Seminar on Algorithms and Geometry – Handout 3

Robert Krauthgamer

April 2, 2009

Today's topics

- Approximation algorithm for Sparsest-Cut via embedding into ℓ_1 (continued from last time).
- Application of Sparsest-Cut to Minimum Bisection
- Distortion lower bounds for embedding a cube into ℓ_2
- Distortion lower bounds for embedding expanders into ℓ_2

Homework

1. Let $G(V,E)$ be an r-regular graph with edge-expansion $\alpha > 0$, i.e. $e(S,\bar{S}) \geq \alpha|S|$ for all $S \subseteq V$ with $0 < |S| \leq |V|/2$. Show that embedding the shortest-path metric of G into ℓ_1 requires distortion $\Omega(c \cdot \log|V|)$ where $c = c(r,\alpha) > 0$ is independent of n.

   Hint: Show a gap between the value of the optimal sparset-cut and that of the LP relaxation.

2. In the Multicut problem the input is a graph G and k demand-pairs $\{s_1,t_1\},\ldots,\{s_k,t_k\}$ (similarly to Sparsest-Cut). The goal is to find a partition of the vertices $V = V_1 \cup V_2 \cup \cdots$ where every s_i must be separated from its corresponding t_i (separated means they belong to different parts V_j), so as to minimize the number of edges cut (an edge is cut if its endpoints belong to different parts). Give an algorithm that approximates the Multicut problem.

   Hint: Use (repeatedly) an algorithm that approximates Sparset-Cut.

3. Let the graph $H = (V,E)$ be the discrete hypercube $\{0,1\}^m$ with an edge (x,y) whenever x differs from y in exactly one coordinate. Prove that for every $f : V \to \ell_2$,

   $$\mathbb{E}_{(x,y)\in V\times V}\|f(x)-f(y)\|^2 \leq O(m)\ \mathbb{E}_{(x,y)\in E}\|f(x)-f(y)\|^2.$$

   Note: The lefthand side has uniform distribution over all pairs, the righthand side has uniform distribution over edges.

   Hint: Use the inequality for "diagonals" shown in class.