Randomized Algorithms 2025A – Problem Set 4

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1. Let C be a weighted cycle, i.e., a cycle graph on vertex set $[n]$ with edge weight $w_e > 0$ on every edge e. Show that there is a probability distribution over the edges (described as a vector $\vec{p}$ whose coordinates $p_e \geq 0$ sum to 1), such that the following holds. The spanning tree (actually a path) T obtained by removing an edge e^* chosen at random according to the distribution $\vec{p}$, satisfies that for all $x, y \in C$,

$$\begin{aligned} d_T(x,y) &\geq d_C(x,y); \text{ and} \\ \mathbb{E}[d_T(x,y)] &\leq 2 \cdot d_C(x,y). \end{aligned}$$

   Remark: We discussed in class the special case where $w_e = 1$ for all edges e, but we did not fully prove it.

2. We saw in class Algorithm A that outputs a random partition P_i of X, and we saw that all its clusters have diameter at most 2^i (always). Show that

$$\forall x, y \in X, \qquad \Pr[x, y \text{ are in different clusters of } P_i] \leq O(\log n) \cdot \tfrac{d(x,y)}{2^i}.$$

   Hint: Assume first that $d(x,y) \leq 2^{i-3}$ and prove the above bound with $O(\log n)$ replaced by $O\big(1 + \log \frac{|B(\{x,y\},2^{i-1})|}{|B(\{x,y\},2^{i-3})|}\big)$, where $B(\{x,y\},r)$ is the set of points within distance at most r from $\{x,y\}$ (analogous to a ball of radius r around a point). You may need the estimate $\sum_{k=1}^{n} \frac{1}{k} = \ln n + \Theta(1)$.