Linear Programming Duality - handout 4

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LP-duality asserts that if the optimal solution of the primal linear program below exists and is bounded, then the same holds for the dual linear program, and both optimal solutions have the same value. (Here A_i denotes row i and A^j denotes column j.)

Primal: *minimize* $c^T x$ subject to
$A_i x \geq b_i$, $1 \leq i \leq h$
$A_i x = b_i$, $h < i \leq m$
$x_j \geq 0$, $1 \leq j \leq \ell$.

Dual: *maximize* $b^T y$ subject to
$(A^j)^T y \leq c_j$, $1 \leq j \leq \ell$
$(A^j)^T y = c_j$, $\ell < j \leq n$
$y_i \geq 0$, $1 \leq i \leq h$.

It may happen that neither the primal LP nor its dual are feasible, or that one is unbounded and the other is not feasible.

The duality theory has many consequences. In particular it places linear programming in NP $\bigcap$ coNP, it implies that finding a feasible solution to a linear program is essentially as difficult as finding an optimal solution, it can be used in order to prove other duality type theorems. One such theorem is Farkas' lemma that says that either $Ax = b$ has a solution with $x \geq 0$, or there is a vector y such that $y^T A \geq 0$ but $y^T b < 0$. Whenever one studies a linear program, it is informative to look at its dual.

In economic theory, dual variables are interpreted as *shadow prices*.

Let x and y be feasible solutions to the primal and dual problem. Then they are optimal iff they satisfy the *complementary slackness* conditions. Namely $y^T(Ax - b) = 0$ and $(c^T - y^T A)x = 0$.

Homework

Helly's theorem says that for any collection of m convex bodies in R^n, if every $n+1$ of them intersect, then there is a point lying in the intersection of all of them. Using linear programming duality, prove Helly's theorem for the special case that the convex bodies are halfspaces. This is equivalent to showing that if a system of inequalities $Ax \geq b$ does not have a solution, then we can select $n+1$ of the inequalities such that the resulting system does not have a solution.

(Hint: Construct a primal LP from this system by choosing a 0 cost vector. Prove that the dual is feasible and unbounded. Prove that $n+1$ nonzero dual variables suffice in order to obtain an unbounded solution. Drop all other dual variables and take the dual of the resulting system. What do you get?)

Remark: the same proof easily extends to polyhedrons. To extend it to arbitrary convex bodies, consider the $\binom{m}{n+1}$ guaranteed points of intersection, and replace each convex body by the convex hull of the points of intersection in which it participates. Of course, there are also other proofs for Helly's theorem.