Advanced Algorithms – Handout 11

Robert Krauthgamer

May 1, 2008

1 Administrative issues

- There will be NO class the next two weeks (May 8 and May 15). Thus there are two more classes, on May 22 and 29.
- The time and/or place of the last class (May 29) is likely to change. Stay tuned!
- Exam: Thursday June 12, 2008, same time and place as the course (2pm in Ziskind 1).

2 Today's topics

Randomized rounding of linear programming relaxations:

- $O(\log n)$ approximation or Set Cover
- 3/4-approximation for MAX-SAT.

3 Homework

The homework deals with the paper "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming", by Goemans and Williamson, JACM 1995. You can find it here:

- `http://dx.doi.org/10.1145/227683.227684`
- `http://www-math.mit.edu/~goemans/maxcut-jacm.pdf`

Read up to section 3.1 (inclusive), and section 4 up to section 4.1 (inclusive). Verify *to yourself* that you understand the following:

- Definition of the MAX-CUT problem
- The program (P), why it is a relaxation of MAX-CUT and how to solve it in polynomial time using semidefinite programming.
- The randomized rounding procedure and how to implement it in polynomial time.
- The approximation guarantee, mostly the analysis of the expected value of the resulting cut, (compared to the value of the SDP).

Some clarifications:

- The unit sphere S_n is the set of all vectors $v \in \mathbb{R}^n$ of unit length (i.e. $\|v\|_2 = 1$).
- Fact: Drawing a vector z uniformly at random from S_n can be done by letting $x \in \mathbb{R}^n$ have each of its coordinates be chosen independently at random according to Gaussian distribution $N(0,1)$ and then taking $z = \frac{x}{\|x\|_2}$. (This is said also in the paragraph before Section 3.) Observation: This distribution is invariant under orthogonal transformations (because it maps the sphere onto itself), and thus in the Gaussian representation we can choose x above according to any orthonormal basis of $\mathbb{R}^n$.

Problem set:

1. Show a MAX-CUT instance for which the program (P) has integrality gap < 0.99. (Recall that integrality gap is the ratio between the optimum max-cut and the SDP value.)
2. Recall that $W_{\text{tot}} = \sum_{i<j} w_{ij}$. Show that in instances where the maximum cut has value at least $(1-\varepsilon)W_{\text{tot}}$, the randomized algorithm (in the paper) obtains approximation factor $(1 - O(\sqrt{\varepsilon}))$.

   Hint: When $\alpha > 0$ is sufficiently small, use the approximation $\cos(\alpha) \approx 1 - \frac{1}{2}\alpha^2$ that follows e.g. from Taylor's expansion.
3. Show that the following constraints (called triangle inequality) are valid for MAX-CUT and thus can be added to (P):

$$(v_i - v_j) \cdot (v_k - v_j) \geq 0 \qquad \forall i,j,k \in V$$
$$(v_i + v_j) \cdot (v_k + v_j) \geq 0 \qquad \forall i,j,k \in V$$

   What does the first constraint say about the angle formed by v_i, v_j, v_k (i.e. the angle has two rays emanating from v_j towards v_i and v_j). Give a similar geometric interpretation for the second constraint?
4. Let (P') be the SDP obtained from (P) with the triangle inequalities. Show that given a solution of P' that lies in two dimensions, i.e. $v_i \in \mathbb{R}^2$ for all $i \in V$, we can find in polynomial time a cut $(S, \bar{S})$ whose value is at least the value of (P'). (In other words, such a solution can be rounded with "loss factor" 1.)

   Hint: show that for 3 such vectors, at least two of them must lie on the same line (hence they are either identical or exactly opposite of each other).