The Ellipsoid algorithm and Positive Definite matrices

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Let Q be an n by n nonsingular real matrix and $t \in R^n$. The mapping $T(x) = Qx + t$ is called an *affine transformation.*

A *unit ball* $S(0, 1)$ in R^n is the set $\{x|x^T x \leq 1\}$.

An *ellipsoid* is the image of a unit ball under an affine transformation.

Observe that $y = Qx + t$ implies $x = Q^{-1}(y - t)$. Hence an ellipsoid is

$$T(S(0,1)) = \{y|(Q^{-1}(y-t))^T Q^{-1}(y-t) \leq 1\} = \{y|((y-t)^T B^{-1}(y-t) \leq 1\}$$

where $B = QQ^T$.

The matrix B is *positive definite* meaning that it is real and symmetric, and satisfies the following conditions (all of which are equivalent):

- $x^T Bx > 0$ for all nonzero $x \in R^n$.
- all its eigenvalues are real and positive.
- all upper left submatrices have positive determinants.
- there exists a matrix Q with linearly independent rows such that $B = QQ^T$. (One possible choice for Q is to have column i equal to $\sqrt{\lambda_i} v_i$, where v_i is the ith eigenvector of B, and λ_i is its eigenvalue.)

The eigenvectors of B are the principle axes of the ellipsoid, the square roots of the eigenvalues are their lengths, and the square root of the determinant gives the volume (scaled by the volume of the unit ball).

In the ellipsoid allgorithm we construct a sequence of ellipsoids $E_k = (B_k, t_k)$. If t_k violates the constraint $a_i^T x \leq b_i$ then we take E_{k+1} to be an ellipsoid that contains $\frac{1}{2}E_k = \{y \in E_k : a_i^T y \leq a_i^T t_k\}$, for which there are the following formulas:

$$t_{k+1} = t_k - \frac{1}{n+1} \frac{B_k a_i}{\sqrt{a_i^T B_k a_i}}$$

$$B_{k+1} = \frac{n^2}{n^2-1}(B_k - \frac{2}{n+1} \frac{B_k a_i a_i^T B_k}{a_i^T B_k a_i})$$

It can be shown that $\text{vol}(E_{k+1}) < e^{-1/2(n+1)}\text{vol}(E_k)$. The proofs start with the simplest ellipsoid, the unit ball, and then use linear transformations.

Positive semidefinite matrices form a convex set, and the ellipsoid algorithm can be used to optimize over them (*positive semidefinite programming*) up to arbitrary precision. As an example, we shall consider the problem of embedding a finite metric space in Euclidean space with minimum distortion.