $\varphi_a = 1, \varphi_b = 0, \varphi_c = x$  and see that

$$\begin{aligned} \bar{d}_{p, G'_1}(a, b)^q &= \min_{x \in \mathbb{R}} \{\alpha^q \cdot |1 - 0|^q + \alpha^q \cdot |0 - x|^q + \alpha^q \cdot |x - 1|^q\} \\ &= \min_{x \in [0, 1]} \{\alpha^q + \alpha^q \cdot x^q + \alpha^q \cdot (1 - x)^q\} \end{aligned}$$

Similarly, we can define  $g(x) = \alpha^q + \alpha^q \cdot x^q + \alpha^q \cdot (1 - x)^q$  and find a minimum in the interval  $[0, 1]$ . By similar computations as above, we deduce that the minimum is achieved in  $x = \frac{1}{2}$ , and thus we get that

$$\bar{d}_{p, G'_1}(a, b)^q = \alpha^q \left(1 + \left(\frac{1}{2}\right)^q + \left(1 - \frac{1}{2}\right)^q\right) = \alpha^q (1 + 2^{1-q}) \quad (3.56)$$

Thus, we can derive the value of the new weights  $\alpha$ .

$$\begin{aligned} \alpha^q \cdot (1 + 2^{1-q}) &= 2^{1-q} \\ \implies \alpha &= \left(\frac{2^{1-q}}{1 + 2^{1-q}}\right)^{1/q} \end{aligned} \quad (3.57)$$

Hence, by examining the  $d_p$  metric on  $G_1$  and  $G'_1$ , we deduce that the weights should be  $\alpha = (1 + 2^{q-1})^{-1/q}$ .

Next, let us focus on the case of  $G_2$ , and we start by computing  $\bar{d}_{p,G_2}(a, b)$ . Note that in order to not pay the extremely large weight of the edges connecting  $v$  to  $b$  and  $c$ , it must hold that any minimizing potentials vector will satisfy  $\varphi_b = \varphi_c = \varphi_v$ . Thus, we get that

$$\begin{aligned}\bar{d}_{p,G_2}(a, b)^q &= \min_{\varphi \in \mathbb{R}^{V_2}: \varphi_a - \varphi_b = 1, \varphi_b = \varphi_c = \varphi_v} \{|\varphi_a - \varphi_r|^q + |\varphi_b - \varphi_r|^q + |\varphi_c - \varphi_r|^q\} \\ &= \min_{x \in \mathbb{R}} \{ |1 - x|^q + |0 - x|^q + |0 - x|^q \} \quad (\varphi_a = 1, \varphi_b = 0, \varphi_r = x) \\ &= \min_{x \in [0,1]} \{ (1 - x)^q + 2x^q \}\end{aligned}$$

Similarly to the previous case, we can define  $f(x) = (1 - x)^q + 2x^q$ , and we wish to find a minimum by considering the derivative of  $f$  and requiring it to be 0 (because again the minimum is obtained when  $x \in [0, 1]$ ).

$$\begin{aligned}f'(x) &= 0 \\ \iff & -(1 - x)^{q-1} + 2x^{q-1} = 0 \\ \iff & 2x^{q-1} = (1 - x)^{q-1} \\ \iff & 2^{1/(q-1)} \cdot x = (1 - x) \\ \iff & x \cdot (2^{1/(q-1)} + 1) = 1 \\ \iff & x = \frac{1}{2^{1/(q-1)} + 1}\end{aligned}$$

Thus, we get that

$$\bar{d}_{p,G_2}(a, c)^q = \left(1 - \frac{1}{2^{1/(q-1)} + 1}\right)^q + 2 \cdot \left(\frac{1}{2^{1/(q-1)} + 1}\right)^q = \frac{2^{q/(q-1)} + 2}{(2^{1/(q-1)} + 1)^q} = 2 \cdot (2^{1/(q-1)} + 1)^{1-q} \quad (3.58)$$

On the other hand, let us compute  $\bar{d}_{p,G'_2}(a, b)$  in terms of  $\alpha$  (note that by our assumption all the weights of edges from  $E_\Delta$  must be equal). Note that again, we do not want to pay the extremely large weight of the edges connecting  $v$  to  $b$  and  $c$ , and thus we must set  $\varphi_b = \varphi_c = \varphi_v$ . Moreover, w.l.o.g we can choose  $\varphi_a = 1, \varphi_b = \varphi_c = 0$  and get that,

$$\begin{aligned}\bar{d}_{p,G'_2}(a, b)^q &= \alpha^q \cdot |1|^q + \alpha^q \cdot |0|^q + \alpha^q \cdot |-1|^q \\ &= 2\alpha^q\end{aligned}$$

Thus, we can derive the value of the new weights  $\alpha$ .

$$\begin{aligned}2 \cdot \alpha^q &= 2 \cdot (2^{1/(q-1)} + 1)^{1-q} \\ \alpha &= (2^{1/(q-1)} + 1)^{\frac{1-q}{q}}\end{aligned}$$

Hence, in this case the new weights should be  $\alpha = (2^{1/(q-1)} + 1)^{-1/p}$ .

Thus, we got that according to the first case, the weights should be  $\alpha = (1 + 2^{q-1})^{-1/q}$ , but on the other hand they should be  $\alpha = (2^{1/(q-1)} + 1)^{-1/p}$ . Thus we ask whether

$$(2^{1/(q-1)} + 1)^{-1/p} \stackrel{?}{=} (1 + 2^{q-1})^{-1/q} \quad (3.59)$$

It is easy to see that the two terms equal when  $p = q = 2$ . We will now show that they are different when  $q \neq 2$ . First, note that (3.59) is equivalent to

$$2^{1/(q-1)} + 1 \stackrel{?}{=} (1 + 2^{q-1})^{p-1} \quad (3.60)$$

by taking power  $p$ .

Next, assume that  $p \geq 2$ , and focus on the RHS. By applying  $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$  for  $f(x) = x^{p-1}$  (since  $p \geq 2$  it is convex) we can see that,

$$\begin{aligned} (1 + 2^{q-1})^{p-1} &\leq 2^{p-1} \cdot \left( \left(\frac{1}{2}\right)^{p-1} + 2^{(q-2)\cdot(p-1)} \right) \\ &= 1 + 2^{p-2(p-1)+(p-1)} \\ &= 3 \end{aligned}$$

Now, plugging this into (3.60) we get

$$\begin{aligned} 2^{1/(q-1)} &\leq 2 \\ \iff \frac{1}{q-1} &\leq 1 \\ \iff 1 &\leq q-1 \\ \iff 2 &\leq q \\ \iff 2 &\geq p \end{aligned} \quad (3.61)$$

Recall that we assumed that  $p \geq 2$  (in order to apply Jensen's inequality) and reached the conclusion that  $p \leq 2$ , thus it is impossible that the two terms in (3.59) equal when  $p > 2$ .

Next, we assume that  $p \leq 2$  (which implies that  $q \geq 2$ ), and note that (3.59) is equivalent to

$$(2^{1/(q-1)} + 1)^{q-1} \stackrel{?}{=} 1 + 2^{q-1} \quad (3.62)$$

by taking power  $q$ . This time we focus on the LHS and similarly to the previous case, we apply Jensen's inequality on it and get that

$$\begin{aligned} (2^{1/(q-1)} + 1)^{q-1} &\leq 2^{q-1} \cdot \left( 2^{(\frac{1}{q-1}-1)\cdot(q-1)} + \left(\frac{1}{2}\right)^{q-1} \right) \\ &= 2^{1-(q-1)+(q-1)} + 1 \\ &= 3 \end{aligned}$$

Now, plugging this into (3.62) we get

$$\begin{aligned} 2^{q-1} &\leq 2 \\ \iff q-1 &\leq 1 \\ \iff q &\leq 2 \\ \iff p &\geq 2 \end{aligned} \quad (3.63)$$

and again, recall that we assumed that  $p \leq 2$ , and reached the conclusion that  $p \geq 2$ , which implies that the two terms in (3.59) equal if and only if  $p = q = 2$ . Thus, we have reached a contradiction, and the claim follows.  $\square$

Note that a corresponding example will also contradict the other direction, i.e. that there is no valid  $\Delta - Y$  transform for any  $p \neq 2$ .

We see that our counter example does not contradict the case of  $p = q = 2$ , since it holds that in both cases  $\alpha = \frac{1}{\sqrt{3}}$ . As a matter of fact, in this case, there exists a proper transform (similarly to the known transform for the effective resistance), and we show it in Appendix [B.2.2](#).

### The Case of Minimum Cuts ( $p = \infty$ )

In this subsection we elaborate on the case of  $p = \infty$ , i.e. minimum cuts. The general case of the  $Y - \Delta$ -transform is called a  $k$ -star-mesh transform, in which a  $k$ -star (a root vertex with  $k$  neighbors) is transformed into a clique (“mesh”) over  $k$  vertices, formally defined as shown in Section [3.3.3](#).

By Theorem 1 from [\[CSWZ00\]](#), and also directly from basic principles, it is easy to conclude that there exists such a transform for  $k = 3$  and for  $k = 2$  (which is essentially the sequential edges reduction case). However, for every  $k > 3$ , using Lemma 4 from [\[CSWZ00\]](#), we can deduce that there does not exist such a transform for removing a vertex of degree  $k$ , stated as follows.

**Theorem 3.24.** *For every  $k > 3$ , there does not exist a local  $k$ -star-mesh transform that preserves  $d_\infty$ .*

This is in fact a corollary from the following Lemma presented in [\[CSWZ00\]](#).

**Lemma 3.25** (Lemma 4 in [\[CSWZ00\]](#)). *There exists a  $k$ -terminal network for which every mimicking network must have at least one non-terminal vertex (in addition to the  $k$  terminals).*

Its proof in [\[CSWZ00\]](#) actually shows the following Lemma (restated using our terminology).

**Lemma 3.26.** *Let  $k > 3$ , and let  $G = (V, E)$  be an unweighted star with a (root) vertex  $r$  of degree  $k$ . Denote by  $G'$  the graph obtained after applying a  $k$ -star-mesh transform on  $G$ . Then, there must exist  $S \subset N(r)$  such that*

$$\text{mincut}_{G'}(S, N(r) \setminus (S)) \neq \text{mincut}_G(S, N(r) \setminus (S)).$$

Note that it does not immediately imply Theorem [3.24](#), as in order to preserve the  $d_\infty$  metric we only need to preserve all *minimum st-cuts*, rather than minimum cuts between all subsets of  $N(r)$ .

*Proof.* (of Theorem [3.24](#)) Assume towards contradiction that there exists a local  $k$ -star-mesh transform that preserves the  $d_\infty$  metric. Let  $G = (V, E)$  be an unweighted  $k$ -star graph, i.e. a root vertex  $r$  with  $k$  neighbors. Then, apply the transform on  $G$  in order to obtain a clique  $G'$ . By Lemma [3.26](#), there is a subset  $S \subset N(r)$  such that

$$\text{mincut}_{G'}(S, N(r) \setminus (S)) \neq \text{mincut}_G(S, N(r) \setminus (S)). \quad (3.64)$$

Next, consider the following graph  $G_S$ . Add to  $G$  two vertices  $v_S$  and  $u_S$ , connect  $v_S$  to every vertex in  $S$ , and connect  $u_S$  to every vertex in  $N(r) \setminus S$ . In addition, define an edge-weight function  $w$  given by

$$w(e) = \begin{cases} 1 & \text{if } e \text{ is incident to } r; \\ \infty & \text{o.w.} \end{cases}$$

i.e. the weights of the star edges remain the same, and the weights of the newly added edges are infinite. An illustration is presented in Figure 3.5.

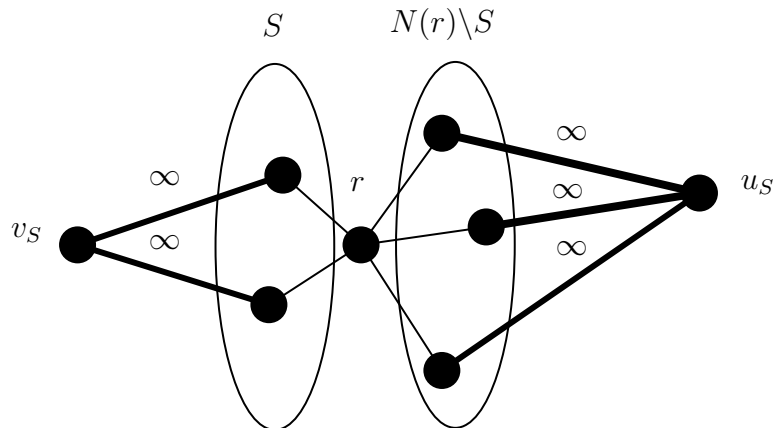


Figure 3.5: Illustration of the graph  $G_S$ .

Observe that  $d_{\infty, G_S}(v_S, u_S) = \text{mincut}_G(S, N(r) \setminus S)$ . Next, apply the transform on  $G_S$  and denote the obtained graph by  $G'_S$ . We assumed the transform preserves  $d_{\infty}$ , and thus in particular  $d_{\infty, G'_S}(v_S, u_S) = d_{\infty, G_S}(v_S, u_S)$ . But this implies that

$$\text{mincut}_{G'}(S, N(r) \setminus (S)) = \text{mincut}_G(S, N(r) \setminus (S)),$$

in contradiction to (3.64). □

Note that the proof in fact shows that there does not exist a local  $k$ -star-mesh transform that preserves  $d_{\infty}$  even for the family of planar graphs, as  $G_S$  is planar. Moreover, the construction of  $G_S$  can be modified such that  $G_S$  will be outer-planar, which will give the same result even for outer-planar graphs. However, for trees there does exist such a transform.

# Chapter 4

## Conclusions and Open Questions

In this chapter we discuss questions that are left open from our work.

**Regarding  $d_p$  sparsifiers.** In Section 3.1 we proved a general lower bound of  $\Omega(n/\sqrt{\varepsilon})$  edges for an  $\varepsilon$ -resistance sparsifier, while there is an upper bound of  $O(n/\varepsilon)$  edges for a resistance sparsifier for the clique. We proved this by showing that every non-complete graph cannot achieve better than  $\left(1 + \frac{1}{O(n^2)}\right)$ -approximation of the resistance distance in an  $n$ -clique. However, we showed the stronger bound of  $\left(1 + \frac{1}{O(n)}\right)$ -approximation for regular graphs, which intuitively, seem to be the best fit for this task. This result suggests that the stronger bound should hold in general, which would prove Conjecture 1.1, stating that in the worst case, an  $\varepsilon$ -resistance sparsifier requires  $\Omega(n/\varepsilon)$  edges.

**Open Question 4.1.** *Is it true that for every non-complete graph,*

$$\frac{\max_{x \neq y \in V} \text{R}_{\text{eff}}(x, y)}{\min_{x \neq y \in V} \text{R}_{\text{eff}}(x, y)} \geq 1 + \frac{1}{O(n)}? \quad (4.1)$$

Another open question is to extend the lower bound on resistance sparsifiers to other values of  $p$  (the solved cases are  $p = 1, \infty$  with matching upper and lower bounds), as our lower bound for effective resistance does not extend to other values of  $p$ .

Additionally, in Section 3.2 we saw that for fixed  $p \in (4/3, \infty]$  with Hölder conjugate  $q$ , there exists  $d_p$ -sparsifiers with  $|E'| = f(n, \varepsilon, p)$  edges, where

$$f(n, \varepsilon, p) = \begin{cases} n - 1 & \text{if } p = \Omega(\varepsilon^{-1} \log n), \\ \tilde{O}(n\varepsilon^{-2}) & \text{if } 2 < p < \infty, \\ \tilde{O}(n^{q/2}\varepsilon^{-5}) & \text{if } \frac{4}{3} < p < 2. \end{cases} \quad (4.2)$$

One question is regarding the number of edges needed to preserve  $d_p$  for  $p$  that is close to 2. Recall that for  $d_2$  there is the construction of Chu et al [CGPSSW18] for resistance sparsifiers with  $\tilde{O}(n/\varepsilon)$  edges. In (4.2) we see that in order to preserve the  $d_p$  metric for  $p > 2$ , then  $\tilde{O}(n \cdot \varepsilon^{-2})$  edges suffice, i.e. we have a gap in the dependence on  $\varepsilon$ . Additionally, for  $\frac{3}{4} < p < 2$ , we see that  $\tilde{O}(n^{q/2} \cdot \varepsilon^{-5})$  edges suffice (where  $q/2 > 1$ ), i.e. there is a gap in both  $\varepsilon$  and  $n$ . Thus, it would be interesting to close these gaps from both sides of  $p = 2$ .

**Open Question 4.2.** *For  $p = 2 \pm 0.01$ , does every graph  $G$  with  $n$  vertices admits a  $d_p$ -sparsifier that achieves  $1 + \varepsilon$  approximation with  $\tilde{O}(n/\varepsilon)$  edges?*

In addition, as  $p$  tends to 1, the  $d_p$  metric tends to the shortest-path metric, where multiplicative spanners are in fact  $d_1$ -sparsifiers [PS89; ADDJS93], for which there exists a lower bound on the number of edges needed in order to preserve the shortest-path distance. This leads to the question whether we can find a lower bound on the number of edges needed to preserve the  $d_p$  metric for values of  $p$  close to 1.

**Open Question 4.3.** *For  $p = 1.1$ , can we prove that a  $d_p$ -sparsifier requires  $n^{1+\Omega(1)}$  edges in order to achieve 1.01 approximation?*

Another interesting question is the gap in the number of needed edges for  $d_p$  sparsifiers for  $p \in (2, \infty)$ . For instance, for  $p = 2$  there exists a resistance sparsifier with  $\tilde{O}(n \cdot \varepsilon^{-1})$  edges [CGPSSW18], for  $p = \Omega(\varepsilon^{-1} \log n)$  there exists a  $d_\infty$ -sparsifier with  $n - 1$  edges, which is clearly the best we can hope for, and yet for any value  $2 < p < \varepsilon^{-1} \log n$  the best result we have so far requires  $\tilde{O}(n \cdot \varepsilon^{-2})$  edges. An explanation for this phenomenon is that essentially our proof for flow metric sparsifiers showed a generalization of spectral sparsifiers, since the sparsifiers preserve the norm of  $\|WB\varphi\|_q$  for every  $\varphi \in \mathbb{R}^V$ , which is stronger than what we need.

**Open Question 4.4.** *For  $2 < p < \varepsilon^{-1} \log n$ , there exists  $d_p$ -sparsifier with  $\tilde{O}(n/\varepsilon^2)$  edges. Can we remove the  $\text{poly}(\log n)$  factors? Can we reduce the dependency in  $\varepsilon$  to (say)  $\varepsilon^{-1}$ ? Can we show that a tree with  $n - 1$  edges is **not** sufficient?*

**Regarding Delta-Wye transform.** It is known that for  $p = 1, 2$  and every  $k \geq 3$ , there exists local  $k$ -star-mesh transforms that preserve  $d_p$ . However, in Section 3.3 we showed that for  $k = 3$ , **local**  $k$ -star-mesh-transform that preserves  $d_p$  exists if and only if  $p = 1, 2, \infty$ . Moreover, we showed that for every  $k > 3$ , there is no **local**  $k$ -star-mesh-transform that preserves  $d_\infty$ . What about transforms such that the weights of the new edges may depend on the rest of the graph (i.e. not local)? For example in the nature of Schur-Complements.

**Open Question 4.5.** *Does there exist a  $k$ -star-mesh-transform that preserves  $d_p$  for  $k = 3$  and  $p \neq 1, 2, \infty$ ? for  $k > 3$  and  $p = \infty$ ?*

**Understanding the geometry of the flow metrics.** An important tool for understanding the structure of the flow metrics is via metric embeddings, i.e. mapping a metric space into another one (specifically in our case into a normed space) while preserving the distances - in which case the mapping is called an isometry, or up to some error - in which case we say that the mapping has distortion  $> 1$ , see e.g. [Mat02; Mat97; Mat13]. Towards this, in Section 2.3 we showed that the  $d_p$  metrics are  $p$ -strong, which gives some information about their structure. For example, for fixed  $p \geq 2$ , in any embedding of  $d_p$  into  $\ell_2$ , no 3 points can lie on the same line.

For the special cases of  $p = 1, 2, \infty$ , it is known that  $d_p$  embeds isometrically into  $\ell_q$  (with  $q$  being the Hölder conjugate of  $p$ ). We conjecture that this should hold in general.

**Conjecture 4.6.** *Fix  $p \in [1, \infty]$  with Hölder conjugate  $q$ , and let  $G = (V, E, w)$  be a graph. Then there exists a mapping  $\Phi : V \rightarrow \ell_q$  such that*

$$\forall s, t \in V, \quad d_p(s, t) = \|\Phi(s) - \Phi(t)\|_q. \quad (4.3)$$



We remark that the  $p$ -strong triangle inequality alone is not enough in order to prove the above, and thus there must be additional properties of the flow metrics that should be used. We present it formally in Appendix C.1.

Furthermore, by using the connection between the resistance distance and the graph Laplacian, it was shown that the resistance distance is isometrically embedded into  $\ell_2^2$  [SS11]. We suspect that this approach can be generalized by using the connection between  $d_p$  and the graph  $q$ -Laplacian (presented in Appendix A.2) in order to show that  $d_p^p$  can be isometrically embedded into  $\ell_2^2$ , which would give some more information about their structure.

**Conjecture 4.7.** *Fix  $p \in (1, \infty)$ , and let  $G = (V, E, w)$  be a graph. Then there exists a mapping  $\Psi : V \rightarrow \ell_2^2$  such that*

$$\forall s, t \in V, \quad d_p(s, t)^p = \|\Psi(s) - \Psi(t)\|_2^2. \quad (4.4)$$

**Small sketches.** Once we understand the geometry of the flow metrics, e.g. which metric spaces they embed into, we would like to find the best trade-off between dimension and approximation of such embeddings. One famous example is the Johnson-Lindenstrauss Lemma [JL84] that states that every  $n$  points in  $\ell_2$  can be embedded with distortion  $1 + \varepsilon$  (for every  $\varepsilon > 0$ ) into a subspace of  $\ell_2$  of dimension  $O(\varepsilon^{-2} \cdot \log n)$ . Once we reduce the dimension, we can design natural small sketches and exploit them to improve running time and storage requirements of algorithms. In particular, Spielman and Srivastava [SS11] utilized such an embedding of the resistance distance in order to construct a data structure that given a query pair of vertices, returns an approximation of the effective resistance between them. If Conjecture 4.7 is true, their approach can be generalized and yield a small sketch for  $d_p^{p/2}$ , and thus also for  $d_p$ .

**Computing all-pairs distances.** Another important line of research is to compute the distance between all pairs of vertices simultaneously, or to construct a data structure that given as query a pair of vertices, returns the exact  $d_p$  distance (or an approximation to it) between them. Such constructions are known for the three special cases. For  $p = \infty$ , there is the Gomory-Hu tree [GH61]; for  $p = 1$ , there are distance oracles [ABCP93; TZ05; Che15], All-Pairs Shortest-Path algorithms [Cha10; Sei95], and spanners [PS89; ADDJS93]; for  $p = 2$ , there are constructions by Spielman and Srivastava [SS11], and later on by Jambulapati and Sidford [JS18] for approximating the effective resistance. It would be interesting to design algorithms that solve this problem for other values of  $p$ , as well as give lower bounds for this problem.

**Capturing properties of the underlying graphs.** As mentioned earlier, effective resistance captures key properties of the underlying graph. For example, the effective resistance between two vertices connected by a unit-weight edge, equals the probability that this edge belongs to a uniformly random spanning tree of the graph, and moreover, it also equals the commute time between them, up to scaling by a factor that depends on the weights in the graph. A natural direction is to extend this characterizations of the effective resistance ( $p = 2$ ) to other values of  $p$ , or to find other properties of the underlying graphs captured by them.

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# Appendix A

## Omitted Proofs from Basic Properties Section

### A.1 Deriving the Connection to the Dual Problem

In this section we show how Claim 2.1 is derived from Proposition 4 in [AvL11], who consider the following optimization problems for a graph  $G = (V, E, w)$  and fixed  $p \in (1, \infty)$  with Hölder conjugate  $q$ .

**Flow problem.**

$$R_p(s, t) = \min \left\{ \sum_{e \in E} \frac{|f(e)|^p}{w(e)} : B^T f = \chi_s - \chi_t \right\}. \quad (\text{A.1})$$

**Potential problem.**

$$C_p(s, t) = \min \left\{ \sum_{xy \in E} w(xy)^{\frac{q}{p}} |\varphi_x - \varphi_y|^q : \varphi_s - \varphi_t = 1 \right\}. \quad (\text{A.2})$$

We remark that  $d_p^p$  is just  $R_p$  on a graph with  $w^p$  as a weight function, and the same holds with  $\bar{d}_p^q$  and  $C_p$ . Moreover, [AvL11] show that  $R_p$  and  $C_p$  are related in the following manner.

**Proposition A.1** (Proposition 4<sup>1</sup> in [AvL11]). *Fix  $p > 1$  with Hölder conjugate  $q$ , and let  $G = (V, E, w)$  be a graph. Then,*

$$\forall s, t \in V, \quad R_p(s, t) = (C_p(s, t))^{-\frac{p}{q}}. \quad (\text{A.3})$$

We can now show how Claim 2.1 is an easy consequence of Proposition A.1.

*Proof.* (Claim 2.1) Let  $G = (V, E, w)$  be a graph, and let  $G^p = (V, E, w^p)$ . Hence, we see that for every  $s \neq t \in V$ ,

$$d_{p,G}(s, t)^p = R_{p,G^p}(s, t) = (C_{p,G^p}(s, t))^{-\frac{p}{q}} = (\bar{d}_{p,G}(s, t)^q)^{-\frac{p}{q}} = \bar{d}_{p,G}(s, t)^{-p}. \quad (\text{A.4})$$

□

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<sup>1</sup>The original statement in [AvL11] is stated with power  $-\frac{q}{p}$ . However, in the supplementary material they proved the version we presented.

## A.2 Connection to the Graph $p$ -Laplacian

In this section we show how Fact 2.4 can be used to relate between the  $d_p$  metric and the graph  $p$ -Laplacian.

Given a graph  $G = (V, E, w)$ , and fix  $p \in (1, \infty)$  with Hölder conjugate  $q$ , denote the  $q$ -Laplacian of  $G$  by  $L_q : \mathbb{R}^V \rightarrow \mathbb{R}^V$ , given by

$$\forall x \in V, \quad (L_q \varphi)_x = \sum_{y \in V} w(xy)^q (\varphi_x - \varphi_y) |\varphi_x - \varphi_y|^{q-2}. \quad (\text{A.5})$$

This is a non-linear generalization of the ordinary Laplacian of the graph. Now, using Fact 2.4, we can see that for a pair  $s, t \in V$ , and minimizing flow  $f^*$  for  $d_p(s, t)$ , and a corresponding potentials vector  $\varphi^*$ , it holds that

$$B^T f^* = L_q \varphi^*. \quad (\text{A.6})$$

In addition, note that  $L_q$  satisfies for any  $\varphi \in \mathbb{R}^V$

$$\begin{aligned} \langle \varphi, L_q \varphi \rangle &= \sum_{x \in V} \sum_{y \in V} w(xy)^q (\varphi_x - \varphi_y) |\varphi_x - \varphi_y|^{q-2} \cdot \varphi_x \\ &= \sum_{xy \in E} \left( w(xy)^q (\varphi_x - \varphi_y) |\varphi_x - \varphi_y|^{q-2} \cdot \varphi_x + w(xy)^q (\varphi_y - \varphi_x) |\varphi_y - \varphi_x|^{q-2} \cdot \varphi_y \right) \\ &= \sum_{xy \in V} w(xy)^q |\varphi_x - \varphi_y|^q \\ &= \|WB\varphi\|_q^q. \end{aligned} \quad (\text{A.7})$$

and in particular, for connected graphs,

$$\langle \varphi, L_q \varphi \rangle = 0 \iff \varphi \in \text{span}\{\mathbf{1}\}. \quad (\text{A.8})$$

This is also shown as Proposition 3.1 in [BH09]. This motivates the definition of the second eigenvalue of the  $q$ -Laplacian.

**Definition A.2.** Let  $p \in (1, \infty)$  with Hölder conjugate  $q$ , and let  $G = (V, E, w)$  be a connected graph with  $q$ -Laplacian  $L_q$ . The second smallest eigenvalue of  $L_q$ , denoted by  $\lambda_q^{(2)}$ , is defined as

$$\lambda_q^{(2)} = \min_{\varphi \perp \mathbf{1}} \frac{\langle \varphi, L_q \varphi \rangle}{\|\varphi\|_2^2}. \quad (\text{A.9})$$

We remark that a consequence of (A.8) (and Proposition 3.1 in [BH09]) is that for connected graphs,  $\lambda_q^{(2)} > 0$ . Next, we show that  $d_p$  and  $\lambda_q^{(2)}$  are related as follows.

**Claim A.3.** Let  $p \in (1, \infty)$  with Hölder conjugate  $q$ , and let  $G = (V, E, w)$  be a connected graph with  $q$ -Laplacian  $L_q$  and second smallest eigenvalue  $\lambda_q^{(2)}$ . Then

$$\forall s, t \in V, \quad d_p(s, t) \leq \left( \frac{2}{\lambda_q^{(2)}} \right)^{1/q}. \quad (\text{A.10})$$

Note that the above claim is tight on the clique for  $p = q = 2$ , since  $\lambda_2^{(2)}(K_n) = n$  ( $\chi_s - \chi_t$  is a corresponding eigenvector for any  $s, t \in V$ ), which yields the exact  $d_2$ -distance.

*Proof.* Fix  $s \neq t \in V$ , and let  $\varphi^* \in \operatorname{argmin}_{\varphi_s - \varphi_t = 1} \|WB\varphi\|_q$ . By subtracting the average of the entries of  $\varphi$ , we may assume that  $\varphi^* \perp \bar{\mathbf{1}}$ . Now, we can see that

$$\begin{aligned}
\frac{1}{d_p(s, t)^q} &= \|WB\varphi^*\|_q^q \\
&= \langle \varphi^*, L_q \varphi^* \rangle \\
&\geq \lambda_q^{(2)} \|\varphi^*\|_2^2 && \text{(by } \varphi^* \perp \bar{\mathbf{1}} \text{ and definition of } \lambda_q^{(2)}) \\
&\geq \lambda_q^{(2)} \cdot \left\| \begin{pmatrix} \varphi_s^* \\ -\varphi_t^* \end{pmatrix} \right\|_2^2 \\
&\geq \lambda_q^{(2)} \cdot \frac{\left| \left\langle \begin{pmatrix} \varphi_s^* \\ -\varphi_t^* \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \right|^2}{\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_2^2} && \text{(by Cauchy-Schwartz)} \\
&= \frac{\lambda_q^{(2)}}{2} && \text{(by } \varphi_s^* - \varphi_t^* = 1).
\end{aligned}$$

□



# Appendix B

## Graph-Size Reductions Appendix

### B.1 Lower Bound on Resistance Sparsifiers

#### Proof of Symmetric Case via Commute Time

Here we present another proof for Claim 3.11 by using the relation between effective resistance and commute time.

First, recall that the hitting time  $h(u, v)$  is the expected number of steps of a random walk starting at vertex  $u$  to reach vertex  $v$  at the first time. i.e.  $h(u, v) = 1 + \mathbb{E}_{x \in N(u)} h(x, v)$ . The commute time  $C(u, v)$  is the expected time that a random walk starting at  $u$  will reach  $v$  and get back to  $u$ , i.e.  $C(u, v) = h(u, v) + h(v, u)$ . In addition, we know that  $C(u, v) = 2w(E) R_{\text{eff}}(u, v)$  where  $w(E) = \sum_{e \in E} w(e)$ . Thus, instead of considering effective resistance, we can work with commute time.

*Proof.* (of Claim 3.11) Let  $x \neq y \in V \setminus \{s, t\}$ , and denote

$$H_0 = h(s, t) = h(t, s), \tag{B.1}$$

$$H_1 = h(x, s) = h(x, t), \tag{B.2}$$

$$H_2 = h(s, x) = h(t, x), \tag{B.3}$$

$$H_3 = h(x, y). \tag{B.4}$$

Note that by symmetry it does not matter which specific vertices  $x, y \in V \setminus \{s, t\}$  we chose when defining the  $H_i$ 's. Now, we can see that the relations between them are as follows.

$$\begin{aligned} H_0 &= 1 + \mathbb{E}_{z \in N(s)} h(z, t) = 1 + H_1, \\ H_1 &= 1 + \mathbb{E}_{z \in N(x)} h(z, t) \\ &= 1 + \frac{\alpha}{2\alpha + (n-3)\beta} H_0 + \frac{(n-3)\beta}{2\alpha + (n-3)\beta} H_1, \\ H_2 &= 1 + \mathbb{E}_{z \in N(s)} h(z, x) \\ &= 1 + \frac{(n-3)\alpha}{(n-2)\alpha} H_3, \\ H_3 &= 1 + \mathbb{E}_{z \in N(y)} h(z, y) \\ &= 1 + \frac{2\alpha}{2\alpha + (n-3)\beta} H_2 + \frac{(n-4)\beta}{2\alpha + (n-3)\beta} H_3. \end{aligned}$$

Let us start with computing  $H_0$  and  $H_1$ .

$$\begin{aligned}
H_1 &= 1 + \frac{\alpha}{2\alpha + (n-3)\beta} (1 + H_1) + \frac{(n-3)\beta}{2\alpha + (n-3)\beta} H_1 \\
&= 1 + \frac{\alpha}{2\alpha + (n-3)\beta} + H_1 \cdot \left( \frac{\alpha + (n-3)\beta}{2\alpha + (n-3)\beta} \right) \\
\Rightarrow \frac{\alpha}{2\alpha + (n-3)\beta} \cdot H_1 &= \frac{3\alpha + (n-3)\beta}{2\alpha + (n-3)\beta} \\
\Rightarrow H_1 &= 3 + (n-3) \frac{\beta}{\alpha}, \\
H_0 &= 4 + (n-3) \frac{\beta}{\alpha}.
\end{aligned}$$

Let us move on to  $H_2$  and  $H_3$ .

$$\begin{aligned}
H_3 &= 1 + \frac{2\alpha}{2\alpha + (n-3)\beta} \left( 1 + \frac{(n-3)}{(n-2)} H_3 \right) + \frac{(n-4)\beta}{2\alpha + (n-3)\beta} H_3 \\
&= 1 + \frac{2\alpha}{2\alpha + (n-3)\beta} + H_3 \left( \left( 1 - \frac{1}{n-2} \right) \cdot \frac{2\alpha}{2\alpha + (n-3)\beta} + \frac{(n-4)\beta}{2\alpha + (n-3)\beta} \right) \\
&= \frac{4\alpha + (n-3)\beta}{2\alpha + (n-3)\beta} + H_3 \left( \frac{2\alpha + (n-4)\beta}{2\alpha + (n-3)\beta} - \frac{2\alpha}{(n-2)(2\alpha + (n-3)\beta)} \right) \\
&= \frac{4\alpha + (n-3)\beta}{2\alpha + (n-3)\beta} + H_3 \left( 1 - \frac{\beta + \frac{2\alpha}{n-2}}{2\alpha + (n-3)\beta} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\Rightarrow H_3 &= \frac{4\alpha + (n-3)\beta}{\beta + \frac{2\alpha}{n-2}} \\
&= (n-2) \left( \frac{4 + (n-3) \frac{\beta}{\alpha}}{2 + (n-2) \frac{\beta}{\alpha}} \right), \\
\Rightarrow H_2 &= 1 + \frac{n-3}{n-2} \left( (n-2) \left( \frac{4 + (n-3) \frac{\beta}{\alpha}}{2 + (n-2) \frac{\beta}{\alpha}} \right) \right) \\
&= 1 + (n-3) \left( \frac{4 + (n-3) \frac{\beta}{\alpha}}{2 + (n-2) \frac{\beta}{\alpha}} \right).
\end{aligned}$$

Denoted  $\gamma = \frac{\beta}{\alpha}$ , and now we conclude that,

$$C(s, t) = 2H_0 = 2(4 + (n-3)\gamma), \quad (\text{B.5})$$

$$C(x, y) = 2H_3 = \frac{2(n-2)(4 + (n-3)\gamma)}{2 + (n-2)\gamma}, \quad (\text{B.6})$$

$$C(s, x) = H_1 + H_2 = (4 + (n-3)\gamma) \left( 1 + \frac{n-3}{2 + (n-2)\gamma} \right) \quad (\text{B.7})$$

$$= \frac{(4 + (n-3)\gamma)(n-1 + (n-2)\gamma)}{2 + (n-2)\gamma}. \quad (\text{B.8})$$

Assume towards contradiction that  $\frac{\max_{x' \neq y' \in V} \text{Reff}(x', y')}{\min_{x' \neq y' \in V} \text{Reff}(x', y')} < 1 + \frac{1}{10n}$ , and note that in particular this implies that for any  $u, v, u', v' \in V$ ,

$$\frac{C(u, v)}{C(u', v')} = \frac{2w(E) \text{Reff}(u, v)}{2w(E) \text{Reff}(u', v')} < 1 + \frac{1}{10n}.$$

Let us compute the ratios.

$$\begin{aligned}\frac{C(s,t)}{C(x,y)} &= 2(4 + (n-3)\gamma) \cdot \frac{2 + (n-2)\gamma}{2(n-2)(4 + (n-3)\gamma)} \\ &= \gamma + \frac{2}{n-2}.\end{aligned}$$

Thus

$$\gamma < 1 - \frac{2}{n-2} + \frac{1}{10n}.$$

But on the other hand we see that,

$$\begin{aligned}\frac{C(x,y)}{C(s,x)} &= \frac{2(n-2)(4 + (n-3)\gamma)}{2 + (n-2)\gamma} \cdot \frac{2 + (n-2)\gamma}{(4 + (n-3)\gamma)(n-1 + (n-2)\gamma)} \\ &= \frac{2(n-2)}{n-1 + (n-2)\gamma} \\ &= \frac{2}{1 + \frac{1}{n-2} + \gamma}.\end{aligned}$$

and thus,

$$\begin{aligned}2 &< \left(1 + \frac{1}{n-2}\right) \cdot \left(1 + \frac{1}{10n}\right) + \gamma \left(1 + \frac{1}{10n}\right) \\ \implies \gamma &> \frac{2 - \left(1 + \frac{1}{n-2}\right) \cdot \left(1 + \frac{1}{10n}\right)}{1 + \frac{1}{10n}} \\ &= 2 \cdot \left(1 - \frac{1}{10n+1}\right) - \left(1 + \frac{1}{n-2}\right) \\ &= 1 - \frac{1}{n-2} - \frac{2}{10n+1}.\end{aligned}$$

and thus we conclude that

$$1 - \frac{1}{n-2} - \frac{2}{10n+1} < \gamma < 1 - \frac{2}{n-2} + \frac{1}{10n}, \tag{B.9}$$

which is a contradiction.  $\square$

## B.2 Transforms for the Flow Metrics

### B.2.1 Another proof for the parallel edges reduction via flows

In this section we present an alternative proof for Claim 3.18 via flows.

*Proof.* (of Claim 3.18) Denote  $f_p(x,t) = \left(\frac{x}{\alpha}\right)^p + \left(\frac{t-x}{\beta}\right)^p$ . For any amount of flow  $0 \leq t \leq 1$  that is shipped to  $a$ , it is the best to minimize  $f_p(x,t)$  where  $0 \leq x \leq t$  (with respect to  $x$  where  $t$  is fixed), and thus choosing how much amount of flow to ship for the top edge and how much to ship from the bottom edge. We will use it to compute the contribution of the discussed edges to the norm of the minimizing flow, and then choose a proper weight  $\gamma$  which will preserve the norm of the flow.

Let us compute the derivative of  $f_p(x, t)$  with respect to  $x$  and equalize it to 0 in order to find the minimizing flow (in the interval  $x \in [0, t]$ ).

$$\begin{aligned}
\frac{d}{dx} f_p(x, t) &= \frac{p}{\alpha} \cdot \left(\frac{x}{\alpha}\right)^{p-1} - \frac{p}{\beta} \cdot \left(\frac{t-x}{\beta}\right)^{p-1} \stackrel{\text{want}}{=} 0 \\
\Rightarrow \frac{x^{p-1}}{\alpha^p} &= \frac{(t-x)^{p-1}}{\beta^p} \\
\Rightarrow x^{p-1} &= \left(\frac{\alpha}{\beta}\right)^p \cdot (t-x)^{p-1} \\
\Rightarrow x &= \left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} \cdot (t-x) \\
\Rightarrow \left(1 + \left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}}\right) \cdot x &= t \cdot \left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} \\
\Rightarrow x_0 &= t \cdot \frac{\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}}}{1 + \left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}}} = t \cdot \frac{\alpha^{\frac{p}{p-1}}}{\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}}} = t \cdot \left(1 - \frac{\beta^{\frac{p}{p-1}}}{\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}}}\right) \\
\Rightarrow \min_{x \in [0, t]} f_p(x, t) &= \left(\frac{t \cdot \alpha^{\frac{1}{p-1}}}{\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}}}\right)^p + \left(\frac{t \cdot \beta^{\frac{1}{p-1}}}{\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}}}\right)^p.
\end{aligned}$$

Thus, we have found the contribution of the discussed edges to the minimizing flow in the graph (between specific vertices), and we wish that the contribution of the new edge will be the same. The contribution of the new edge is  $\left|\frac{t}{\gamma}\right|^p$ , and hence setting

$$\gamma = \frac{1}{\left(\left(\frac{\alpha^{\frac{1}{p-1}}}{\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}}}\right)^p + \left(\frac{\beta^{\frac{1}{p-1}}}{\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}}}\right)^p\right)^{1/p}} = \left(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

will give us the desired outcome. In other words, if we take  $q = \frac{p}{p-1}$  the Hölder conjugate of  $p$ , we get the following rule:

$$\gamma^q = \alpha^q + \beta^q.$$

which is the same conclusion as before. □

## B.2.2 Proof of Y- $\Delta$ transform for $p=2$

In this section we show the existence of a Y- $\Delta$  transform analogue for  $d_2$ .

**Claim B.1.** *There exists a local Y- $\Delta$  transform that preserves  $d_2$ .*

Consider the case presented in figure 3.3. Recall that in the case of effective resistance, the rule is as follows:

$$\begin{aligned}
\alpha &= \frac{w_b \cdot w_c}{w_a + w_b + w_c}, \\
\beta &= \frac{w_a \cdot w_c}{w_a + w_b + w_c}, \\
\gamma &= \frac{w_a \cdot w_b}{w_a + w_b + w_c}.
\end{aligned} \tag{B.10}$$

But in our case, recall that  $d_2$  is in fact the squareroot of the resistance distance in the same graph but with squared edge weights. Thus, we conclude that for  $p = 2$  the rule should be

$$\begin{aligned}\alpha &= \sqrt{\frac{w_b^2 \cdot w_c^2}{w_a^2 + w_b^2 + w_c^2}}, \\ \beta &= \sqrt{\frac{w_a^2 \cdot w_c^2}{w_a^2 + w_b^2 + w_c^2}}, \\ \gamma &= \sqrt{\frac{w_a^2 \cdot w_b^2}{w_a^2 + w_b^2 + w_c^2}}.\end{aligned}$$

Below are the algebraic computations that show that the numbers add up. Throughout the proof we use the same notations as presented for the proof of Claim 3.23.

*Proof.* (of Claim B.1) We will show that this transform indeed holds for  $d_2$ . In fact, we will show that for any potential function  $\varphi$  over the terminals  $V_T$ , the new weights satisfy

$$\min_{x \in \mathbb{R}} \{w_a |\varphi_a - x|^q + w_b |\varphi_b - x|^q + w_c |\varphi_c - x|^q\} = \gamma^q |\varphi_a - \varphi_b|^q + \alpha^q |\varphi_b - \varphi_c|^q + \beta^q |\varphi_c - \varphi_a|^q. \quad (\text{B.11})$$

In particular, this will hold for the minimizing potential function, and will lead to the desired outcome.

Let us compute the LHS of (B.11), in the case where  $q = p = 2$  to verify the rule. Note that the minimizing  $x$  in the LHS of (B.11) is the weighted average of the potentials (weighted by the weights of the edges that connect them to the center of the star). Define a random variable  $X$  by

$$\Pr(X = x) = \begin{cases} \frac{w_a^2}{w_a^2 + w_b^2 + w_c^2} & \text{if } x = \varphi_a, \\ \frac{w_b^2}{w_a^2 + w_b^2 + w_c^2} & \text{if } x = \varphi_b, \\ \frac{w_c^2}{w_a^2 + w_b^2 + w_c^2} & \text{if } x = \varphi_c; \end{cases}$$

and now we can view the LHS of (B.11) as

$$\min_{x \in \mathbb{R}} \left\{ w_a^2 \cdot |\varphi_a - x|^2 + w_b^2 \cdot |\varphi_b - x|^2 + w_c^2 \cdot |\varphi_c - x|^2 \right\} = \min_{x \in \mathbb{R}} \left\{ (w_a^2 + w_b^2 + w_c^2) \cdot \mathbb{E} \left[ |X - x|^2 \right] \right\}.$$

The minimum of the RHS in the above is exactly the variance of  $X$ , and thus

$$x = \mathbb{E}[X] = \frac{w_a^2}{w_a^2 + w_b^2 + w_c^2} \cdot \varphi_a + \frac{w_b^2}{w_a^2 + w_b^2 + w_c^2} \cdot \varphi_b + \frac{w_c^2}{w_a^2 + w_b^2 + w_c^2} \cdot \varphi_c.$$

Let us now plug this into the LHS of (B.11) and compute.

$$\begin{aligned}
& LHS(B.11) = \\
& = w_a^2 \cdot \left| \varphi_a - \frac{w_a^2 \cdot \varphi_a + w_b^2 \cdot \varphi_b + w_c^2 \cdot \varphi_c}{w_a^2 + w_b^2 + w_c^2} \right|^2 \\
& + w_b^2 \cdot \left| \varphi_b - \frac{w_a^2 \cdot \varphi_a + w_b^2 \cdot \varphi_b + w_c^2 \cdot \varphi_c}{w_a^2 + w_b^2 + w_c^2} \right|^2 \\
& + w_c^2 \cdot \left| \varphi_c - \frac{w_a^2 \cdot \varphi_a + w_b^2 \cdot \varphi_b + w_c^2 \cdot \varphi_c}{w_a^2 + w_b^2 + w_c^2} \right|^2 \\
& = \left( \frac{w_a}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot |(w_b^2 + w_c^2) \cdot \varphi_a - w_b^2 \cdot \varphi_b - w_c^2 \cdot \varphi_c|^2 \\
& + \left( \frac{w_b}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot |(w_a^2 + w_c^2) \cdot \varphi_b - w_a^2 \cdot \varphi_a - w_c^2 \cdot \varphi_c|^2 \\
& + \left( \frac{w_c}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot |(w_a^2 + w_b^2) \cdot \varphi_c - w_a^2 \cdot \varphi_a - w_b^2 \cdot \varphi_b|^2 \\
& = \left( \frac{w_a}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot |w_b^2 \cdot (\varphi_a - \varphi_b) + w_c^2 \cdot (\varphi_a - \varphi_c)|^2 \\
& + \left( \frac{w_b}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot |w_a^2 \cdot (\varphi_b - \varphi_a) + w_c^2 \cdot (\varphi_b - \varphi_c)|^2 \\
& + \left( \frac{w_c}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot |w_a^2 \cdot (\varphi_c - \varphi_a) + w_b^2 \cdot (\varphi_c - \varphi_b)|^2 \\
& = \left( \frac{w_a}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot \left( w_b^4 \cdot (\varphi_a - \varphi_b)^2 + 2 \cdot w_b^2 \cdot w_c^2 \cdot (\varphi_a - \varphi_b) \cdot (\varphi_a - \varphi_c) + w_c^4 \cdot (\varphi_a - \varphi_c)^2 \right) \\
& + \left( \frac{w_b}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot \left( w_a^4 \cdot (\varphi_b - \varphi_a)^2 + 2 \cdot w_a^2 \cdot w_c^2 \cdot (\varphi_b - \varphi_a) \cdot (\varphi_b - \varphi_c) + w_c^4 \cdot (\varphi_b - \varphi_c)^2 \right) \\
& + \left( \frac{w_c}{w_a^2 + w_b^2 + w_c^2} \right)^2 \cdot \left( w_a^4 \cdot (\varphi_c - \varphi_a)^2 + 2 \cdot w_a^2 \cdot w_b^2 \cdot (\varphi_c - \varphi_a) \cdot (\varphi_c - \varphi_b) + w_b^4 \cdot (\varphi_c - \varphi_b)^2 \right) \\
& = \frac{w_a^2 \cdot w_b^2 \cdot (w_a^2 + w_b^2) \cdot (\varphi_a - \varphi_b)^2 + w_a^2 \cdot w_c^2 \cdot (w_a^2 + w_c^2) \cdot (\varphi_a - \varphi_c)^2 + w_b^2 \cdot w_c^2 \cdot (w_b^2 + w_c^2) \cdot (\varphi_b - \varphi_c)^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& + 2 \cdot w_a^2 \cdot w_b^2 \cdot w_c^2 \cdot \frac{(\varphi_a - \varphi_b) \cdot (\varphi_a - \varphi_c) + (\varphi_b - \varphi_a) \cdot (\varphi_b - \varphi_c) + (\varphi_c - \varphi_a) \cdot (\varphi_c - \varphi_b)}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& = \frac{w_a^2 \cdot w_b^2 \cdot (w_a^2 + w_b^2) \cdot (\varphi_a - \varphi_b)^2 + w_a^2 \cdot w_c^2 \cdot (w_a^2 + w_c^2) \cdot (\varphi_a - \varphi_c)^2 + w_b^2 \cdot w_c^2 \cdot (w_b^2 + w_c^2) \cdot (\varphi_b - \varphi_c)^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& + 2 \cdot w_a^2 \cdot w_b^2 \cdot w_c^2 \cdot \frac{\varphi_a^2 - \varphi_a \cdot \varphi_c - \varphi_a \cdot \varphi_b + \varphi_b \cdot \varphi_c + \varphi_b^2 - \varphi_b \cdot \varphi_c - \varphi_a \cdot \varphi_b + \varphi_a \cdot \varphi_c + \varphi_c^2 - \varphi_a \cdot \varphi_c - \varphi_b \cdot \varphi_c + \varphi_a \cdot \varphi_b}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& = \frac{w_a^2 \cdot w_b^2 \cdot (w_a^2 + w_b^2) \cdot (\varphi_a - \varphi_b)^2 + w_a^2 \cdot w_c^2 \cdot (w_a^2 + w_c^2) \cdot (\varphi_a - \varphi_c)^2 + w_b^2 \cdot w_c^2 \cdot (w_b^2 + w_c^2) \cdot (\varphi_b - \varphi_c)^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& + 2 \cdot w_a^2 \cdot w_b^2 \cdot w_c^2 \cdot \frac{\varphi_a^2 + \varphi_b^2 + \varphi_c^2 - \varphi_a \cdot \varphi_c - \varphi_b \cdot \varphi_c - \varphi_a \cdot \varphi_b}{(w_a^2 + w_b^2 + w_c^2)^2}.
\end{aligned}$$

Now let us compute the RHS of (B.11) when applying the rule.

$$\begin{aligned}
& RHS(B.11) = \\
& = \left( \sqrt{\frac{w_a^2 \cdot w_b^2}{w_a^2 + w_b^2 + w_c^2}} \right)^2 \cdot |\varphi_a - \varphi_b|^2 + \left( \sqrt{\frac{w_b^2 \cdot w_c^2}{w_a^2 + w_b^2 + w_c^2}} \right)^2 \cdot |\varphi_b - \varphi_c|^2 + \left( \sqrt{\frac{w_a^2 \cdot w_c^2}{w_a^2 + w_b^2 + w_c^2}} \right)^2 \cdot |\varphi_c - \varphi_a|^2 \\
& = \frac{w_a^2 \cdot w_b^2}{w_a^2 + w_b^2 + w_c^2} \cdot |\varphi_a - \varphi_b|^2 + \frac{w_b^2 \cdot w_c^2}{w_a^2 + w_b^2 + w_c^2} \cdot |\varphi_b - \varphi_c|^2 + \frac{w_a^2 \cdot w_c^2}{w_a^2 + w_b^2 + w_c^2} \cdot |\varphi_c - \varphi_a|^2 \\
& = \frac{w_a^2 \cdot w_b^2 \cdot (w_a^2 + w_b^2 + w_c^2) \cdot |\varphi_a - \varphi_b|^2 + w_b^2 \cdot w_c^2 \cdot (w_a^2 + w_b^2 + w_c^2) \cdot |\varphi_b - \varphi_c|^2 + w_a^2 \cdot w_c^2 \cdot (w_a^2 + w_b^2 + w_c^2) \cdot |\varphi_c - \varphi_a|^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& = \frac{w_a^2 \cdot w_b^2 \cdot (w_a^2 + w_b^2) \cdot (\varphi_a - \varphi_b)^2 + w_a^2 \cdot w_c^2 \cdot (w_a^2 + w_c^2) \cdot (\varphi_a - \varphi_c)^2 + w_b^2 \cdot w_c^2 \cdot (w_b^2 + w_c^2) \cdot (\varphi_b - \varphi_c)^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& + w_a^2 \cdot w_b^2 \cdot w_c^2 \cdot \frac{(\varphi_a - \varphi_b)^2 + (\varphi_a - \varphi_c)^2 + (\varphi_b - \varphi_c)^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& = \frac{w_a^2 \cdot w_b^2 \cdot (w_a^2 + w_b^2) \cdot (\varphi_a - \varphi_b)^2 + w_a^2 \cdot w_c^2 \cdot (w_a^2 + w_c^2) \cdot (\varphi_a - \varphi_c)^2 + w_b^2 \cdot w_c^2 \cdot (w_b^2 + w_c^2) \cdot (\varphi_b - \varphi_c)^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& + w_a^2 \cdot w_b^2 \cdot w_c^2 \cdot \frac{\varphi_a^2 - 2 \cdot \varphi_a \cdot \varphi_b + \varphi_b^2 + \varphi_a^2 - 2 \cdot \varphi_a \cdot \varphi_c + \varphi_c^2 + \varphi_b^2 - 2 \cdot \varphi_b \cdot \varphi_c + \varphi_c^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& = \frac{w_a^2 \cdot w_b^2 \cdot (w_a^2 + w_b^2) \cdot (\varphi_a - \varphi_b)^2 + w_a^2 \cdot w_c^2 \cdot (w_a^2 + w_c^2) \cdot (\varphi_a - \varphi_c)^2 + w_b^2 \cdot w_c^2 \cdot (w_b^2 + w_c^2) \cdot (\varphi_b - \varphi_c)^2}{(w_a^2 + w_b^2 + w_c^2)^2} \\
& + 2 \cdot w_a^2 \cdot w_b^2 \cdot w_c^2 \cdot \frac{\varphi_a^2 + \varphi_b^2 + \varphi_c^2 - \varphi_a \cdot \varphi_b - \varphi_a \cdot \varphi_c - \varphi_b \cdot \varphi_c}{(w_a^2 + w_b^2 + w_c^2)^2}.
\end{aligned}$$

Thus we got that both quantities are the same, and hence we have a proper  $Y - \Delta$  transform for  $d_2$ .  $\square$

# Appendix C

## Embedding Conjecture Appendix

In Chapter 4, we discussed the geometry of the flow metrics, which we would like to better understand. We mentioned Conjecture 4.6, which we restate here.

**Conjecture C.1.** *Let  $p \in [1, \infty]$  with Hölder conjugate  $q$ , and let  $G = (V, E, w)$  be a graph. Then, there exists a mapping  $\Phi : V \rightarrow \ell_q$  such that,*

$$\forall s, t \in V, \quad d_p(s, t) = \|\Phi(s) - \Phi(t)\|_q. \quad (\text{C.1})$$

We remark that for the special cases  $p = 1, 2, \infty$  it is known to hold.

**Effective Resistance.** Spielman and Srivastava [SS11] showed that the resistance distance can be isometrically embedded into  $\ell_2^2$ . Thus, since  $d_2$  is the squareroot of the effective resistance, we conclude that  $d_2$  can be isometrically embedded into  $\ell_2$  as desired (since  $p = q = 2$  in this special case).

**Shortest-path.** Note that every finite metric space embeds isometrically into  $\ell_\infty$  [Mat02], and thus  $d_1$  does as well.

**Minimum Cuts.** We remark that  $d_\infty$  is in fact an ultrametric, and thus it embeds isometrically into  $\ell_1$ .

### C.1 The $p$ -strong Triangle Inequality is not Enough

In this section we show that even though the flow metrics satisfy a stronger version of the triangle inequality (Theorem 2.15), it is not enough in order to prove Conjecture C.1. We first recall the relevant definitions.

**Definition C.2.** *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and let  $\Phi : X \rightarrow Y$  be a mapping (which we call an embedding). The distortion of  $\Phi$  is the minimum  $D \geq 1$  for which there exists a scaling factor  $\alpha > 0$ , such that*

$$\forall x, x' \in X, \quad d_X(x, x') \leq \alpha \cdot d_Y(\Phi(x), \Phi(x')) \leq D \cdot d_X(x, x'). \quad (\text{C.2})$$

**Definition C.3.** *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and let  $D \geq 1$ . We say that  $(X, d_X)$   $D$ -embeds into  $(Y, d_Y)$  if there exists an embedding of  $(X, d_X)$  into  $(Y, d_Y)$  with distortion  $D$ .*

Our focus in this section is to show the following claim.



**Claim C.4.** For all  $p \in [1, \infty)$ ,  $q \in [1, \infty)$ , and  $n \geq 2$ , there exists an  $n$ -point metric space  $(X, d)$ , that satisfies the  $p$ -strong triangle inequality, but  $D$ -embeds into  $\ell_q$  only for  $D = \Omega\left(\frac{1}{q} \cdot (\log n)^{1/p}\right)$ .

In other words, the above claim says that the fact that a metric space is  $p$ -strong isn't enough to guarantee isometric embedding into  $\ell_q$ , for any  $q \in [1, \infty)$ . We remark that  $p$  and  $q$  in the statement are not necessarily Hölder conjugates of each other. But, in the specific case where they are, and  $p \in [2, \infty)$ , it holds that  $q \in [1, 2]$ . Thus,  $D = \Omega\left((\log n)^{1/p}\right)$  and as a consequence, we will need to use additional properties of the family of the flow-metrics if we desire to prove the Conjecture C.1.

Our proof of Claim C.4 relies on the following theorem, presented by Matoušek [Mat97].

**Theorem C.5.** There exists constants  $c_1 > 0$  and  $n_0 \in \mathbb{N}$  such that for any  $p \geq 1$  and any  $n \geq n_0$  there exists an  $n$ -point metric space which  $D$ -embeds into  $\ell_p$  only for  $D \geq \frac{c_1}{p} \cdot \log n$ .

*Proof.* (Claim C.4) Theorem C.5 was proved via expanders - the metric space that is promised to exist in the theorem arises from this family of graphs. Let  $G$  be an expander on  $n$  vertices, and let  $(X, d)$  be the metric space that is derived from the shortest-path metric on  $G$ . Let  $p \in (1, \infty)$ , and define  $d' : X \times X \rightarrow \mathbb{R}^+$  by

$$\forall x, y \in X, \quad d'(x, y) = (d(x, y))^{1/p}.$$

We would like to show the following.

1.  $d'$  is a metric.
2.  $d'$  is  $p$ -strong.
3. Embedding  $(X, d')$  in  $\ell_q$  requires large distortion (or at least larger than 1).

First, note that property 2 is immediate, since it simply says that

$$\begin{aligned} \forall x, y, z \in X, \quad & (d'(x, y))^p \leq (d'(x, z))^p + (d'(z, y))^p \\ \iff & d(x, y) \leq d(x, z) + d(z, y). \end{aligned}$$

where the last line is just the triangle inequality which is clearly satisfied since  $(X, d)$  is a metric space.

For showing property 1, we recall Claim 2.17.

**Claim C.6.** Let  $a_1, \dots, a_n \geq 0$  and let  $p > 0$ , then:

1. if  $p \leq 1$ :

$$\sum_{i=1}^n a_i^p \geq \left( \sum_{i=1}^n a_i \right)^p.$$

2. if  $p \geq 1$ :

$$\sum_{i=1}^n a_i^p \leq \left( \sum_{i=1}^n a_i \right)^p.$$

To prove property 1, let  $x, y, z \in X$ , and observe that

$$\begin{aligned} d'(x, y) &= (d(x, y))^{1/p} \\ &\leq (d(x, z) + d(z, y))^{1/p} && (d \text{ is a metric}) \\ &\leq (d(x, z))^{1/p} + (d(z, y))^{1/p} && (1 \leq p \text{ and Claim 2.17}) \\ &= d'(x, z) + d'(z, y). \end{aligned}$$

We are ready to show that property 3 holds. Assume that  $\Phi : (X, d') \rightarrow \ell_q$  is a  $D$ -embedding. Thus, there exists a scaling factor  $\alpha > 0$  such that,

$$\forall x, y \in X, \quad d'(x, y) \leq \alpha \cdot \|\Phi(x) - \Phi(y)\|_q \leq D \cdot d'(x, y).$$

Hence

$$\forall x, y \in X, \quad d(x, y)^{1/p} \leq \alpha \cdot \|\Phi(x) - \Phi(y)\|_q \leq D \cdot d(x, y)^{1/p} \quad (\text{C.3})$$

Next, denote  $\rho = \text{diam}(G)$ , and let us consider an embedding  $\Psi$  defined by

$$\Psi(x) = \rho^{\frac{p-1}{p}} \cdot \alpha \cdot \Phi(x).$$

Fix  $x, y \in X$ , thus

$$\|\Psi(x) - \Psi(y)\|_q = \rho^{\frac{p-1}{p}} \cdot \alpha \cdot \|\Phi(x) - \Phi(y)\|_q.$$

But now we can see that on the one hand

$$\begin{aligned} \|\Psi(x) - \Psi(y)\|_q &\geq \rho^{\frac{p-1}{p}} \cdot (d(x, y))^{1/p} \\ &\geq (d(x, y)^{p-1} \cdot d(x, y))^{1/p} \\ &= d(x, y), \end{aligned} \quad (\text{C.4})$$

and on the other hand, since  $d(x, y) \geq 1$ ,

$$\begin{aligned} \|\Psi(x) - \Psi(y)\|_q &\leq \rho^{\frac{p-1}{p}} \cdot (D \cdot (d(x, y))^{1/p}) \\ &\leq (D \cdot \rho^{\frac{p-1}{p}}) \cdot d(x, y). \end{aligned} \quad (\text{C.5})$$

Combining equations (C.4) and (C.5) together, we can conclude that  $\Psi$  is a  $(D \cdot \rho^{\frac{p-1}{p}})$ -embedding of  $(X, d)$  (the original metric space) into  $\ell_q$ . But according to Theorem C.5,  $D \cdot \rho^{\frac{p-1}{p}} \geq \frac{c_1}{q} \cdot \log n$  and  $\rho = \text{diam}(G) = O(\log n)$ , which implies that  $D = \Omega((\log n)^{1/p} / q)$  as claimed.  $\square$