Randomized Algorithms 2023A – Lecture 8a*
Probabilistic Embedding into Dominating Trees (cont'd)

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1 Probabilistic Embedding into Dominating Trees (cont'd)

We shall complete the proof of the following theorem.

Theorem 2 [Bartal'96, Fakcharoenphol-Rao-Talwar'03]: Every n-point metric admits a probabilistic embedding into dominating trees with distortion $O(\log n)$.

1.1 Randomized Decomposition

Intuition: We start with a randomized algorithm to create one partition P_i, and then extend this algorithm to produce a hierarchical decomposition.

Algorithm A (partitioning X at a given scale 2^i):

1. choose a random permutation $\pi : [n] \to X$ and a random $\beta \in [1, 2]$

4. let $P_i \leftarrow \emptyset$

5. for $l = 1$ to n do

7. add to P_i a new cluster consisting of all point in X that are within distance $\beta 2^{i-2}$ from $\pi(l) \in X$ and are not already in any cluster of P_i.

Observations:

a) Every cluster has a "center" point $\pi(l)$, but it need not contain the center.

b) Every cluster has diameter at most $2\beta_i \leq 2^i$.

c) We can think of line 7 as if each vertex in X assigns itself to the first center, according to the order π, within distance $\beta 2^{i-2}$.

d) The algorithm may create empty clusters but we can discard them.

*These notes summarize the material covered in class, usually skipping proofs, details, examples and so forth, and possibly adding some remarks, or pointers. The exercises are for self-practice and need not be handed in. In the interest of brevity, most references and credits were omitted.

Algorithm B (hierarchical partitioning of X):

1. choose a random permutation $\pi : [n] \to X$ and a random $\beta \in [1, 2]$
2. initialize $P_L \leftarrow \{X\}$
3. for $i = L - 1$ down to 0 do
4. let $P_i \leftarrow \emptyset$
5. for $l = 1$ to n do
6. for every cluster $S \in P_{i+1}$
7. add to P_i a new cluster consisting of all points in S that are within distance $\beta 2^{i-2}$ from $\pi(l)$ and are not already in any cluster of P_i.

Observation: This is like applying Algorithm A recursively to partition each $S \in P_{i+1}$, except that the "centers" are taken from all of X a(not only from S) and all scales use the same randomness (π and β).

Analysis: We say that a center w *separates a pair* $\{x, y\} \subset X$ *at level* i if x, y are in the same cluster of P_{i+1} but in different clusters of P_i, and w is the center that "caused" this, i.e., the first point (according to π) that "captures" exactly one of x, y (at level i).

Lemma 6: For every $x, y \in X$,

$$\mathbb{E}[d_T(x, y)] \le \sum_{i=0}^{L} \sum_{w \in X} \Pr[w \text{ separates } \{x, y\} \text{ at level } i] \cdot 2^{i+2}.$$

Proof: As seen in class, it follows easily form Lemma 3.

Lemma 7 (contribution of a single center): Fix $x, y \in X$. Arrange $X = \{w_1, \dots, w_n\}$ in order of increasing distance from the set $\{x, y\}$ (breaking ties arbitrarily). Then

$$\forall s \in [n], \quad \sum_i \Pr[w_s \text{ separates } \{x, y\} \text{ at level } i] \cdot 2^{i+2} \le O(\tfrac{1}{s}) \cdot d(x, y).$$

Completing the proof of Theorem 2: Putting together Lemmas 6 and 7, we get

$$\forall x, y \in X, \quad \mathbb{E}[d_T(x, y)] \le \sum_{s=1}^{n} O(\tfrac{1}{s}) \cdot d(x, y) \le O(\log n) \cdot d(x, y).$$

QED.

Proof of Lemma 7: Was seen in class by carefully breaking the event into two events, roughly one about β and one about the ordering π.

Exer: Prove that Algorithm A outputs a partition P_i of X where all clusters have diameter at most 2^i, and

$$\forall x, y \in X, \qquad \Pr[x, y \text{ are in different clusters of } P] \le O(\log n) \cdot \tfrac{d(x,y)}{2^i}.$$

Observe that this is the bound required in Lemma 5 with $\alpha = O(\log n)$ (although it is not a hierarchical decomposition).

Hint: Start by assuming that $d(x, y) \leq 2^{i-3}$ and proving a similar bound except that $O(\log n)$ is replaced by $O\big(1 + \log \frac{|B(\{x,y\},2^{i-1})|}{|B(\{x,y\},2^{i-3})|}\big)$, where $B(\{x, y\}, r)$ is the set of points within distance at most r from $\{x, y\}$ (analogous to a ball of some radius around a point). You may need the estimate $\sum_{k=1}^{n} \frac{1}{k} = \ln n + \Theta(1)$.