PROBLEMS RELATED TO INVARIANT THEORY OF TORUS AND FINITE GROUPS

By

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$\mathbb{P}^1(\mathbb{C})$= **Flag variety of** $SL_2(\mathbb{C})$

A Thesis in Mathematics submitted to the

Chennai Mathematical Institute

in partial fulfillment of the requirements

for the degree of Doctor of Philosophy

cmi

March 2011

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DECLARATION

I declare that the thesis entitled *"Problems related to Invariant theory of torus and finite groups"* submitted by me for the Degree of Doctor of Philosophy in Mathematics is the record of academic work carried out by me during the period from October 2005 to March 2011 under the guidance of Professor S. Senthamarai Kannan and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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Certificate

I certify that the thesis entitled *"Problems related to Invariant theory of torus and finite groups"* submitted for the degree of Doctor of Philosophy in Mathematics by Santosha Kumar Pattanayak is the record of research work carried out by him during the period from October 2005 to March 2011 under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

Chennai Mathematical Institute
Date: March, 2011

Professor S. Senthamarai Kannan
Thesis Supervisor

Acknowledgement

First of all I would like to express my gratefulness to my advisor Professor S. Senthamarai Kannan for introducing me to the research area of Invariant theory. I am specially grateful to him for introducing me to the problem in a beautiful way at the time when I was an absolute novice; for his efforts to understand my mindset and thereby advising me appropriately; for his patience in dealing with me; for his continuous encouragement. I deeply appreciate his way of working, start from basics, and must say that it has had a great influence on me. I owe much more to him than I can write here.

I had the opportunity to talk Mathematics with several people. I am especially grateful to Pranab Sardar, Dr. P. Vanchinathan and Professor K. N. Raghavan who were always welcoming, happy to talk about mathematics and answer my questions. I am grateful to Professor V. Balaji and Dr. Clare D'Cruz for their encouragement and constant help. I have had very helpful mathematical discussions with Professor R. Sridharan, Dr. Suresh Nayak and Rohith Varma. I am grateful to all of them.

I am really grateful to Dr. Peter O'Sullivan for carefully reading all my papers in arxiv and for giving me many valuable remarks which helped me to improve my thesis tremendously. I would like to thank Professor Pramathanath Sastry and Dr. M. Sundari for many valuable comments on my thesis-synopsis.

I take this opportunity to thank many of my colleagues, especially Abhishek and Prakash for helping me to write the C-program given in appendix-A. I would like to thank Mahender Singh with whom I have shared many good times and for helping me in many ways to improve this thesis.

I would like to thank NBHM (National Board for Higher Mathematics) and the Chennai Mathematical Institute for supporting me financially throughout the Ph.D program. In particular, I would like to thank the Chennai Mathematical Institute for providing me with various facilities and an environment to do independent work.

I would like to thank the office staff of the Chennai Mathematical Institute (CMI) for their alltime ready-to-help attitude in any kind of official matter. I would also like to thank all the academic staff at the Chennai Mathematical Institute who have been very friendly to me.

Finally this endeavor would not have existed without support from my parents. There can not be any substitute for the unconditional support and love of my parents, who have given me complete freedom over the years.

Dedicated to my advisor

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Chapter 0

Introduction

One of the classical problems in invariant theory is the study of binary quantics. The main object is to give an explicit description of the ring $K[V]^{SL_2}$, where V is the space of all homogeneous forms of degree n in two variables and study the geometric properties of SL_2 quotients of projective space for a suitable choice of linearization. The natural generalization of this classical problem is the following;

Let K be an algebraically closed field. Let G be a semi-simple algebraic group over K, T a maximal torus of G, B a Borel subgroup of G containing T, N the normalizer of T in G and, $W = N/T$ the Weyl group. For a parabolic subgroup Q of G containing B, consider the quotient variety $N\backslash\backslash(G/Q)$. In the case when $G = SL_n(K)$, the special linear group and Q is the maximal parabolic subgroup of $SL_n(K)$ associated to the simple root α_2, one knows that G/Q is the Grassmannian $G_{2,n}$ of two- dimensional subspaces of an n dimensional vector space. One also has an isomorphism:

$$N\backslash\backslash(G/Q)^{ss}(\mathcal{L}_2) = N\backslash\backslash(G_{2,n})^{ss}(\mathcal{L}_2) \simeq SL_2\backslash\backslash(\mathbb{P}(V))^{ss},$$

where V is the vector space of homogeneous polynomials of degree n in two variables and $\mathcal{L}_2$ is the line bundle associated to the fundamental weight ϖ_2, and the variety $SL_2\backslash\backslash\mathbb{P}(V)^{ss}$ is precisely the space of binary quantics, (for example, see the proof of Theorem-1 and the proof of Theorem-4 of [100]). More generally one has the following isomorphism;

$$T\backslash\backslash(G/P)^{ss}(\mathcal{L}_r) = T\backslash\backslash(G_{r,n})^{ss}(\mathcal{L}_r) \simeq SL_r\backslash\backslash(\mathbb{P}^{r-1})^n,$$

where $G = SL_n(K)$, P is the maximal parabolic subgroup associated to the simple root α_r, $G_{r,n}$ is the Grassmannian of r-dimensional subspaces of an n dimensional vector space and $\mathcal{L}_r$ is the line bundle on $G/P = G_{r,n}$ associated to ϖ_r.

One direction of our work is the study of projective normality of GIT quotient varieties for finite group actions and another direction is to study the semi-stable points for a maximal torus action on the homogeneous space G/P, where G is a semi-simple simply connected algebraic group and P is a parabolic subgroup of G. Both studies arose out of an attempt to understand the quotient $S_n\backslash(T\backslash\backslash G_{2,n})^{ss}$.

0.1 General Layout of the Thesis

We now describe the organization of this thesis. The thesis consists of four chapters. A conscious effort is made to make this thesis self-contained and reader-friendly. Chapters 1 and 2 are preliminary in nature and are intended to introduce most of the basic concepts used in this thesis. We do not aim to give a complete account of these topics but try to give most of the definitions and results used later and provide appropriate references for these results. Then while using these results we refer to the first two chapters instead of referring to the original papers, which we have anyway referred to in the introductory chapters. Chapters 3 and 4 report the work done by the author.

In Chapter 1 we give a brief account of the theory of algebraic groups. In this chapter, we introduce some definitions and terminologies which we keep using throughout this thesis. For a detailed study of the theory of algebraic groups, we refer the reader to [2], [46], [115].

Chapter 2 is a survey of computational invariant theory of finite groups as well as reductive algebraic groups. In this chapter we present many classical as well as modern results in invariant theory. In the last section of this chapter "Geometric invariant theory" is introduced.

Chapter 3 is about torus action on G/P. Mainly under the action of a maximal torus we describe all the minimal-dimensional Schubert varieties in G/P admitting semi-stable points with respect to an ample line bundle, where G is a semi-simple simply connected algebraic group and P is a maximal parabolic subgroup of G. In this chapter we also describe all Coxeter elements $w \in W$ for which the corresponding Schubert variety $X(w)$ admits a semi-stable point for the action of a maximal torus with respect to a non-trivial line bundle on G/B.

In Chapter 4 we investigate the projective normality of GIT quotient varieties for the action of finite groups. At the end of this chapter, we also take the opportunity to describe some of the questions that remain to be answered.

At the end of this thesis we have included two appendices, named as Appendix-A and Appendix-B. In Appendix-A we give a C-program that is used in Chapter 3. Appendix-B collects the most important pieces of information about the Lie algebras associated to semi-simple algebraic groups.

Chapter 1

Algebraic Groups

This chapter is the most basic and at the same time the most essential part of this thesis. Here we define all the required terms and review the basic results (without proof) that are needed later in this thesis. The theory of linear algebraic groups is a well-developed topic and there are many excellent books available on it. Mostly, we refer to [2, 46, 115] for simplicity.

1.1 Basic Definitions and Properties

1.1.1 Definition and Examples

An affine algebraic group is a group G equipped with a structure of an affine variety such that the multiplication map $\mu : G \times G \to G, \mu(g_1, g_2) = g_1 g_2$ and the inverse map $i : G \to G, i(g) = g^{-1}$ are morphisms of affine varieties.

Since any variety has atleast one smooth point and the action of G on itself by left translation is transitive, G is a smooth variety.

A homomorphism of algebraic groups $\phi : G_1 \to G_2$ is a homomorphism of groups and also a morphism of varieties. An isomorphism of algebraic groups is a bijective homomorphism $\phi : G_1 \to G_2$ such that ϕ^{-1} is also a morphism of varieties. An isomorphism from G to itself is called an automorphism.

Example : An example of an affine algebraic group is the group GL_n of $n \times n$ invertible matrices. Indeed, we have

$$GL_n = \{ \begin{pmatrix} X & 0 \\ 0 & x_{n+1} \end{pmatrix} : det(X)x_{n+1} = 1\}$$

and for $X, Y \in GL_n$, the entries of the product XY are polynomial functions in the entries of X and Y. We call GL_n, the general linear group.

Remark: In this chapter we consider only affine algebraic groups, so the adjective ”affine” will be sometimes omitted.

A closed subgroup of GL_n is called a linear algebraic group. It is easy to see that if H is a subgroup of an algebraic group G and also a closed subvariety of G, then H is an algebraic subgroup. So we have several examples of algebraic subgroups of GL_n. We list some of them below:

Example: D_n: the group of invertible diagonal matrices.
B_n: the group of upper triangular matrices.
U_n: the group of unipotent upper triangular matrices.
$SL_n = \{X \in GL_n : det(X) = 1\}$: the special linear group.
$O_n = \{X \in GL_n : X^t X = I_n\}$: the orthogonal group, (where X^t denotes the transpose of the matrix X and I_n is the identity matrix in GL_n).
$SO_n = SL_n \cap O_n$: the special orthogonal group. This group can also be defined as $\{X \in GL_n : X^t JX = J\}$, where $J \in GL_n$ is the matrix $\begin{pmatrix} 0 & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0 \end{pmatrix}$ if n is even and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_{\frac{n-1}{2}} \\ 0 & I_{\frac{n-1}{2}} & 0 \end{pmatrix}$ if n is odd (char $K \neq 2$).
$Sp_{2n} = \{X \in GL_{2n} : X^t JX = J\}$: the symplectic group, where $J \in GL_{2n}$ is the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Example : (Finite groups). Any finite set X with n elements admits a canonical structure of an affine algebraic variety (over K). This variety has n irreducible one-point components and the algebra of regular functions $K[X]$ is the direct sum of n copies of the field K: $K[X] = K \oplus \cdots \oplus K$. In particular, any K-valued function on X is regular, and any map $X \to Y$ to another affine variety Y is a morphism. This shows that any finite group G has a canonical structure of an affine algebraic group.

Example : (Additive and multiplicative groups). The additive group G_a is the affine line K^1 with group law $\mu(x, y) = x + y$ and $i(x) = -x$. The multiplicative group G_m is the affine open subset $K^\times \subset K$ with $\mu(x, y) = xy$, $i(x) = x^{-1}$. Clearly, they are commutative one-dimensional algebraic groups. The group G_m may be realized as GL_1, but for a matrix realization of G_a one needs 2×2-matrices:

$$\{\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : c \in K\}.$$

Infact G_a and G_m are the only connected one-dimensional algebraic groups.

Again, the direct product of two affine algebraic groups has a canonical structure of an affine algebraic group. So, we can construct many examples of algebraic groups, for example: the direct product $T = G_m^k$ is a commutative algebraic group called an algebraic torus.

For an algebraic group G the connected component G^0 containing the identity element is a closed normal subgroup of finite index and coincide with the irreducible component containing identity. So, the notions of irreducibility and connectedness coincide for affine algebraic groups. Since GL_n is an open subset of $M_{n\times n}$, it is irreducible. So, GL_n is connected. One can check that the commutator subgroup $[G, G]$ of a connected algebraic group G is connected. In particular, SL_n is connected being commutator of GL_n.

1.1.2 Actions and Representations of Algebraic Groups

Let X be an algebraic variety and G an algebraic group. A morphism $G \times X \to X$ is said to be an algebraic action, if it satisfies the following properties:

(i) $e.x = x$ for any $x \in X$;

(ii)$g_1.(g_2.x) = (g_1 g_2).x$ for any $g_1, g_2 \in G, x \in X$.

Example: There are three actions of a group G on itself, which are considered most often. Namely, $g.g_1 = gg_1,\ g.g_1 = g_1 g^{-1},\ g.g_1 = gg_1g^{-1}$.

We refer to [46] for the definition of orbit, stabilizer and the set of fixed points of an action. The subset X^G of G-fixed points is closed in X and for any $x \in X$ the stabilizer G_x is a closed subgroup of G. Further, the orbit $G.x$ is a smooth locally closed subvariety of X and orbit of the smallest dimension is closed in X. Moreover, $dim(G) = dim(G_x) + dim(G.x)$.

Definition : A rational representation of an algebraic group G in a finite dimensional vector space V is a homomorphism $\rho : G \to GL(V)$ of algebraic groups. Here V is said to be a rational G-module.

Any representation $\rho : G \to GL(V)$ defines an action $G \times V \to V,\ g.v = \rho(g)v$. Such actions are called linear.

Remark : A rational representation of GL_n is a homomorphism $\rho : GL_n \to GL(V)$ such that the matrix entries of $\rho(A)$ are polynomials in $a_{ij},\ \frac{1}{det(A)}$, where V is a finite dimensional vector space. The presence of $\frac{1}{det(A)}$ motivates the term "rational".

Remark : Standard constructions of representation theory (restrictions to invariant subspaces, quotient and dual representations, direct sums, tensor products, symmetric and exterior powers etc.) allow to produce numerous rational G-modules from given ones.

Remark : Any rational representation $\rho : G \to GL(V)$ defines a natural algebraic action on the projective space $P(V);\ g.[v] := [\rho(g)v]$.

If X is an affine G-variety then, there is a natural action of G on the algebra of regular functions $K[X]$:

$$(g.f)(x) := f(g^{-1}.x);\ f \in K[X],\ x \in X,\ g \in G.$$

The G-module $K[X]$ is locally finite i.e. any element $f \in K[X]$ is contained in a finite dimensional rational submodule.

The next theorem explains why we call an affine algebraic group linear.

Theorem 1.1.1. *Any affine algebraic group is isomorphic to a closed subgroup of GL_n for some $n \in \mathbb{N}$.*

1.2 Jordan Decomposition in Linear Algebraic Groups

In this section we will assume that K is an algebraic closed field.
A matrix $x \in M_n(K)$ is semi-simple if x is diagonalizable: there is a $g \in GL_n(K)$ such that gxg^{-1} is a diagonal matrix. Also, x is unipotent if $x - I_n$ is nilpotent: $(x - I_n)^k = 0$ for some natural number k. For given $x \in GL_n(K)$, there exist elements x_s and x_u in $GL_n(K)$ such that x_s is semi-simple, x_u is unipotent, and $x = x_s.x_u = x_u.x_s$. Furthermore, x_s and x_u are uniquely determined (see [46, pg. 96]). Now suppose that G is an affine algebraic group. We can choose n and an injective homomorphism $\phi : G \to GL_n(K)$ of algebraic groups. If $g \in G$, the semi-simple and unipotent parts $\phi(g)_s$ and $\phi(g)_u$ of $\phi(g)$ lie in $\phi(G)$.The elements g_s and g_u such that $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$ depend only on g and not on the choice of ϕ (or n). The elements g_s and g_u are called the semi-simple and unipotent part of g, respectively. An element $g \in G$ is semi-simple if $g = g_s$, and unipotent if $g = g_u$.

Theorem 1.2.1. *(Jordan-Chevalley Decomposition ([46, pg. 99]). If $g \in G$, there exist unique elements g_s and g_u in G such that $g = g_s.g_u = g_u.g_s$, g_s is semi-simple, and g_u is unipotent. Further, Jordan decompositions are preserved by homomorphisms of algebraic groups.*

For any algebraic group G the set $G_u = \{g_u : g \in G\}$ is a closed subset of G. An algebraic group G is called unipotent if all of its elements are unipotent. For example G_a is unipotent.

A solvable (resp. nilpotent) algebraic group is an algebraic group which is solvable (resp. nilpotent) as an abstract group. Now let G be an arbitrary connected algebraic group. Suppose that A and B are two closed connected normal solvable subgroups of G. Then AB is again a closed connected normal solvable subgroup of G containing both A and B. It follows that G contains a unique closed connected normal solvable subgroup of maximal dimension. This is called the radical of G, denoted by $R(G)$. An algebraic group is called *semi-simple* if its radical $R(G) = e$. Similarly the unipotent radical of G, denoted by $R_u(G)$ is the unique closed connected normal unipotent subgroup of maximal dimension. An algebraic group is called *reductive* if its unipotent radical $R_u(G) = e$. Since $R_u(G)$ is unipotent, it is nilpotent, hence solvable. Thus $R_u(G) \subseteq R(G)$. So semi-simple groups are reductive.

For example, SL_n is a semi-simple group but GL_n has a one dimensional radical consisting of the scalar matrices. Thus GL_n is not semi-simple, but since scalar matrices are semi-simple, its unipotent radical is trivial, so GL_n is reductive.

1.3 Lie Algebra of an Algebraic Group

Let G be a linear algebraic group. The tangent bundle $T(G)$ of G is the set $Hom_{K-alg}(K[G], K[t]/(t^2))$ of K-algebra homomorphisms from the affine algebra $K[G]$ of G to the algebra $K[t]/(t^2)$. If $g \in G$, the evaluation map $f \mapsto f(g)$ from $K[G]$ to K is a K-algebra isomorphism. This results in a bijection between G and $Hom_{K-alg}(K[G], K)$. Composing elements of $T(G)$ with the map $a + bt + (t^2) \mapsto a$ from $K[t]/(t^2)$ to K results in a map from $T(G)$ to $G = Hom_{K-alg}(K[G], K)$. The tangent space $T_1(G)$ of G at the identity element 1 of G is the fibre of $T(G)$ over 1. If $X \in T_1(G)$ and $f \in K[G]$, then $X(f) = f(1) + td_X(f) + (t^2)$ for some $d_X(f) \in K$. This defines a map $d_X : K[G] \to K$ which satisfies:

$$d_X(f_1 f_2) = d_X(f_1)f_2(1) + f_1(1)d_X(f_2),\ \ f_1, f_2 \in K[G]$$

Let $\mu^* : K[G] \to K[G] \otimes K[G]$ be the K-algebra homomorphism which corresponds to the multiplication map $\mu : G \times G \to G$. Set $\delta_X = (1 \otimes d_X) \circ \mu^*$. The map $\delta_X : K[G] \to K[G]$ is a K-linear map and a derivation:

$$\delta_X(f_1 f_2) = \delta_X(f_1)f_2 + f_1\delta_X(f_2), f_1, f_2 \in K[G].$$

Furthermore, δ_X is left-invariant: $l_g\delta_X = \delta_X l_g$ for all $g \in G$, where $(lgf)(g') = f(g^{-1}g'), f \in K[G]$. The map $X \mapsto \delta_X$ is a K-linear isomorphism of $T_1(G)$ onto the vector space of K-linear maps from $K[G]$ to $K[G]$ which are left-invariant derivations.

Let $\mathfrak{g} = T_1(G)$. Define $[X, Y] \in \mathfrak{g}$ by $\delta_{[X,Y]} = \delta_X \circ \delta_Y - \delta_Y \circ \delta_X$. Then $\mathfrak{g}$ is a vector space over K and the map $[\cdot, \cdot]$ satisfies:

(1) $[\cdot, \cdot]$ is bilinear

(2) $[X, X] = 0$ for all $X \in \mathfrak{g}$

(3) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for all $X, Y, Z \in \mathfrak{g}$ (Jacobi identity).

Therefore $\mathfrak{g}$ is a Lie algebra over K. We call it the Lie algebra of G.

Example : If $G = GL_n(K)$, then $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}l_n(K)$ which is $M_n(K)$ equipped with the Lie bracket $[X, Y] = XY - YX$,$X, Y \in M_n(K)$.

Example : The Lie algebras of the algebraic groups SL_n, SO_n, and SP_{2n} are $\mathfrak{s}l_n$:= the trace zero matrices, $\mathfrak{s}o_n$:= the anti-symmetric matrices and $\mathfrak{s}p_{2n} := \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2n} : A = -D^T, B = B^T, C = C^T\}$ respectively.

Let $\phi : G \to G'$ be a homomorphism of linear algebraic groups. Composition with the algebra homomorphism $\phi^* : K[G'] \to K[G]$ results in a map $T(\phi) : T(G) \to T(G')$. The differential $d\phi$ of ϕ is the restriction $d\phi = T(\phi)|_{\mathfrak{g}}$ of $T(\phi)$ to $\mathfrak{g}$. It is a K-linear map from $\mathfrak{g}$ to

$\mathfrak{g}'$, and satisfies

$$d\phi([X,Y]) = [d\phi(X), d\phi(Y)],\ X, Y \in \mathfrak{g}.$$

That is, $d\phi$ is a homomorphism of Lie algebras. If ϕ is bijective, then ϕ is an isomorphism if and only if $d\phi$ is an isomorphism of Lie algebras. If K has characteristic zero, any bijective homomorphism of linear algebraic groups is an isomorphism.

In characteristic zero the correspondence between algebraic groups and their Lie algebras is very nice. If H is a closed subgroup of a connected linear algebraic group G, then (via the differential of inclusion) the Lie algebra $\mathfrak{h}$ of H is isomorphic to a Lie subalgebra of $\mathfrak{g}$. Infact the correspondence $H \mapsto \mathfrak{h}$ is $1-1$ and inclusion preserving between the collection of closed connected subgroups H of G and the collection of their Lie algebras, regarded as subalgebras of $\mathfrak{g}$. And H is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ ($[X,Y] \in \mathfrak{h}$ whenever $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$). If G is solvable (resp. nilpotent), then $\mathfrak{g}$ is solvable (resp. nilpotent). If G is semi-simple (resp. reductive), then $\mathfrak{g}$ is semi-simple (resp. reductive). Recall that a Lie algebra $\mathfrak{g}$ is said to be semi-simple if $rad(\mathfrak{g})$: the maximal solvable ideal is 0 and reductive if $rad(\mathfrak{g}) = Z(\mathfrak{g})$.

If $g \in G$, then $Int_g : G \to G, Int_g(g_0) = gg_0g^{-1}, g_0 \in G$, is an isomorphism of algebraic groups. So, $Ad(g) := d(Int_g) : \mathfrak{g} \to \mathfrak{g}$ is an isomorphism of Lie algebras and the map $Ad : G \to GL(\mathfrak{g})$ is a homomorphism of algebraic groups, called the *adjoint representation* of G.

Jordan decomposition in the Lie algebra: We can define semi-simple and nilpotent elements in $\mathfrak{g}$ in a manner analogous to definitions of semi-simple and unipotent elements in G (as $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{g}l_n(K)$ for some n). If $X \in \mathfrak{g}$, there exist unique elements X_s and $X_n \in \mathfrak{g}$ such that $X = X_s + X_n,\ [X_s, X_n] = [X_n, X_s] = 0,\ X_s$ is semi-simple, and X_n is nilpotent.

1.4 Homogeneous Spaces

Let G be an affine algebraic group and H a closed subgroup of G. The set of left cosets G/H admits a natural transitive G-action: $g.g_1H = gg_1H$. The following celebrated theorem of Chevalley gives a structure of an algebraic variety on G/H such that the action above becomes algebraic.

Theorem 1.4.1. *(Chevalley [14] (1951)). Let G be an affine algebraic group and H a closed subgroup of G. Then,*

(1) There is a rational representation $\rho : G \to GL(V)$ and a non-zero vector $v \in V$ such that $H = \{g \in G : \rho(g)v \in K.v\}$.

(2) If the subgroup H is normal, then there is a representation $\rho' : G \to GL(V')$ such that $H = Ker(\rho')$.

Now, the induced action of G on $P(V)$ is algebraic, and there exist $[v] \in \mathbb{P}(V)$ such that the stabilizer of $[v]$ coincides with H. The orbit $G[v]$ is open in its closure and thus has a structure of a quasi-projective variety with an algebraic transitive G-action. The orbit map $G \to \mathbb{P}(V),\ g \mapsto g.[v]$ defines a bijection $G/H \to G[v]$, and induces a structure of a quasi-projective variety on G/H such that the natural action of G on G/H is algebraic. Infact we have more;

Corollary 1.4.2. *The set G/H of left cosets admits a unique structure of a quasi-projective algebraic variety such that the natural action of G on G/H is algebraic. In addition if H is a closed normal subgroup of G, then the quotient group G/H has a unique structure of an affine algebraic group such that the projection $G \to G/H$ is a homomorphism of algebraic groups.*

Let G be a unipotent group and choose n such that G is a closed subgroup of GL_n, then there is a $g \in GL_n$ such that $gGg^{-1} \subset U_n$. In particular G is nilpotent. If V is a non-zero rational G-module, then $V^G \neq 0$. For an affine G-variety X, the G-orbits are closed. For if there exists $x \in X$ such that $Z = G.x$ is not closed in X, then $Y = \bar{Z} - Z$ is a non-empty closed subset of $\bar{Z}$. So there exists $f \in I(Y) \setminus \{0\}$ such that $g.f = f$ for all $g \in G$. Hence $f(g.x) = f(x)$ and f is constant on Z, and so on $\bar{Z}$. But f is zero on Y, and so $f = 0$, a contradiction.

It follows that if H is a closed subgroup of an unipotent group G, then the variety G/H is affine (see [118, pg. 397]).

The following proposition is sometimes helpful for computing invariants.

Proposition 1.4.3. *Let G be an algebraic group and H a closed subgroup. Then the projection morphism $G \to G/H$ is open and $K[G/H] = K[G]^H := \{f \in K[G] : f(gh) = f(g)$ for any $g \in G, h \in H\}$.*

For a reductive algebraic group, the following theorem called the "Matsushima criterion" gives a necessary and sufficient condition for a homogeneous space to be affine.

Theorem 1.4.4. *(Matsushima [75]) If G is reductive then G/H is affine if and only if H is reductive.*

We end this section with some examples of homogeneous spaces.

Example : (Grassmannians and Flag Varieties). The group GL_n acts transitively on the set of k-dimensional subspaces of $V = K^n (1 \leq k \leq n)$. The stabilizer of the standard K-subspace $\langle e_1, e_2, \cdots, e_k \rangle$ is

$$P(k,n) := \{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} : A \in GL_k, C \in GL_{n-k}, B \in M_{k \times n-k}\}.$$

Hence the homogeneous space $GL_n/P(k,n)$ is isomorphic to the Grassmannian $Gr(k,n)$ of k-dimensional subspaces in V. Now consider the subgroup $B_n \subset GL_n$. It is the stabilizer of the standard complete flag

$$\{0\} \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \cdots, e_n \rangle = K^n$$

in K^n. Since GL_n acts transitively on the set of complete flags, we again have that GL_n/B_n is isomorphic to the flag variety $\mathcal{F}(V)$. In this case the homogeneous spaces $Gr(k,n)$ and $\mathcal{F}(V)$ are projective.

Example (Homogeneous spaces for $G = SL_2$)
(1) Let $G = SL_2$ and $H = B := \{A \in T_2 : det(A) = 1\}$. In order to apply Chevalley's theorem, consider the tautological SL_2-module $V = K^2$ and the first standard vector $e_1 \in V$. Clearly, $B = \{A \in SL_2 : A.e_1 \in K.e_1\}$. Since SL_2 acts transitively on one dimensional subspaces in V, the homogeneous space SL_2/B is isomorphic to the projective line $\mathbb{P}^1$.

(2) Let $G = SL_2$ and $H = U := U_2$. Again consider $V = K^2$ and $v = e_1$, and note that $U = \{A \in SL_2 : A.e_1 = e_1\}$. Thus, SL_2/U is isomorphic to the orbit of e_1 in V. This is a quasi-affine (non-affine) variety $K^2 \setminus \{0\}$.

(3) Finally, take $G = SL_2$ and $H = T := \{A \in D_2 : det(A) = 1\}$. Let V be the three-dimensional space of 2×2-matrices with trace zero, where SL_2 acts by conjugation: $A.C = ACA^{-1}$. Set

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The stabilizer of v coincides with T, and the orbit Gv consists of matrices with eigenvalues 1 and -1. This orbit is defined in V by the equation $det(C) = -1$. Thus SL_2/T is an affine quadric in A^3.

1.5 Tori

A torus is a linear algebraic group which is isomorphic to the direct product $G_m^d = G_m \times \cdots \times G_m$ (d times), where d is a positive integer. It is easy to see that a linear algebraic group G is a torus if and only if G is connected and abelian, and every element of G is semi-simple.

A character of a torus T is a homomorphism of algebraic groups from T to G_m. The set $X(T)$ of characters of T is a free abelian group. A one-parameter subgroup of T is a homomorphism of algebraic groups from G_m to T. The set $Y(T)$ of one-parameter subgroups of G is also a free abelian group. If $T \simeq G_m$, then $X(T) = Y(T)$ and the only characters are of the form $x \mapsto x^r$, where $r \in \mathbb{Z}$. In general, $T \simeq G_m^d$ for some positive integer d, so $X(T) \simeq X(G_m)^d \simeq \mathbb{Z}^d \simeq Y(T)$. We have a pairing

$$\langle \cdot, \cdot \rangle : X(T) \times Y(T) \to \mathbb{Z}; \quad \langle \chi, \eta \rangle \mapsto r \text{ where } \chi \circ \eta(x) = x^r, \ x \in G_m.$$

Let G be a linear algebraic group which contains at least one torus. Then the set of tori in G has maximal elements, relative to inclusion. Such maximal elements are called maximal tori of G. All the maximal tori in G are conjugate. The *rank* of G is defined to be the dimension of a maximal torus in G. The *semi-simple rank* of G is defined to be the rank of $G/R(G)$, and the *reductive rank* of G is the rank of $G/R_u(G)$.

Now suppose that G is a linear algebraic group and T is a torus in G. Recall that the adjoint representation $Ad : G \to GL(\mathfrak{g})$ is a homomorphism of algebraic groups. Therefore $Ad(T)$ consists of commuting semi-simple elements and so is diagonalizable. Given $\alpha \in X(T)$, let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : Ad(t)X = \alpha(t)X,\ \forall\ t \in T\}$. The nonzero $\alpha \in X(T)$ such that $\mathfrak{g}_\alpha \neq 0$ are the *roots* of G relative to T. The set of roots of G relative to T will be denoted by $\phi(G, T)$.

The centralizer $Z_G(T)$ of T in G is the identity component of the normalizer $N_G(T)$ of T in G. The *Weyl group* $W(G, T)$ of T in G is the (finite) quotient $N_G(T)/Z_G(T)$. Because $W(G, T)$ acts on T, $W(G, T)$ also acts on $X(T)$, and $W(G, T)$ permutes the roots of T in G. When T is a maximal torus, $Z_G(T) = T$ and, hence $W(G, T) = N_G(T)/T$. Since any two maximal tori in G are conjugate, their Weyl groups are isomorphic. The Weyl group of any maximal torus is referred to as the Weyl group of G.

1.6 Solvable Groups and Borel Subgroups

Assume in this section that K is algebraically closed. As in the theory of finite groups, solvable groups are well studied in the theory of algebraic groups, we start with the structure theorem of these groups.

Theorem 1.6.1. *Let G be a connected solvable group. Then, the set G_u of all unipotent elements of G is a closed connected normal subgroup of G. All the maximal tori of G are conjugate, and if T is any one of them, then G is the semi-direct product of T acting on G_u. If G is abelian, then the set of semi-simple elements G_s is also a closed subgroup, and $G \simeq G_s \times G_u$.*

Remark: If G is a unipotent group then the map $exp : Lie(G) \to G$ is an isomorphism of algebraic varieties. So G is connected and is isomorphic (as an variety) to an affine space. If in addition G is commutative then the above map is an isomorphism, $Lie(G)$ is considered as an additive group of the underlying vector space. It follows that any commutative unipotent algebraic group is isomorphic to G_a^m.

Definition: A variety Z is complete if for every variety Y the projection map $\pi : Y \times Z \to Y$ is a closed map, i.e., it takes closed sets to closed sets. All projective varieties are complete. A complete and quasi-projective variety is projective.

The following celebrated fixed point theorem is due to Borel.

Theorem 1.6.2. *(Borel's Fixed Point Theorem ([46, pg. 134] (1956)). Let a connected solvable algebraic group G acts on a complete variety X. Then G has a fixed point in X.*

Let $\rho : G \to GL(V)$ be a finite dimensional rational representation of a connected solvable algebraic group. Then G acts on $\mathbb{P}(V)$ and hence G has a fixed point. This fixed point corresponds to a G-stable line L in V, and G acts on L via some character χ of G. So, we have;

Theorem 1.6.3. *(Lie-Kolchin Theorem ([46, pg. 113]) (1948)). Let G be a connected solvable algebraic group and $\rho : G \to GL(V)$ be a rational representation. Then there is a non-zero vector $v \in V$ such that $\rho(g)v = \chi(g)v$ for some $\chi \in X(G)$ and any $g \in G$.*

Note that the above theorem is analogous to Lie's theorem for a solvable Lie algebra, which says that if $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$, V finite dimensional, then V contains a common eigen vector for all the endomorphisms in $\mathfrak{g}$.

Again let G be a connected solvable algebraic group and let $\rho : G \to GL(V)$ be a rational representation. Since the flag variety $\mathcal{F}(V)$ of complete flags in V is projective, the natural action of G on $\mathcal{F}(V)$ has a fixed point. By taking a basis in V compatible with a G-fixed flag we have $A\rho(G)A^{-1} \subseteq T_n$, for some $A \in GL(V)$, $n = dimV$.

A *Borel subgroup* of an algebraic group G is a connected solvable subgroup of G which is maximal in the partial order on closed subgroups given by inclusion of subsets.

Let B be a Borel subgroup of G and B_0 be a Borel subgroup of maximal dimension. By Borel's fixed point theorem, B has a fixed point on G/B_0, or, equivalently, there is a $g \in G$ with $gBg^{-1} \subseteq B_0$. By maximality of B, $gBg^{-1} = B_0$. Now take two maximal tori T_1 and T_2 in G. Since T_1 and T_2 are connected and solvable, there are Borel subgroups B_1 and B_2 with $T_1 \subset B_1$, $T_2 \subset B_2$. Since, $gB_1g^{-1} = B_2$ for some $g \in G$, T_1 and T_2 are conjugate. Similarly the maximal unipotent subgroups of G are all conjugate. So we have;

Theorem 1.6.4. *In an algebraic group the maximal tori (resp. Borel subgroups, maximal connected unipotent subgroups) are conjugate.*

Let B be a Borel subgroup of largest possible dimension in an algebraic group G. By Chevalley's theorem there exists a rational G-module V and a non-zero $v \in V$ such that $B = \{g \in G : g.v \in K.v\}$. Let $\mathcal{F}_0$ be the closed subvariety of the flag variety $\mathcal{F}(V)$ consisting of complete flags with the first element $K.v$. The subvariety $\mathcal{F}_0$ is B-invariant, and by Borel's fixed point theorem B has a fixed point $F \in \mathcal{F}_0$. Hence the stabilizer $G_F = B$ and the G-orbit of F is closed in $\mathcal{F}(V)$, since it is of minimal dimension. So, $G/B \simeq G.F$ is closed in the projective variety $\mathcal{F}(V)$, thus is projective too. This gives the following important theorem;

Theorem 1.6.5. *Let G be an algebraic group and B a Borel subgroup of G. Then the homogeneous space G/B is projective.*

The following theorem shows that conjugates of B cover the whole group G.

Theorem 1.6.6. *Let B be a Borel subgroup of a connected algebraic group G, then $G = \cup_{g\in G} gBg^{-1}$, $N_G(B) = B$, $Z(B) = Z(G)^0$. Further if B is nilpotent, then $B = G$. In particular G is nilpotent.*

A *parabolic* subgroup P of G is any closed subgroup of G such that G/P is a projective variety. Let B be a Borel subgroup of G. It acts on G/P with a fixed point, say $BgP = gP$.

This implies that $g^{-1}Bg \subseteq P$, i.e., P contains a Borel subgroup. Conversely, suppose P contains a Borel subgroup B. Then the map $G/B \to G/P$ is surjective and, G/B is complete. This shows, the variety G/P is complete and quasi-projective. So, G/P is projective.

Let P be a parabolic subgroup of G. Then P contains a Borel subgroup B of G. Let $x \in N_G(P)$. Then both B and xBx^{-1} are Borel subgroups of P^0, so they are conjugate by an element $y \in P^0$ and hence $yx \in N_G(B) = B$. Thus $x \in P^0$, i.e., $P^0 = P = N_G(P)$. So the parabolic subgroups are self-normalizing, connected. Further, if P, Q are two conjugate parabolic subgroups of G containing a Borel subgroup B, then $P = Q$.

1.7 Root Systems and Semi-simple Theory

An abstract root system in a Euclidean space (a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric bilinear form $(\cdot, \cdot)$) V, is a subset Φ of V that satisfies the following axioms:

(R1): Φ is finite, Φ spans V and $0 \notin \Phi$.

(R2): If $\alpha \in \Phi$, then there exists a reflection s_α relative to α such that $s_\alpha(\Phi) \subset \Phi$. (A reflection relative to α is a linear transformation sending α to $-\alpha$ that restricts to the identity map on a subspace of co-dimension one).

(R3): If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) - \beta$ is an integer multiple of α.

A root system is *reduced* if it has the property that if $\alpha \in \Phi$, then $\pm\alpha$ are the only multiples of α which belong to Φ. The rank of Φ is defined to be $dim(V)$. The abstract Weyl group $W(\Phi)$ is the subgroup of $GL(V)$ generated by the set $\{s_\alpha : \alpha \in \Phi\}$. Note that $W(\Phi)$ is finite, since it permutes the finite set Φ.

Example: Let G be a connected reductive group. Let T be a torus in G and let $\Phi = \Phi(G, T)$. Let $\mathbb{Z}\Phi$ be the subgroup of $X(T)$ generated by Φ and let $V = \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. Then the set Φ is a subset of the vector space V and is a root system. If T is a maximal torus in G, then $\Phi = \Phi(G, T)$ is a root system in $V = \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$, and it is reduced. The rank of Φ is equal to the semi-simple rank of G, and the abstract Weyl group $W(\Phi)$ is isomorphic to $W = W(G, T)$.

A base of Φ is a subset $\Delta = \{\alpha_1, \cdots, \alpha_l\}$, such that Δ is a basis of V and each $\alpha \in \Phi$ is uniquely expressed in the form $\alpha = \sum_{i=1}^{l} c_i\alpha_i$, where the c_i's are all integers, no two of which have different signs. The elements of Δ are called simple roots and $Card(\Delta)$ is the *rank* of Φ. The set of positive roots Φ^+ is the set of $\alpha \in \Phi$ such that the coefficients of the simple roots in the expression for α as a linear combination of simple roots, are all nonnegative. Similarly, Φ^- consists of those $\alpha \in \Phi$ such that the coefficients are all non-positive. Clearly Φ is the disjoint union of Φ^+ and Φ^-. Given $\alpha \in \Phi$, there exists a base Δ containing α. Given a base Δ, the set $\{s_\alpha : \alpha \in \Delta\}$ generates $W = W(\Phi)$. The reflections s_α, $\alpha \in \Delta$ are called simple reflections. For simplicity we set $s_i = s_{\alpha_i},\ 1 \leq i \leq l$. The length function on W relative to $s_1, s_2, \cdots, s_l$

is given by

$$l(w) = min\{k : w = s_{i_1}s_{i_2}\cdots s_{i_k},\ 1 \leq i_1, \cdots, i_k \leq l\}.$$

If $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ with $k = l(w)$, this is called a *reduced expression* for w. There is a unique element w_0 of largest length in W, called the *longest element* of W. The element w_0 has the property that $w_0(\alpha) < 0$ for all $\alpha > 0$, i.e., $w_0(\Phi^+) = \Phi^-$. There is a partial order in W, called the *Bruhat order*, with $w' \leq w$ if there exists a sequence $\{\alpha_{i_i}, \cdots, \alpha_{i_k}\}$ of simple reflections such that $w'\alpha_{i_i}\cdots\alpha_{i_k} = w$.

If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) = \beta - (2(\beta,\alpha)/(\alpha,\alpha))\alpha$. A *Weyl chamber* in V is a connected component in the complement of the union of the hyperplanes orthogonal to the roots. The set of Weyl chambers in V and the set of bases of Φ correspond in a natural way, and W permutes each of them simply transitively.

If $\alpha \in \Phi$, define $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. The set $\Phi^\vee$ of elements $\alpha^\vee$ (called co-roots) forms a root system in V, called the dual of Φ. The Weyl group $W(\Phi^\vee)$ is isomorphic to $W(\Phi)$, via the map $s_\alpha \mapsto s_\alpha^\vee$.

A root system Φ is said to be irreducible if Φ cannot be expressed as the union of two mutually orthogonal proper subsets. In general, Φ can be partitioned uniquely into a union of irreducible root systems in subspaces of V.

Let Φ be a root system in an Euclidean space V with Weyl group W. Let

$$\Lambda = \{\lambda \in V : \langle\lambda,\alpha\rangle := 2(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{Z},\ \alpha \in \Phi\}.$$

Then Λ is a lattice (abelian subgroup generated by a basis of V) called weight lattice and elements of Λ are called weights. Note that Λ contains Φ. Let Λ_r be the lattice generated by Φ, called root lattice. Fix a basis Δ of Φ. An element $\lambda \in \Lambda$ is called dominant if $\langle\lambda,\alpha\rangle \geq 0$, $\forall\alpha \in \Delta$ and strongly dominant if $\langle\lambda,\alpha\rangle > 0$, $\forall\alpha \in \Delta$. We denote by Λ^+ the set of dominant weights. Each weight is conjugate under W to one and only one dominant weight. If λ is dominant, then $\sigma(\lambda) \leq \lambda$, for all $\sigma \in W$. Moreover for $\lambda \in \Lambda^+$ the number of dominant weights $\mu < \lambda$ is finite.

Let $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$, then the vectors $2\alpha_i/(\alpha_i,\alpha_i)$ also form a basis of V. Let $\varpi_1, \varpi_2, \cdots, \varpi_l$ be the dual basis, i.e., $2(\varpi_i,\alpha_j)/(\alpha_i,\alpha_i) = \delta_{ij}$. Note that ϖ_i's are dominant weights called fundamental dominant weights. Every element $\lambda \in V$ can be written as $\lambda = \sum m_i\varpi_i$, where $m_i = \langle\lambda,\alpha_i\rangle$. Therefore, $\Lambda = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_l$ and $\Lambda^+ = \mathbb{Z}_{\geq 0}\varpi_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\varpi_l$. Since Λ and Λ_r are of same rank, the group Λ/Λ_r is a finite group, called the *fundamental group* of Φ. The set of fundamental weights and the associated fundamental group for each type of simple Lie algebras are listed in appendix-B.

1.7.1 Classification of Root Systems

Let (V,Φ) be a root system and let $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ be a base of Φ. The Cartan matrix $A = (a_{i,j})_{1\leq i,j\leq l}$ is the matrix with $a_{i,j} = \langle\alpha_i, \alpha_j^\vee\rangle$, where $\langle\alpha,\beta^\vee\rangle := 2(\beta,\alpha)/(\alpha,\alpha)$. Since all the bases are conjugate under the action of W, the Cartan matrix is an invariant of the root system (up to simultaneous permutation of rows/columns). Here are some basic properties about this matrix:

(C1) $a_{i,i} = 2$.

(C2) For $i \neq j$, $a_{i,j} \in \{0, -1, -2, -3\}$.

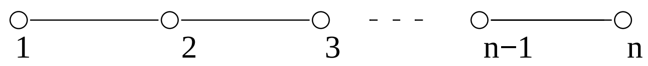
(C3) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

We can completely recover the form $\langle \cdot, \cdot \rangle$ on V up to a scalar multiple from the Cartan matrix. We can also recover Φ since the Cartan matrix contains enough information to compute the reflection s_{α_i} for each $i = 1, \cdots, l$ and $\Phi = W.\Delta$. So an irreducible root system is completely determined up to isomorphism by its Cartan matrix.

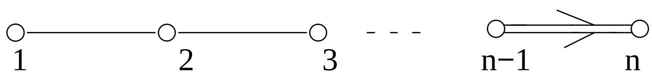
A convenient shorthand for Cartan matrices is given by the Dynkin diagram. This is a graph with vertices labelled by $\alpha_1, \cdots, \alpha_l$. There are $a_{i,j}.a_{j,i}$ edges joining vertices α_i and α_j, with an arrow pointing towards α_i if $(\alpha_i, \alpha_i) < (\alpha_j, \alpha_j)$ (equivalently, $a_{i,j} = -1, a_{j,i} = -2, -3$). Clearly the Cartan matrix, hence the root system can be recovered from the Dynkin diagram. Now we have a classification theorem for root systems.

Theorem 1.7.1. *If Φ is an irreducible root system of rank l, then its Dynkin diagram is one of the following:*

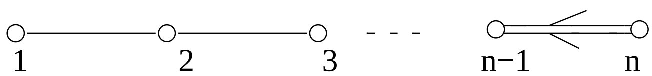
An ($n \geqslant 1$) :



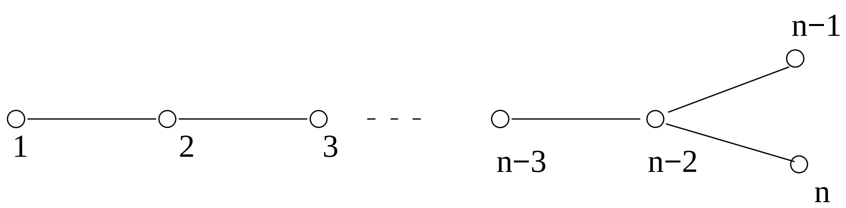
Bn($n \geqslant 3$) :



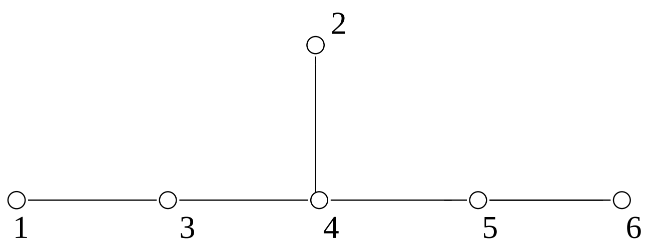
Cn ($n \geqslant 3$) :



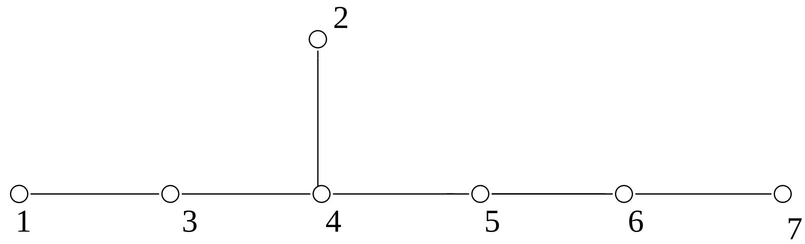
Dn ($n \geqslant 4$):



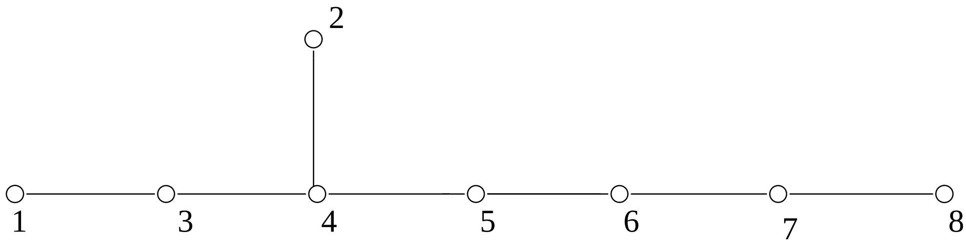
E_6 :



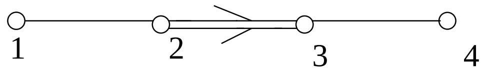
E_7 :



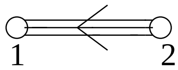
E_8 :



F_4 :



G_2 :



1.7.2 Classification of Semi-simple Lie Algebras and Algebraic Groups

First we start with a semi-simple Lie algebra and build a root system out of it, and vice versa. Let us begin with a finite dimensional semi-simple Lie algebra $\mathfrak{g}$ over an algebraically closed field K. Then $\mathfrak{g}$ possesses a non-degenerate invariant symmetric bilinear form $\kappa(X,Y) = Tr(adXadY)$ called the Cartan Killing form, where invariant here means $\kappa([X,Y],Z) = \kappa(X,[Y,Z])$. Note that if $\mathfrak{g}$ is simple, there is a unique such form upto a non-zero scalar.

A *maximal toral subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ (also called the *Cartan subalgebra*) is a maximal abelian subalgebra, all of whose elements are semi-simple. It turns out that in a semi-simple Lie algebra, maximal toral subalgebras are non-zero, and they are all conjugate under automorphisms of $\mathfrak{g}$. Now fix a maximal toral sub-algebra $\mathfrak{h}$. Firstly, the restriction of the invariant form κ on $\mathfrak{g}$ to $\mathfrak{h}$ is still non-degenerate. So we can define a map

$$\mathfrak{h}^* \to \mathfrak{h}, \alpha \mapsto t_\alpha,$$

where $t_\alpha \in \mathfrak{h}$ is the unique element satisfying $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. Now we can even lift the non-degenerate form on $\mathfrak{h}$ to $\mathfrak{h}^*$, by defining $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$.

For $\alpha \in \mathfrak{h}^*$, define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [h,X] = \alpha(h)X \text{ for every } h \in \mathfrak{h}\}.$$

Clearly, $\mathfrak{g} = \mathfrak{h} \oplus_\alpha \mathfrak{g}_\alpha$. Set $\Phi = \{0 \neq \alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\}$. Then we have the Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

The $\mathfrak{g}_\alpha$ are one dimensional and Φ satisfies all the properties of a root system. Let V be the real vector subspace of $\mathfrak{h}^*$ spanned by Φ. The restriction of the form on $\mathfrak{h}^*$ to V turns out to be real valued, and makes V into a Euclidean space.

Now start with a reduced root system Φ and choose a basis $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ of Φ. For each $i \in \{1, 2 \cdots, l\}$ we associate three symbols x_i, y_i, h_i and let $\bar{\mathfrak{g}}$ be the free Lie algebra with generators x_i, y_i, h_i $(i \in \{1, 2 \cdots, l\})$. Consider the ideal I of $\bar{\mathfrak{g}}$ generated by $[h_i h_j],\ [x_i y_i] - h_i,\ [x_i y_j]\ (i \neq j),\ [h_i x_j] - \langle \alpha_j, \alpha_i \rangle x_j,\ [h_i y_j] + \langle \alpha_j, \alpha_i \rangle y_j,\ (ad(x_i))^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j)(i \neq j)$ and $(ad(y_i))^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j)(i \neq j)$ and define $\mathfrak{g} = \bar{\mathfrak{g}}/I$. It turns out that the Lie algebra $\mathfrak{g}$ is semi-simple and has root system isomorphic to the given Φ. The above relations in $\mathfrak{g}$ among the generators are called Chevalley-Serre generators and relations. Now we have a map from the category of semi-simple Lie algebras to the set of Dynkin Diagrams (root systems) and vice versa. So we have;

Theorem 1.7.2. *The map from semi-simple Lie algebras to Dynkin diagrams gives a bijection between isomorphism classes of semi-simple Lie algebras and Dynkin diagrams. The decomposition of a semi-simple Lie algebra as a direct sum of simple Lie algebras corresponds to the decomposition of the Dynkin diagram into connected components.*

Assume that the ground field is of characteristic 0. Then a connected algebraic group G is semi-simple if and only if $\mathfrak{g}$ is semi-simple. In that case, $Ad\ G = G/Z(G)$. Note that for semi-simple G, $Z(G)$ is finite. So theorem (1.7.2) almost classifies the semi-simple algebraic groups in characteristic 0: the isomorphism type of $G/Z(G)$ at least is classified by the isomorphism type of $\mathfrak{g}$. Since the latter are classified by Dynkin diagrams, so are the centerless semi-simple groups.

Recall that if $\mathfrak{g}$ is simple then G is simple over an arbitrary field. But SL_n in characteristic dividing n gives us an example where G is simple but $\mathfrak{g}$ is not. So theorem (1.7.2) is not true in positive characteristic.

The following theorem classifies semi-simple algebraic groups in terms of fundamental groups.

Theorem 1.7.3. *([46, pg. 196]). If G and G' are simple algebraic groups having isomorphic root systems and isomorphic fundamental groups, then G and G' are isomorphic, unless the root system is D_l ($l \geq 6$) and the fundamental group has order 2, in which case there may be two distinct isomorphic types.*

Remark: The group G is simple (or almost simple) if G contains no proper nontrivial closed connected normal subgroup, equivalently $\mathfrak{g}$ is a simple Lie algebra. Note that a simple algebraic group G may contain a proper normal subgroup. For example take SL_2 but $G/Z(G)$ is simple as an abstract group. When G is semi-simple and connected, then G is simple if and only if Φ is irreducible.

The reduced irreducible root systems are those of type $A_n, n \geq 1, B_n, n \geq 2, C_n, n \geq 3, D_n, n \geq 4, E_6, E_7, E_8, F_4$, and G_2. For each $n \geq 1$ there is one irreducible non-reduced root system, BC_n. If $n \geq 2$, the root system of $GL_n(K)$ (relative to any maximal torus) is of type A_{n-1}. The root system of $Sp_{2n}(K)$ is of type C_n, if $n \geq 3$, and of type A_1 and B_2 for $n = 1$ and 2 respectively.

1.7.3 Weights and Representations

Throughout this subsection the ground field K is an algebraic closed field of characteristic 0. A classical theorem of Herman Weyl says that all finite dimensional representations of a semi-simple algebraic group (Lie algebra) are semi-simple. So we need to consider only finite dimensional irreducible representations. Finite-dimensional representations of semi-simple Lie algebras over K have been well-studied, from various points of view: the algebraic "highest weight" theory of E. Cartan (see [11]), the compact group viewpoint of H. Weyl, the geometric viewpoint of A. Borel, A. Weil, R. Bott (see [19, 20]). We will discuss here the highest weight theory and the representations.

Universal Enveloping Algebra : Let $\mathfrak{g}$ be a Lie algebra. Any associative algebra A can be made into a Lie algebra by the operation $[xy] = xy - yx$ for $x, y \in A$. Roughly speaking, to a Lie algebra $\mathfrak{g}$ we will associate an associative algebra $U(\mathfrak{g})$ which contains $\mathfrak{g}$ and the Lie algebra operation on $\mathfrak{g}$ becomes usual bracket operation in $U(\mathfrak{g})$. An associative algebra $U(\mathfrak{g})$ with a map $i : \mathfrak{g} \to U(\mathfrak{g})$ which is a Lie algebra homomorphism is called the *universal enveloping algebra* if it satisfies the following universal property: for any associative algebra A if we have a Lie algebra map $\phi : \mathfrak{g} \to A$, then there exists an algebra homomorphism $\bar{\phi} : U(\mathfrak{g}) \to A$ such that $\phi = \bar{\phi} \circ i$.

Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$. Consider the ideal J generated by elements $[xy] - (x \otimes y - y \otimes x)$ in $T(\mathfrak{g})$ for $x, y \in \mathfrak{g}$. Define $U(\mathfrak{g}) = T(\mathfrak{g})/J$ and the map $i : \mathfrak{g} \to U(\mathfrak{g})$ by sending elements of $\mathfrak{g}$ to in the 1st component of the tensor algebra. Then $U(\mathfrak{g})$ is the required universal enveloping algebra. Note that if $\mathfrak{g}$ is abelian then $U(\mathfrak{g})$ is the symmetric algebra. The map i in the definition of $U(\mathfrak{g})$ is injective and hence $\mathfrak{g}$ can be identified with its image. A theorem of Poincare-Birkhoff-Witt (see [45, pg. 92]) says if $\mathfrak{g}$ has countable dimension with a basis $\{x_1, x_2, \cdots\}$, then $\{1, x_{i_1} x_{i_2} \cdots x_{i_m} : m \in \mathbb{Z}^+, i_1 \leq i_2 \leq \cdots \leq i_m\}$ is a basis of $U(\mathfrak{g})$. It is easy to see that any representation of $\mathfrak{g}$ is a representation of $U(\mathfrak{g})$ and vice-versa.

Let V be a $\mathfrak{g}$ module. Then the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts diagonally on V. For $\lambda \in \mathfrak{h}^*$, let $V_\lambda = \{v \in V : h.v = \lambda(h)v,\ h \in \mathfrak{h}\}$. Whenever $V_\lambda \neq 0$, we call λ a weight of V and V_λ the weight space corresponding to λ. If V is finite dimensional, then $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. Write the root space decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Then, $\mathfrak{g}_\alpha$ maps V_λ into $V_{\lambda+\alpha}$ ($\lambda \in \mathfrak{h}^*,\ \alpha \in \Phi$). It follows that, in respective of dimension, the sum V' of all weight spaces V_λ is a $\mathfrak{g}$ submodule of V.

Choose a basis $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ of Φ. A maximal vector (of weight λ) in a $\mathfrak{g}$-module V is a non-zero vector $v^+ \in V_\lambda$ such that $\mathfrak{g}_\alpha.v^+ = 0$ ($\alpha \in \Delta$). If $dim(V)$ is finite, then the *Borel subalgebra* $\mathfrak{b}(\Delta) := \mathfrak{h} \oplus_{\alpha>0} \mathfrak{g}_\alpha$ has a common eigen vector by Lie'e theorem, and this is a maximal vector in V.

In order to study finite dimensional irreducible $\mathfrak{g}$-modules, it is useful to study first the larger class of $\mathfrak{g}$-modules generated by a maximal vector. If $V = U(\mathfrak{g}).v^+$ for a maximal vector v^+ (of weight λ), we say that V is *standard cyclic* (of weight λ) and we call λ the *highest weight* of V. In this case V is the direct sum of its weight spaces and the weights are of the form $\mu = \lambda - \sum_{i=1}^{l} k_i\alpha_i$ ($k_i \in \mathbb{Z}_{\geq 0}$). This justifies the terminology highest weight for λ, since $\mu \leq \lambda$. Again V is an indecomposable $\mathfrak{g}$-module, with a unique proper maximal submodule and a corresponding unique irreducible quotient. If further, V itself was irreducible, then v^+ is the unique maximal vector in V, up to non-zero scalar multiples. It is easy to check that such a cyclic module is unique upto isomorphism if it exists.

For the existence, there are two ways to construct a cyclic $\mathfrak{g}$-module of highest weight λ for any $\lambda \in \mathfrak{h}^*$. The first way of construction is to consider the one dimensional vector space $D_\lambda = K.v^+$ and define an action of $\mathfrak{b} = \mathfrak{b}(\Delta) = \mathfrak{h} \oplus_{\alpha>0} \mathfrak{g}_\alpha$ on D_λ by $h.v^+ = \lambda(h)v^+, \quad x_\alpha.v^+ = 0$. Consider the $U(\mathfrak{g})$-module $Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} D_\lambda$. Then, $Z(\lambda)$ is a standard cyclic module of weight λ and the element $1 \otimes v^+$ is a highest weight vector of weight λ (see [45, Ch. 6]). The other way of construction is the Verma module. Consider the left ideal $I(\lambda)$ in $U(\mathfrak{g})$ generated by $\{x_\alpha,\ \alpha \in \Phi^+\}$ and $\{h_\alpha - \lambda(h_\alpha).1,\ \alpha \in \Phi\}$. Then $U(\mathfrak{g})/I(\lambda)$ is a $\mathfrak{g}$-module with highest weight λ. There is a canonical homomorphism of left $U(\mathfrak{g})$-modules $U(\mathfrak{g})/I(\lambda) \to Z(\lambda)$ sending the coset of 1 onto the maximal vector v^+. Again using PBW basis of $U(\mathfrak{g})$ it is easy to see that the above map is infact an isomorphism. The standard cyclic module $Z(\lambda)$ of weight λ has a unique maximal submodule $Y(\lambda)$ and therefore, $V(\lambda) = Z(\lambda)/Y(\lambda)$ is an irreducible standard cyclic $\mathfrak{g}$-module of weight λ.

We now discuss the following: (1) For which λ, the $V(\lambda)$ are finite dimensional. (2) Determine for such $V(\lambda)$, exactly which weights μ occur and give the formula for multiplicity of $V(\lambda)_\mu$ in $V(\lambda)$.

Suppose V is a finite dimensional irreducible $\mathfrak{g}$-module. Then V has atleast one maximal vector, of uniquely determined weight λ, and the submodule it generates must be all of V by irreducibility. Therefore, V is isomorphic to $V(\lambda)$. By considering V as an $\mathfrak{sl}_2$-module it is easy to see that $\lambda(h_i)$ are nonnegative integers, i.e, the highest weight λ is dominant. More generally, if V is any finite dimensional $\mathfrak{g}$-module and μ is a weight of V, then $\mu(h_i) = \langle \mu, \alpha_i \rangle \in \mathbb{Z}$. An element $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) \in \mathbb{Z}$ is called an integral weight and if all $\lambda(h_i)$ are nonnegative integers then it is called a dominant integral weight. As before we denote Λ^+ by the set of dominant integral weights. The converse of the above result is also true, that is if $\lambda \in \mathfrak{h}^*$ is dominant integral, then the irreducible $\mathfrak{g}$-module $V(\lambda)$ is finite dimensional. The Weyl group W permutes the set of weights occurring in V and $dim(V_\mu) = dim(V_{\sigma(\mu)})$ for $\sigma \in W$. Let us summerize;

Theorem 1.7.4. *(1) For any $\lambda \in \mathfrak{h}^*$, there exists an unique (upto isomorphism) irreducible standard cyclic module of weight λ.*

(2) If $\lambda \in \mathfrak{h}^*$ is dominant integral, then the irreducible $\mathfrak{g}$-module $V(\lambda)$ is finite dimensional.

(3) Every finite dimensional irreducible $\mathfrak{g}$-module V is isomorphic to $V(\lambda)$ for some dominant integral weight λ.

(4) The Weyl group W permutes the set of weights occurring in V and $dim(V_\mu) = dim(V_{\sigma(\mu)})$ for $\sigma \in W$.

Corollary 1.7.5. *The map $\lambda \mapsto V(\lambda)$ induces a one-one correspondence between Λ^+ and the isomorphism classes of finite dimensional irreducible $\mathfrak{g}$-modules.*

The representation theory of semi-simple Lie algebras and semi-simple algebraic groups is same. We just need to translate the Lie-algebra language to group theoretic settings.

Let G be a semi-simple algebraic group over K. Let T be a maximal torus of G and B be a Borel subgroup G containing T. Let $\rho : G \to GL(V)$ be a finite dimensional rational irreducible representation of G. Then V may be regarded as a T-module. Then the complete reducibility of T implies that V is the direct sum of weight spaces, i.e., $V = \oplus_{\chi \in X(T)} V_\chi$, where $V_\chi = \{v \in V : t.v = \chi(t)v, t \in T\}$. We say $\chi \in X(T)$ is a weight in V if $V_\chi \neq 0$.

By Lie-Kolchin theorem there is a one dimensional subspace V_1 of V stable under $\rho(B)$. The generator v of the one-dimensional subspace V_1 is called a maximal vector; equivalently, $0 \neq v$ lies in some weight space V_λ and fixed by all U_α, where U_α is the unique connected T-stable (relative to conjugation by T) subgroup of G having Lie algebra $\mathfrak{g}_\alpha$. Let V' be the G submodule of V generated by v. Then the irreducibility of V implies $V' = V$. It turns out that v is unique (upto a scalar multiple) and λ is a dominant weight with multiplicity one called the *highest weight* of V. Conversely, for any $\lambda \in X(T)$, dominant there exists an irreducible G-module $V(\lambda)$ of highest weight λ. Again any two irreducible G modules are isomorphic if and only if their highest weights are same. Then we have the following theorem (see [46, Ch. 9]);

Theorem 1.7.6. *There is a bijection between irreducible rational G modules and dominant weights.*

For a connected semi-simple group G with Lie algebra $\mathfrak{g}$ one considers the categories $Repr(G)$ of the representations of G on finite dimensional vector spaces over K and $Repr(\mathfrak{g})$, the category of the representations of $\mathfrak{g}$ on finite dimensional vector spaces over K. Any representation of G on a vector space induces a representation of $\mathfrak{g}$ on the same vector space. This defines a functor $\mathfrak{F} : Repr(G) \to Repr(\mathfrak{g})$, which is fully faithful, i.e., $Hom_G(V_1, V_2) \to Hom_{\mathfrak{g}}(V_1, V_2)$ is a bijection. Further, G is simply connected if and only if $\mathfrak{F}$ is an equivalence.

The representation theory of a semi-simple Lie algebra (algebraic group) over an algebraic closed field of positive characteristic can be found in [44].

Weyl Dimension Formula: Suppose $V(\lambda)$ is an irreducible representation of a semi-simple algebraic group G with highest weight λ. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Then the dimension of $V(\lambda)$ is given by

$$dim(V(\lambda)) = \frac{\prod_{\alpha\in\Phi^+}\langle\alpha, \lambda+\rho\rangle}{\prod_{\alpha\in\Phi^+}\langle\alpha, \rho\rangle}.$$

Weyl Character Formula: Sometimes it is convenient to write the elements of $X(T)$ multiplicatively. So, we introduce symbols e^{λ} for $\lambda \in X(T)$ subject to the rule $e^{\lambda}.e^{\mu} = e^{\lambda+\mu}$. The character of an irreducible representation $V(\lambda)$ is given by

$$ch(V(\lambda)) = \frac{\sum_{w\in W}(-1)^{l(w)}(e^{w(\lambda+\rho)})}{e^{\rho}\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})}.$$

Kostant Multiplicity Formula: Suppose μ is an element of the root lattice. Let $p(\mu)$ denote the number of ways that μ can be expressed as a linear combination of positive roots with non-negative integer co-efficients. The function p is called the *Kostant partition function*.

Suppose that $V(\lambda)$ is a finite dimensional irreducible representation of a semi-simple algebraic group G with highest weight λ. If μ is a weight of $V(\lambda)$, then the multiplicity $m_{\mu}(\lambda)$ is given by

$$m_{\mu}(\lambda) = \sum_{w\in W}(-1)^{l(w)}p(w.(\lambda+\rho)-(\mu+\rho)).$$

1.8 Reductive Group

Recall that, a linear algebraic group G is said to be reductive if its unipotent radical $R_u(G)$ is trivial. If G is connected, then $R(G)$ is a torus. The following theorem reduces the study of reductive groups to the study of semi-simple groups and tori.

Theorem 1.8.1. *Let G be a connected reductive algebraic group. Then we have $R(G) = Z(G)^0$, $G = R(G)[G, G]$, and the subgroup $[G, G]$ is semi-simple.*

Let G be a connected reductive group. Let T be a torus in G. Then $Z_G(T)$ is reductive. This fact is useful for inductive arguments. Let $\langle\Phi\rangle$ be the subgroup of $X(T)$ generated by Φ and let $V = \langle\Phi\rangle \otimes_{\mathbb{Z}} \mathbb{R}$. Then the set Φ is a subset of the vector space V and is a root system. If T is a maximal torus, then the root system is reduced. The rank of Φ is equal to the semi-simple rank of G, and the abstract Weyl group $W(\Phi)$ is isomorphic to $W = W(G, T)$.

Now let us assume that T is a maximal torus. Let $\mathfrak{t}$ be the Lie algebra of T and let $\Phi = \Phi(G, T)$. Then

(1) $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi}\mathfrak{g}_{\alpha}$ and $dim\mathfrak{g}_{\alpha} = 1$ for all $\alpha \in \Phi$.

(2) If $\alpha \in \Phi$, let $T_{\alpha} = (Ker\alpha)^0$. Then T_{α} is a torus of co-dimension one in T.

(3) If $\alpha \in \Phi$, let $Z_\alpha = Z_G(T_\alpha)$. Then Z_α is a reductive group of semi-simple rank 1, and the Lie algebra $\mathfrak{z}_\alpha$ of Z_α satisfies $\mathfrak{z}_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. The group G is generated by the subgroups Z_α, $\alpha \in \Phi^+$.

(4) The centre $Z(G)$ of G is equal to $\cap_{\alpha\in\Phi} T_\alpha$.

(5) If $\alpha \in \Phi$, there exists a unique connected T-stable (relative to conjugation by T) subgroup U_α of G having Lie algebra $\mathfrak{g}_\alpha$. Also, $U_\alpha \subset Z_\alpha$.

(6) Let $n \in N_G(T)$, and let w be the corresponding element of $W = W(G,T)$. Then $nU_\alpha n^{-1} = U_{w(\alpha)}$ for all $\alpha \in \Phi$.

(7) Let $\alpha \in \Phi$. Then there exists an isomorphism $\epsilon : G_a \to U_\alpha$ such that $t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x), t \in T, x \in G_a$.

(8) The groups $U_\alpha,\ \alpha \in \Phi$, together with T, generate the group G.

The Bruhat Decomposition: Let B be a Borel subgroup of G, and let T be a maximal torus of G contained in B. Then G is the disjoint union of the double cosets BwB, as w ranges over a set of representatives in $N_G(T)$ of the Weyl group W ($BwB = Bw'B$ if and only if $w = w'$ in W), i.e.,

$$G = \sqcup_{w\in W} BwB.$$

Remark: More generally the Bruhat decomposition holds for a group with BN-pair. A BN-pair in a group G is a datum (B,N,S) consisting of sub-groups B and N, such that $B \cap N$ is normal in N, and a set of involutions S in the quotient group $W = N/(B\cap N)$. The datum satisfies the following properties:

(1) The set $B \cup N$ generates G.

(2) The set S generates W.

(3) For any $s \in S$, and $w \in W$ we have $sBw \subset BwB \cup BswB$.

(4) For any $s \in S$ we have $sBs \not\subset B$.

The group W is called the Weyl group of the BN-pair (see [3, pg. 15]). It follows from these properties that (W,S) is in fact a Coxeter system, and moreover the third property can be refined to

$$BsBwB = \begin{cases} BswB & \text{if } \ l(sw) = l(w)+1. \\ BwB \cup BswB & \text{if } \ l(sw) = l(w)-1. \end{cases}$$

Let G be a connected reductive group. Then if B is a Borel subgroup and T is a maximal torus in B, the pair of subgroups B and $N = N_G(T)$ is a BN-pair for G, where the set S is equal to $\{s_\alpha : \alpha \in \Delta\} \subset W$.

1.8.1 Classification of Reductive Algebraic Groups

Like semi-simple algebraic groups are classified by root systems, the reductive algebraic groups are classified by an invariant called *root datum*. An *abstract root datum* is a quadruple $\Psi = (X, Y, \Phi, \Phi^\vee)$, where X and Y are free abelian groups such that there exists a bilinear mapping $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$ inducing isomorphisms $X \simeq Hom(Y, \mathbb{Z})$ and $Y \simeq Hom(X, \mathbb{Z})$, and $\Phi \subset X$ and $\Phi^\vee \subset Y$ are finite subsets, and there exists a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$. The following two axioms must be satisfied:

(RD1): $\langle \alpha, \alpha \rangle = 2$

(RD2): If $s_\alpha : X \to X$ and $s_{\alpha^\vee} : Y \to Y$ are defined by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$, and $s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$, then $s_\alpha(\Phi) \subset \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$ (for all $\alpha \in \Phi$). (see [115, pg. 124])

If $\Phi \neq \emptyset$, then Φ is a root system in $V = \langle \Phi \rangle \otimes_{\mathbb{Z}} \mathbb{R}$, where $\langle \Phi \rangle$ is the subgroup of X generated by Φ. The set $\Phi^\vee$ is the dual of the root system. The quadruple $\Psi^\vee = (Y, X, \Phi^\vee, \Phi)$ is also a root datum, called the dual of Ψ. A root datum is *reduced* if it satisfies a third axiom

(RD3): $\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi$.

Let G be a connected reductive group and let T be a maximal torus in G. Then the quadruple $\Psi(G, T) = (X, Y, \Phi, \Phi^\vee) = (X(T), Y(T), \Psi(G, T), \Psi^\vee(G, T))$ is a root datum and it is reduced.

An isomorphism of a root datum $\Psi = (X, Y, \Phi, \Phi^\vee)$ onto a root datum $\Psi' = (X', Y', \Phi', \Phi'^\vee)$ is a group isomorphism $f : X \to X'$ which induces a bijection of Φ onto Φ' and whose dual induces a bijection of $\Phi'^\vee$ onto $\Phi^\vee$. If G' is a linear algebraic group which is isomorphic to G, and T' is a maximal torus in G', then the root data $\Psi(G, T)$ and $\Psi(G', T')$ are isomorphic.

If Ψ is a reduced root datum, there exists a connected reductive group G and a maximal torus T in G such that $\Psi = \Psi(G, T)$. The pair (G, T) is unique up to isomorphism. So we have (see [115, Ch. 9, 10]);

Theorem 1.8.2. *For every root datum, there exists a corresponding reductive algebraic group. Further, any two reductive algebraic groups are isomorphic if and only if their root datums (relative to some maximal tori) are isomorphic.*

1.9 Parabolic Subgroups

Recall that a parabolic subgroup of G is a closed subgroup Q of G such that G/Q is a projective variety. Note that a subgroup Q of G is parabolic if and only if it contains a Borel subgroup.

Let Q be a parabolic subgroup containing a Borel subgroup B. Let $R(Q)$ be the radical of Q, namely, the connected component through the identity element of the intersection of all the Borel subgroups of Q. Let $R_u(Q)$ be the unipotent radical of Q and let Φ_Q^+ be the subset of Φ^+ defined by $\Phi^+ \setminus \Phi_Q^+ = \{\alpha \in \Phi^+ : U_\alpha \subset R_u(Q)\}$. Let $\Phi_Q^- = -\Phi_Q^+,\ \Phi_Q = \Phi_Q^+ \cup \Phi_Q^-$ and $\Delta_Q = \Delta \cap \Phi_Q$. Then Φ_Q is a subroot system of Φ called the root system associated to Q, with Δ_Q as a set of simple roots and Φ_Q^+ (resp. Φ_Q^-) as the set of positive (resp. negative) roots of Φ_Q relative to Δ_Q. On the other hand, given a subset J of Δ, the subgroup Q of G generated by B and $U_{-\alpha},\ \alpha \in \Phi_J^+ = \{\sum_{\beta \in J} a_\beta \beta : a_\beta \geq 0\} \cap \Phi^+$ is a parabolic subgroup of G containing B. Thus the set of parabolic subgroups containing B is in bijection with the power set of Δ. In particular for $Q = B$ (resp. G), Δ_Q is the empty set (resp. the whole set Δ). The subgroup of Q generated by T and $\{U_\alpha : \alpha \in \Phi_Q\}$ is called the *Levi subgroup* associated to Δ_Q, and is denoted by L_Q. We have that Q is the semidirect product of $R_u(Q)$ and L_Q called the *Levi decomposition* of Q. The set of maximal parabolic subgroups containing B is in one-to-one correspondence with Δ. Namely given $\alpha \in \Delta$, the parabolic subgroup Q where $\Delta_Q = \Delta \setminus \{\alpha\}$ is a maximal parabolic subgroup, and conversely. We shall denote the maximal parabolic subgroup Q, where $\Delta_Q = \Delta \setminus \{\alpha_i\}$ by P_i.

1.9.1 The Weyl Group of a Parabolic Subgroup

Given a parabolic subgroup Q, let W_Q be the subgroup of W generated by $\{s_\alpha : \alpha \in \Delta_Q\}$. W_Q is called the Weyl group of Q. Note that $W_Q \simeq N_Q(T)/T$, where $N_Q(T)$ is the normalizer of T in Q. In each coset $wW_Q \in W/W_Q$, there exists a unique element of minimal length. Let W_Q^{min} be the set of minimal length representatives of W/W_Q . We have $W_Q^{min} = \{w \in W : l(ww') = l(w) + l(w'), \text{for all } w' \in W_Q\}$. In other words, each element $w \in W$ can be written uniquely as $w = uv$ where $u \in W_Q^{min}, v \in W_Q$ and $l(w) = l(u) + l(v)$. The set W_Q^{min} can also be characterized as $W_Q^{min} = \{w \in W : w(\alpha) > 0, \text{for all } \alpha \in \Delta_Q\}$. W_Q^{min} is also denoted by W^Q. Similarly in each coset $wW_Q \in W/W_Q$, there exists a unique element of maximal length and the set W_Q^{max} of maximal length representatives of W/W_Q is equal to $\{w \in W : w(\alpha) < 0, \text{for all } \alpha \in \Delta_Q\}$. Further if w_Q is the unique element of maximal length in W_Q, then we have $W_Q^{max} = \{ww_Q : w \in W_Q^{min}\}$. If Q is the parabolic subgroup corresponding to a subset I of Δ, then W_Q (resp. W^Q) is also denoted by W_I (resp. W^I).

1.10 Schubert Varieties

Let G be a semi-simple algebraic group over an algebraically closed field K. Let T be a maximal torus of G and B be a Borel subgroup of G containing T. The projective variety G/P is called a generalised flag variety. For the left action of T on G/P, there are only finitely many fixed points $\{e_w := wW_P : w \in W/W_P\}$. For $w \in W/W_P$, the B-orbit $C_P(w) := Be_w = BwP/P$ in G/P is a locally closed subset of G/P, called the *Schubert cell*. The Zariski closure of $C_P(w)$ with the canonical reduced structure is the *Schubert variety* associated to w, and is denoted by $X_P(w)$. Thus Schubert varieties in G/P are indexed by W^P. Note that if $P = B$, then $W_P = \{id\}$, and the Schubert varieties in G/B are indexed by the elements of W. We denote the Schubert variety corresponding to $w \in W$ by $X(w)$.

Dimension of $X_P(w)$**:** If $P = B$, then for $w \in W$, the isotropy subgroup in G at the T fixed point e_w in G/B is wBw^{-1}; hence, the isotropy subgroup in the unipotent radical B_u (of B) at e_w is generated by the root subgroups $\{U_\alpha, \alpha \in \Phi^+ : U_\alpha \subset wBw^{-1}\}$, i.e., $\{U_\alpha, \alpha \in \Phi^+ : w^{-1}(\alpha) > 0\}$. Hence we get an identification

$$C_B(w) \simeq \prod_{\{\alpha \in \Phi^+ : w^{-1}(\alpha) < 0\}} U_\alpha$$

Since $|\{\alpha \in \Phi^+ : w^{-1}(\alpha) < 0\}| = l(w)$, $C_B(w)$ is isomorphic to the affine space $K^{l(w)}$. Hence we have

$$dim X_B(w) = dim C_B(w) = l(w).$$

For a general parabolic P, consider $w \in W/W_P$ and denote the unique representative for w in W_P^{min} (resp. W_P^{max}) by w_P^{min} (resp. w_P^{max}). Now under the canonical projection $\pi_P : G/B \to G/P$, $X_B(w_P^{min})$ maps birationally onto $X_P(w)$, and $X_B(w_P^{max}) = \pi_P^{-1}(X_P(w))$. Hence we obtain

$$dim X_P(w) = dim X_B(w_P^{min}) = l(w_P^{min}).$$

Note that, $G/B = X(w_0)$, w_0 being the longest element in W. The cell $C_B(w_0)$ is the unique cell of maximal dimension (= $l(w_0) = |\Phi^+|$); it is affine, open and dense in G/B, called the *big cell* of G/B. It is denoted as $\mathcal{O}$. Let $B^- = w_0 B w_0^{-1}$ be the opposite Borel subgroup to B. The B^- orbit $B^- e_{id}$ is again affine, open and dense in G/B, and is called the *opposite big cell* of G/B, and it is denoted as $\mathcal{O}^-$. For a $w \in W$, $Y_B(w) = X_B(w) \cap \mathcal{O}^-$ is called the opposite cell in $X_B(w)$.

There is a partial order on W_P, known as the Bruhat order, induced by the partial order on the set of Schubert varieties given by inclusion, namely, for $w_1, w_2 \in W_P, w_1 \geq w_2 \iff X_P(w_1) \supseteq X_P(w_2)$. Taking $Q = B$, we obtain a partial order on W.

The Bruhat decomposition of G/P and $X_P(w)$ are induced by the Bruhat decomposition of G/B. They are $G/P = \bigsqcup_{w \in W^P} Be_w P(modP)$ and $X_P(w) = \bigsqcup_{\{w \in W^P,\ e_{w'} \in X_P(w)\}} Be_{w'} P(modP)$ respectively.

Example: Let $G = SL_{n+1}$ and $P = P_{\alpha_1}$. The semi-simple part of this P is just SL_n. Thus $W_P = S_n$ and has the longest element $(w_0)_P = s_n(s_{n-1}s_n)\cdots(s_2\cdots s_n)$. Note that $w_0 = (w_0)_P(s_1\cdots s_n)$. The number of Schubert varieties in G/P is then $[W : W_P] = n+1$. These are given by the sequence $w_i = \begin{cases} id & \text{if } i = 0; \\ s_i \cdots s_1 & \text{if } 1 \leq i \leq n. \end{cases}$

1.10.1 Line Bundles on G/P

For the study of G/B, there is no loss in generality in assuming that G is simply connected; in particular, the character group $X(T)$ coincides with the weight lattice Λ. Henceforth, we shall suppose that G is simply connected. The canonical projection $\pi : G \to G/B$ is a principal B-bundle with B as both the structure group and fiber. Any $\lambda \in X(T)$ defines a character

$\lambda_B : B \to G_m$ obtained by composing the natural map $B \to T$ with $\lambda : T \to G_m$. Then we have an action of B on K, namely $b.k = \lambda_B(b)k,\ b \in B,\ k \in K$. Set $E = G \times K/\sim$, where $\sim$ is the equivalence relation defined by $(gb, b.k) \sim (g, k), g \in G, b \in B, k \in K$. Then E is the total space of a line bundle over G/B. We denote by $\mathcal{L}(\lambda)$, the line bundle associated to λ. Thus we obtain a map

$$\mathcal{L} : X(T) \to Pic(G/B),\ \lambda \mapsto \mathcal{L}(\lambda),$$

where $Pic(G/B)$ is the Picard group of G/B which is by definition, the group of isomorphism classes of line bundles on G/B. By a theorem of Chevalley [16] the above map is in fact an isomorphism of groups since G is simply connected.

On the other hand, consider the irreducible divisors $X(w_0s_i)$, $1 \leq i \leq l$ on G/B. Let $\mathcal{L}_i = \mathcal{O}_{G/B}(X(w_0s_i))$ be the line bundle defined by $X(w_0s_i)$, $1 \leq i \leq l$. The Picard group $Pic(G/B)$ is a free abelian group generated by the $\mathcal{L}_i$'s, and under the isomorphism $\mathcal{L} : X(T) \simeq Pic(G/B)$, we have $\mathcal{L}(\varpi_i) = \mathcal{L}_i$, $1 \leq i \leq l$ (see [16]). Thus for $\lambda = \sum_{i=1}^{l} \langle \lambda, \alpha_i \rangle \varpi_i$, we have $\mathcal{L}(\lambda) = \otimes_{i=1}^{l} \mathcal{L}_i^{\otimes \langle \lambda, \alpha_i \rangle}$.

For a general parabolic P, any $\lambda \in X(T)$ can not be lifted to a character of P always. To be a character of P the weight λ must be orthogonal to the positive roots of P. Therefore, λ must be an integral linear combination of the fundamental weights, $\varpi_1, \cdots, \varpi_r$ dual to the simple roots in Δ_P. We call $\varpi_1, \cdots, \varpi_r$ the fundamental weights of P and the sublattice $\Lambda_P \subset \Lambda$ they generate the weights of P.

A line bundle $\mathcal{L}$ on an algebraic variety X is very ample if there exists an immersion $i : X \hookrightarrow \mathbb{P}^n$ such that $i^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{L}$. A line bundle $\mathcal{L}$ on X is ample if $\mathcal{L}^m$ is very ample for some positive integer $m \geq 1$. A line bundle $\mathcal{L}$ on X is said to be *numerically effective*, if the degree of the restriction to any algebraic curve in X is non-negative.

The following theorem summarizes some well-known facts about line bundles on G/P (for example see [113]).

Theorem 1.10.1. *Let $X = G/P$, where G is a semi-simple algebraic group and P is a parabolic subgroup. Let $\varpi_1, \cdots, \varpi_r$ be the fundamental weights of P and let $\mathcal{L}$ be a line bundle on X defined by $\lambda = \sum_{j=1}^{r} m_j \varpi_j \in \Lambda_P$. Then*
(1) $X = X_1 \times \cdots \times X_s$, where $X_i = G_i/P_i$, G_i is a simple algebraic group and P_i is a parabolic subgroup of G_i, $i = 1, \cdots, s$.
(2) $\mathcal{L} = pr_1^*\mathcal{L}_1 \otimes \cdots \otimes pr_s^*\mathcal{L}_s$, where $\mathcal{L}_i$ is a line bundle on X_i, $i = 1, \cdots, s$.
(3) $Pic(X) \simeq \Lambda_P$. In particular, $Pic(X) \simeq \mathbb{Z}$ if P is a maximal parabolic subgroup of G.
(4) $\mathcal{L}$ is numerically effective if and only if λ is dominant.
(5) $\mathcal{L}$ is very ample if and only if λ is a regular dominant weight ($\langle \lambda, \alpha_i \rangle > 0$, for all $\alpha_i \in \Delta$).

As we have described above, let E denote the total space of the line bundle $\mathcal{L}(\lambda)$, over G/B. Let $\sigma : E \to G/B$ be the canonical map $\sigma(g, c) = gB$. Let

$$M_\lambda = \{f \in K[G] : f(gb) = \lambda(b)f(g),\ g \in G,\ b \in G\}.$$

Then M_λ can be identified with the space of sections $H^0(G/B, L(\lambda)) := \{s : G/B \to E : \sigma \circ s = id_{G/B}\}$ as follows. To $f \in M_\lambda$, we associate a section $s : G/B \to E$ by setting $s(gB) = (g, f(g))$. To see that s is well defined, consider $g' = gb,\ b \in B$. Then $(g', f(g')) = (gb, f(gb)) = (gb, \lambda(b)f(g)) = (gb, bf(g)) \sim (g, f(g))$. From this, it follows that s is well defined. Conversely, given $s \in H^0(G/B, L(\lambda))$, consider $gB \in G/B$. Let $s(gB) = (g', f(g'))$, where $g' = gb$ for some $b \in B$ (note that $g'B = gB$, since $\sigma \circ s = id_{G/B}$). Now the point $(g', f(g'))$ may also be represented by $(g, \lambda(b)^{-1}f(gb))$ (since $(g', f(g')) = (gb, f(gb)) \sim (g, \lambda(b)^{-1}f(gb))$. Thus given $g \in G$, there exists a unique representative of the form $(g, f(g))$ for $s(gB)$. This defines a function $f : G \to K$. Further, this f has the property that for $b \in B$, $f(g) = \lambda(b)^{-1}f(gb)$, i.e., $f(gb) = \lambda(b)f(g),\ b \in B,\ g \in G$. Thus we obtain an identification

$$M_\lambda = H^0(G/B, \mathcal{L}(\lambda)).$$

It can be easily verified that the above identification preserves the respective G-module structures.

1.10.2 Weyl Module

In this section we assume that the ground field is $\mathbb{C}$. Let G be a semi-simple algebraic group over $\mathbb{C}$ with root system Φ and the set of simple roots Δ. Let $\mathfrak{g} = Lie(G)$ and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let $U^+(\mathfrak{g})$ be the subalgebra of $U(\mathfrak{g})$ generated by $\{X_\alpha : \alpha \in \Delta\}$, and $U^+_{\mathbb{Z}}(\mathfrak{g})$ be the Kostant $\mathbb{Z}$-form of $U^+(\mathfrak{g})$, which is by definition the $\mathbb{Z}$-subalgebra of $U^+(\mathfrak{g})$ generated by $\{\frac{X_\alpha^n}{n!}, \alpha \in \Phi^+, n \in \mathbb{N}\}$.

Let λ be a dominant weight, and $V(\lambda)$ be the irreducible G-module over $\mathbb{C}$. Fix a highest-weight vector u_λ in V of weight λ; we have that the weight λ in $V(\lambda)$ has multiplicity one. For $w \in W$, fix a representative n_w for w in $N_G(T)$, and set $u_{w,\lambda} = n_w u_\lambda$, known as an *extremal weight vector*; it is a weight vector in $V(\lambda)$ of weight $w(\lambda)$, and is unique up to scalars. Having fixed λ, we shall denote u_λ (resp. $u_{w\lambda}$) by just u (resp. u_w). Set $V_{w,\mathbb{Z}}(\lambda) = U^+_{\mathbb{Z}}(\mathfrak{g})u_w$. For any field K, let $V_{w,\lambda} = V_{w,\mathbb{Z}}(\lambda) \otimes K$, $w \in W$. Then $V_K(\lambda) := V_{w_0,\lambda} = V_{w_0,\mathbb{Z}}(\lambda) \otimes K$ is the *Weyl module* with highest weight λ, and for $w \in W$, $V_{w,\lambda}$ is the *Demazure module* corresponding to w and λ (see [72, pg. 25]). The vectors $u_{w,\lambda}$ for $w \in W$ are also called extremal weight vectors in $V_K(\lambda)$. Then the following theorem can be found in [49].

Theorem 1.10.2. *$H^0(G/B, \mathcal{L}(\lambda)) \simeq V_K(\lambda)^*$ and $H^0(X(w), \mathcal{L}(\lambda)) \simeq V^*_{w,\lambda}$.*

Chapter 2

Invariant theory

This chapter is a brief survey of invariant theory of finite groups as well as reductive algebraic groups. Here we present many classical as well as modern results in invariant theory mostly on computational aspects. In the last section of this chapter "Geometric invariant theory" is introduced.

2.1 Introduction

Invariant theory served as one of the major motivations for the development of commutative algebra: from Hilbert's basis theorem to Noetherian rings and modules. It is primarily concerned with the study of group actions, their fixed points and their orbits. The actions are usually on algebras of various sorts, the fixed points are subalgebra, and the orbits form a variety of groups on rings and the invariants of the action, e.g. the fixed subring and related objects. The basic object to study is the ring of invariants. In this chapter we deal with only linear actions. If G is a finite group acting linearly on a vector space V over a field K, then the action may be extended to $K[V]$, the algebra of polynomial functions on V, by the formula $(gf)(v) := f(g^{-1}.v)$ for all $v \in V$ and the ring of G-invariant polynomials is $K[V]^G := \{f \in K[V] : gf = f \ \forall \ g \in G\}$. If G is a linear algebraic group acting on an affine variety X, then the same formula above defines an action on the coordinate ring $K[X]$ of X and $K[X]^G := \{f \in K[X] : gf = f \ \forall \ g \in G\}$. In this section we focus on the case when $X = V$ is a representation of V and, when we talk about algebraic group actions, the base field K is algebraic closed, unless stated otherwise. The G action on $K[V]$ preserves degree and $K[V]^G \subseteq K[V]$ inherits the grading.

The basic question in invariant theory is when is $K[V]^G$ finitely generated ? If it is finitely generated then find the generators and relations for $K[V]^G$(fundamental systems of invariants), find the degree bounds for the generators. When is $K[V]^G$ a polynomial ring ? If not then what is the distance of $K[V]^G$ from being a polynomial ring ? and, what is the distance of $K[V]^G$ from being free as module over a homogeneous system of parameters ? What is the cohomological co-dimension: $depth(K[V]^G)$?

2.2 Finite Generation

The question, whether there is always a finite set of fundamental invariants for arbitrary groups was considered to be one of the most important problems in 19'th century algebra. It was proved to be true, using explicit calculations, by P Gordan for $K = \mathbb{C}$ and $G = SL_2(\mathbb{C})$ in the 1860/70's. In 1890 David Hilbert introduced new methods in invariant theory, which still today are basic tools of modern algebra (Hilbert's basis theorem). Applying these he was able to prove finite generation for the invariants of the general linear groups $GL_n(\mathbb{C})$.

In the year 1900, on the occasion of the international congress of mathematics in Paris, Hilbert posed the general question of finite generation of invariant rings for arbitrary groups as the 14'th of the now famous "Hilbert problems". In general the answer to this question is negative: In 1958 Nagata gave a counterexample to this.

For finite groups, Hilbert's 14'th problem has been solved affirmatively: In 1916 Emmy Noether had considered the problem specifically for $K = \mathbb{C}$, where she was able to find constructive procedures to compute fundamental systems explicitly. Ten years later (1926) she proved that the invariant ring is finitely generated, if G is a finite group and K is an arbitrary field. The price one has to pay for this generality is, that the proof is not constructive and does not provide an immediate algorithm how to compute a finite set of fundamental invariants.

Theorem 2.2.1. *(Hilbert [40], 1890; Noether [88, 89], 1916, 1926). Let G be a finite group. Then the ring extension $K[V]^G \subseteq K[V]$ is finite and $K[V]^G$ is a finitely generated K-algebra.*

In the general setting, finite generation no longer holds for $K[V]^G$. There are many counter examples. This is the Nagata's famous counter example to Hilbert's 14'th problem.

Nagata's counter example [79]: Let G' be the subgroup of G_a^n equal to the set of solutions $(t_1, \cdots, t_n)$ of a system of linear equations $\sum_{j=1}^n a_{ij}x_j = 0,\ i = 1, 2, 3$. The group G' acts on the affine space $\mathbb{C}^{2n}$ by the formula $(t_1, \cdots, t_n)(x_1, y_1, \cdots, x_n, y_n) = (x_1 + t_1y_1, y_1, \cdots, x_n + t_ny_n, y_n)$. Now consider the subgroup $H = \{(h_1, \cdots, h_n) \in G_a^n : \prod_{i=1}^n h_i = 1\}$ of G_m^n. It acts on $\mathbb{C}^{2n}$ by the formula $(h_1, \cdots, h_n).(x_1, y_1, \cdots, x_n, y_n) = (h_1x_1, h_1y_1, \cdots, h_nx_n, h_ny_n)$. Both of these groups are identified naturally with subgroups of SL_{2n} and we enlarge G' by considering the group $G = G'H$. Then Nagata showed that, in an appropriate choice of a_{ij}'s and the number n the algebra of invariants $K[x_1, \cdots, x_n, y_1, \cdots, y_n]^G$ is not finitely generated. For example, taking $n = 16$ and a_{ij}'s are algebraically independent over $\mathbb{Q}$, the invariant algebra is not finitely generated (see [22, pg. 43]).

Remark: Here are some more counter examples involving the additive group G_a. Let K be an algebraically closed field of characteristic 0. Roberts [94] found a non-linear action of G_a on K^7 such that the invariant ring is not finitely generated. Recently, Daigle and Freudenburg [18] found the following counter example in dimension 5. Consider the action of G_a on K^5 defined by $g.(a, b, x, y, z) = (a, b, x + ga^2, y + g(ax + b) + \frac{1}{2}g^2a^3, z + gy + \frac{1}{2}g^2(ax + b) + \frac{1}{6}g^3a^3)$. Then $\mathbb{C}[a, b, x, y, z]^{G_a}$ is not finitely generated. However for $n \leq 3$, Zariski [125] showed that for any rational action of G_a on K^n that the invariant ring is finitely generated and if V is a representation of G_a over a field K of characteristic 0, Weitzenböck [122] proved that $K[V]^G$ is finitely generated.

Reductivity, Linear Reductivity and Geometric Reductivity: We first recall the definition of a reductive algebraic group from last chapter. A linear algebraic group G is called reductive if its unipotent radical $R_u(G)$ is trivial, i.e., the largest connected normal unipotent subgroup of G is trivial. The examples are GL_n, all semi-simple groups, tori, finite groups, etc. A linear algebraic group G is called linearly reductive if for any rational representation V and any nonzero invariant vector $v \in V$ there exists a linear G-invariant function f on V such that $f(v) \neq 0$. Equivalently every rational representation V of G is completely reducible. A linear algebraic group G is called geometrically reductive if for any rational representation V and any nonzero invariant vector $v \in V$ there exists a homogeneous G-invariant polynomial f on V such that $f(v) \neq 0$.

In characteristic 0, an algebraic group is reductive $\Longleftrightarrow$ linear reductive $\Longleftrightarrow$ geometrically reductive. For any characteristic reductivity $\Longleftrightarrow$ geometric reductivity. Clearly in any characteristic linear reductivity $\Longrightarrow$ geometric reductivity. The converse is not true, though. For example a non-trivial finite p-group in characteristic p is geometrically reductive but not linearly reductive. Nagata [80] proved that in characteristic p a linear algebraic group G is linearly reductive if and only if G^0 is a torus and $|G/G^0|$ is not divisible by p. These groups are completely classified: finite groups whose order is not divisible by p, tori, and extensions of tori by finite groups whose order is not divisible by p.

Example: Define a regular action of G_a on K^2 by $g.(x, y) = (x+g.y, y)$, $g \in G_a$, $(x, y) \in K^2$. Then $K[x, y]^G = K[y]$. If $v \in K \setminus \{0\} = (K^2)^{G_a}$, then every invariants vanishes on v. The group is therefore not geometrically reductive.

In 1964 Nagata proved the following finiteness theorem for geometrically reductive groups.

Theorem 2.2.2. *(Nagata [81]). If X is an affine G-variety and G is a geometrically reductive group, then $K[X]^G$ is finitely generated. In particular if V is a representation of G, then $K[V]^G$ is a finitely generated K-algebra.*

The converse is also true. Popov proved the following.

Theorem 2.2.3. *(Popov [90]). If $K[X]^G$ is a finitely generated algebra for every affine G-variety X, then G must be reductive (Geometrically reductive).*

2.3 Construction of Invariants

Let V be a finite dimensional representation of a finite group G over a field K. For an orbit $B \subset V^*$ define the orbit polynomial $\phi_B(X) = \prod_{b \in B}(X + b) \in K[V]^G[X]$. Note that $\phi_B(X)$ is a polynomial in X of degree $|B|$ and expanding we get $\phi_B(X) = \sum_{i+j=|B|} c_i(B).X^j$. The defining classes $c_i(B) \in K[V]^G$ are called *orbit Chern classes* of the orbit B. The *first orbit*

Chern class $c_1(B)$ is the sum of the orbit elements. If $|B| = k$ then $c_k(B)$ is the product of all the elements in the orbit B and called the *top Chern class* of the orbit. If $b \in B$ then the top Chern class of B is also referred to as the *norm* of b. The first Chern class is additive and the norm is multiplicative.

Theorem 2.3.1. *(L. Smith and R.E. Stong [108]). Let V be a representation of a finite group G over a field K. Suppose either the field K is of characteristic zero or that the order of G is less than the characteristic of K. Then $K[V]^G$ is generated by orbit Chern classes. If b is the size of the largest orbit of G acting on V^* then $K[V]^G$ is generated by classes of degree at most b.*

In [23], Dickson showed that for a 2-dimensional representation of the quaternion group Q_8 over the field $K = F_3$, $K[x, y]^{Q_8}$ is not generated by orbit Chern classes. So the assumption that the characteristic of K is zero or larger than the order of $|G|$ in the above theorem cannot be relaxed to the assumption that $|G|$ is prime to the characteristic of K.

There are many other cases where the orbit Chern classes generate the ring of invariants: if G is solvable and the characteristic of K does not divide $|G|$ (see [110]); if $G = A_n$ is the alternating group and the characteristic of K is prime to $|A_n|$ (see [111, 112]); if G is a Coxeter group containing no factor of the form E_6, E_7, E_8 (see [71]).

2.4 Hilbert Series of an Invariant Algebra

Let V be a linear representation of a group G over a field K. We would like to compute the dimension of the homogeneous component $K[V]_j^G$ consisting of invariant polynomials of degree j. These numbers may be conveniently arranged in a generating function called the Hilbert series (Poincaré Series). The Hilbert series of a graded K-algebra $R = \oplus_{d \in \mathbb{Z}_{\geq 0}} R_d$ is the formal power series defined by $H(R, t) = \sum_{d \in \mathbb{Z}_{\geq 0}} dim_K(R_d) t^d$.

When G is finite, there is a beautiful theorem about the Hilbert series of invariant ring in non modular case and in the modular case when the group action is by permutation. Assume that characteristic of K does not divide $|G|$ and let V be a finitely generated KG-module. It is known from representation theory (for example see [99, Ch. 18]) of finite groups, that there is a “Brauer lift” $\hat{V}$ to zero characteristic: in brief terms, this is an $\mathcal{O}G$-module which is free as an $\mathcal{O}$-module, where $\mathcal{O}$ is a suitable discrete valuation ring with quotient field F of characteristic zero and a maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ such that $\mathcal{O}/\mathfrak{m} \simeq K$ and $\hat{V} \otimes_{\mathcal{O}} K \simeq V$. In particular for each $g \in G$ there is a “lift” of $\det(g|_V) \in K$ to $\det(g|_{\hat{V}}) \in \mathcal{O}$.

Theorem 2.4.1. *(Molien [77]). Let G be a finite group acting on a finite dimensional vector space V over a field K of characteristics does not divide $|G|$, then one has*

$$H(K[V]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{det(1 - g|_{\hat{V}^*} t)}.$$

If characteristic of K is 0, then $det(1 - g|_{\hat{V}^*} t)$ can be taken as $det(1 - g|_{V^*} t)$.
If K is arbitrary and $G \leq S_n$, then $H(K[K^n]^G, t) = H(\mathbb{C}[\mathbb{C}^n]^G, t)$.

The Molien's formula can be generalized to arbitrary reductive groups as well. Assume that $K = \mathbb{C}$. Since G always contains a maximal compact subgroup C, we can choose a Haar measure $d\mu$ on C and normalize it such that $\int_C d\mu = 1$. Let V be a finite dimensional rational representation of G. Then the Hilbert series is given by (see [22, pg. 180])

$$H(\mathbb{C}[V]^G, t) = \int_C \frac{d\mu}{det(1 - g|_V t)}$$

For a r-dimensional torus T and V a rational representation of T, there is a simple formula for the Hilbert series of the invariant ring. Choose generators $z_1, z_2, \cdots, z_r$ of $X(T) \simeq \mathbb{Z}^r$. The action of T on V^* is diagonal and given by the matrix $diag(m_1(z), m_2(z), \cdots, m_n(z))$, where $m_1, m_2, \cdots, m_n$ are Laurent monomials in $z_1, z_2, \cdots, z_r$. Then the Hilbert series of $K[V]^T$ is given by (see [22, pg. 184]) the co-efficient of $z_1^0 z_2^0 \cdots z_r^0 = 1$ in

$$\frac{1}{(1 - m_1(z)t)(1 - m_2(z)t) \cdots (1 - m_n(z)t)}$$

Let V be a rational representation of a connected reductive group G over an algebraic closed field K of characteristic 0. There is a formula by Weyl for computing Hilbert series of the invariant ring $K[V]^G$. Fix a maximal torus T of G and a Borel subgroup B of G containing T. Let W be the Weyl group of G and $\lambda_1, \lambda_2, \cdots, \lambda_r$ be the fundamental weights. Let us denote the character of T associated to a weight $\lambda \in X(T)$ by z^λ. Then every character of T is a Laurent monomial in z^{λ_i}'s. The action of T on V^* is diagonal and given by the matrix $diag(m_1(z), m_2(z), \cdots, m_n(z))$, where $m_1(z), m_2(z), \cdots, m_n(z)$ are Laurent monomials in z^{λ_i}'s. Then the Hilbert series (see [22, pg. 186]) is the coefficient of z^ρ in

$$\frac{\sum_{w \in W} (-1)^{l(w)} z^{w(\rho)}}{(1 - m_1(z)t)(1 - m_2(z)t) \cdots (1 - m_n(z)t)}$$

where ρ is the half sum of positive roots.

2.5 UFD and Polynomial Algebra

In this section we investigate when is a ring of invariants a UFD and when is it a polynomial algebra. We begin with an example where the invariant ring fails to be both.

Example: Let K be a field of characteristic not equal 2. Consider the action of $\mathbb{Z}_2$ on $K[x, y]$ by algebra automorphisms: $x \mapsto -x$ and $y \mapsto -y$. Then the ring of invariants $K[x, y]^{\mathbb{Z}_2} = K[x^2, y^2, xy]/((xy)^2 - x^2y^2)$, which is not a UFD (and not a polynomial ring).

As a positive result we have the following theorem;

Theorem 2.5.1. *([86, pg. 27]). Let $\rho : G \to GL(V)$ be a representation of a finite group G over a field K. If there is no non-trivial homomorphisms $G \to K^*$, then $K[V]^G$ is a UFD.*

The above theorem covers several interesting cases, such as: G is a simple non-abelian group, G is perfect (equals to its own commutator), $char(K) = p$ and G is a finite p-group,

$|K| = q$ and G is a group such that $gcd(|G|, q-1) = 1$, $K = \mathbb{Q}$ and G is a group of odd order, $K = F_2$ and G is finite or $K = F_3$ and G has odd order.

Let V be a vector space of dimension n over a field K. A linear transformation $\sigma : V \to V$ is called a pseudo-reflection, if it fixes pointwise a subspace of co-dimension one. Let $G \leq GL(V)$ be a finite group acting linearly on V. We say that G is a pseudo-reflection group if G is generated by pseudo-reflections.

The list above is the most special cases of the following characterization of representations with UFD ring of invariants due to Nakajima.

Theorem 2.5.2. *(Nakajima [83]). Let $\rho : G \to GL(V)$ be a representation of a finite group G over a field K. Denote by H the subgroup of G generated by all the pseudo-reflections in G. Then the following are equivalent:*

1. $K[V]^G$ is a UFD.

2. G/H has no non-trivial one-dimensional representation.

3. $K[V]^G_\lambda := \{f \in K[V] : g.f = \lambda(g).f \ \forall g \in G\}$ is a free $K[V]^G$-module for every one-dimensional representation $\lambda : G \to K^*$.

G.C. Shephard and J.A. Todd showed in 1954 that the ring of invariants of a finite group over a field of characteristic zero is a polynomial ring if and only if the action of G on V is generated by pseudo-reflections. Their proof of the "if" half of this theorem was by classification. In 1955, C. Chevalley gave an elegant algebraic proof that, if the action of a finite group G on V is generated by reflections (still assuming $char(K) = 0$), then $K[V]^G$ is a polynomial ring. Soon thereafter, J.P. Serre observed that Chevalley's proof was also valid for actions generated by pseudo-reflections and that it therefore provided an algebraic proof of the "if" half of the Shephard-Todd theorem.

Theorem 2.5.3. *(Chevalley-Serre-Shephard-Todd, [15, 98, 106]). Let V be a finite dimensional representation of a finite group G over a field K. Assume that the order of G is relatively prime to the characteristic of K. Then G is generated by pseudo-reflections if and only if $K[V]^G$ is a polynomial algebra. In this case one also has $|G| = \prod_{i=1}^n deg(f_i)$, where $f_1, f_2, \cdots, f_n$ are the generators of $K[V]^G$.*

Recently (2007) Broer gave an extension of the above theorem to positive characteristic.

Theorem 2.5.4. *(Broer [5]). Suppose V is an irreducible representation of a finite group G over a field K, then $K[V]^G$ is a polynomial algebra if and only if G is generated by pseudo-reflections and there is a surjective $K[V]^G$ linear map $\pi : K[V] \to K[V]^G$.*

The following criterion due to Kemper is valid over any field.

Theorem 2.5.5. *(Kemper [60]). Let V be a n-dimensional representation of a finite group G over a field K. Then $K[V]^G$ is a polynomial ring if and only if there is a homogeneous* *system of parameter $h_1, \cdots, h_n$ of $K[V]^G$ with $|G| = \prod_{i=1}^n deg(h_i)$. In particular $K[V]^G = K[h_1, h_2, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n deg(h_i)$.*

If $G \leq GL(V)$ is generated by pseudo-reflections, but the characteristic of K divides the order of G, then the ring of invariants $K[V]^G$ need not be a polynomial algebra. For example ([86, pg. 193]) for the Weyl group W of type F_4, the ring of invariants is not a polynomial algebra at the prime 3 (3 divides $|W| = 1152$).

The following theorem shows that for a pseudo-reflection group, the ring of invariants is always a UFD.

Theorem 2.5.6. *(A. Dress [28]). Let $\rho : G \to GL(n, K)$ be a representation of a finite group over a field K. If $\rho(G)$ is generated by pseudo-reflections then $K[V]^G$ is a unique factorization domain.*

An element $\eta \in GL(V)$ is called a transvection if $Ker(I - \eta) \subset V$ has co-dimension 1 and $Im(I - \eta) \subset Ker(I - \eta)$. The hyperplane $H_\eta = Ker(I - \eta)$ is called the hyperplane of η. We say that $G \subset GL(V)$ is a transvection group if G is generated by transvections (see [109, pg. 242]).

Theorem 2.5.7. *(Nakajima [82]). Let V be a finite dimensional irreducible representation of a finite group G over a field K. Assume that $char(K) \neq 2$, $dim(V) \geq 3$ and that G is generated by transvections. Then $K[V]^G$ is a polynomial algebra if and only if G is conjugate on $GL(V)$ to $SL(n, F_q)$, where F_q is a finite field of characteristic p.*

Definition: Assuming that the ground field K is of positive characteristic p, a p-subgroup G of $GL(V)$ is called a Nakajima p-group (on V) if there is a basis $B = \{z_1, z_2, \cdots, z_n\}$ of V such that under this basis G is upper triangular and such that $G = G_1 \cdots G_n$, where each subgroup $G_i := \{g \in G | gz_j = z_j \text{ for } j \neq i\}$. The basis B is called a Nakajima basis (see [10, pg. 141]).

Obviously Nakajima p-groups are modular reflection p-groups. But a reflection p-group may not be a Nakajima p-group (see [124, pg. 4]).

The following important result concerns Nakajima p-groups.

Theorem 2.5.8. *(Nakajima [85]). Let V be a finite-dimensional vector space over the prime field F_p, and P a p-subgroup of $GL(V)$. Then P is a Nakajima p-group if and only if $F_p[V]^P$ is a polynomial ring.*

The above result does not extend to other fields of characteristic p as is shown by an example due to Stong (see [86, pg. 164]). However when K is a field of characteristic p, in 1983 Nakajima proved that if P is a Nakajima group then $K[V]^P = K[N(z_1), \cdots, N(z_n)]$, where $N(z_i) = \prod_{g \in P} g.z_i$, the orbit product of z_i. Recently (2008) Y. Wu in his Ph.D thesis proved the converse.

Theorem 2.5.9. *(Nakajima [85], Wu [124]). P is a Nakajima-group with respect to B if and only if $K[V]^P$ is a polynomial ring. In that case $K[V]^P$ is generated by $N(z_i)$'s.*

The following theorem shows that for an abelian pseudo-reflection group G the property that $K[V]^G$ be a polynomial algebra is controlled by a p-Sylow subgroup in characteristic p.

Theorem 2.5.10. *(Nakajima [84]). Let $\rho : G \to GL(V)$ be a pseudo-reflection representation of an abelian group G over a field K of characteristic p. Then, $K[V]^{Syl_p(G)}$ is a polynomial algebra if and only if $K[V]^G$ is a polynomial algebra, where $Syl_p(G)$ denote a Sylow p-subgroup of G.*

Kemper and Malle classified all irreducible representations V of G such that $K[V]^G$ a polynomial ring.

Theorem 2.5.11. *(Kemper-Malle [61]). Let V be a finite dimensional irreducible representation of a finite group G over a field K. Then $K[V]^G$ is a polynomial ring if and only if G is a reflection group and $K[V]^{G_W}$ is a polynomial ring for every non-zero linear subspace W of V, where $G_W = \{g \in G : g.w = w$ for every $w \in W\}$.*

Very recently (2010), Fleischmann and Woodcock [36] showed that if $|G| = p^n$ and $char(K) = p$ then there is a non-linear faithful action of G on $K[x_1, \cdots, x_n]$ such that $K[x_1, \cdots, x_n]^G$ is a polynomial ring.

A method for classifying those reductive groups having a polynomial ring of invariants was suggested in 1976 by V. Kac, V. Popov and E. Vinberg [50]. Using this method G. Schwarz [97] in 1978, and independently O. Adamovich and E. Golovina [1] in 1979, enumerated the representations of complex connected simple algebraic groups having a polynomial ring of invariants. In 1989, P. Littelmann [74] listed (up to the equivalence relation called castling) the irreducible representations of connected semi-simple complex groups whose rings of invariants are polynomial rings. While the result of Shephard and Todd gives simple conditions for $K[V]$ to be a polynomial algebra if G is finite, there is no known similar simple characterization for a semi-simple algebraic group G.

When $G = T$ a torus, in 1994, D. Wehlau [121] gave two constructive criteria each of which determines those representations of T for which the ring of invariants is a polynomial ring.

2.6 Cohen-Macaulay Property

In a graded Noetherian K-algebra $R = \oplus_{d \in \mathbb{Z}_{\geq 0}} R_d$, a sequence $f_1, f_2, \cdots, f_r$ of homogeneous elements of R is called a *homogeneous system of parameters* if $f_1, f_2, \cdots, f_r$ are algebraically independent and R is finitely generated as a module over the subring $A = K[f_1, f_2, \cdots, f_r]$, i.e., if there exist $g_1, g_2, \cdots, g_m \in R$ such that $R = Ag_1 + Ag_2 + \cdots + Ag_m$. The Noether normalization lemma asserts that R always has a homogeneous system of parameters. The number r is called the *Krull dimension* of R. The ring R is *Cohen-Macaulay* if R is a free $K[h_1, h_2, \cdots, h_r]$-module for some homogeneous system of parameters, $h_1, h_2, \cdots, h_r$. It can be shown that if R is Cohen-Macaulay then R is a free $K[h_1, h_2, \cdots, h_r]$-module for every homogeneous system of parameters, $h_1, h_2, \cdots, h_r$. If $R = K[V]^G$ then the elements of a homogeneous system of parameters, $f_1, f_2, \cdots, f_r$ are called *primary invariants* and the module generators $g_1, g_2, \cdots, g_m \in R$ are called *secondary invariants*. Together, primary and secondary invariants generate $K[V]^G$. Of course there are many choices for primary invariants and secondary invariants.

Theorem 2.6.1. *(Hochster and Eagon [41]). Let V be a finite dimensional representation of a finite group G over a field K. If $char(K) \nmid |G|$, then $K[V]^G$ is Cohen-Macaulay.*

The above theorem is false in the modular case. In 1980, Ellingsrud-Skjelbred [30] showed that $F_p[V_{reg}]^{C_p}$ is not CM for all $p \geq 5$.

The following theorem is due to Campbell, Hughes, Kemper, Shank, Wehlau [9].

Theorem 2.6.2. *Let $char(K) = p > 0$ and let N be a normal subgroup of G with cyclic factor group $G/N \simeq C_p$. Then for every representation V of G, the ring $K[mV]^G$ is not Cohen-Macaulay for $m \geq 3$.*

The following two theorems due to Kemper give simple criteria for an invariant ring to be Cohen-Macaulay in positive characteristic.

Theorem 2.6.3. *(Kemper [63]). Let $char(K) = p$ and let V be a finite dimensional representation of a p group G over K such that $K[V]^G$ is Cohen-Macaulay, then G is generated by bi-reflections, i.e., by $g \in G$ which fix a subspace $U \subset V$ of co-dimension 2.*

Theorem 2.6.4. *([22, pg. 98]). Let V be a finite dimensional faithful representation of a finite group G over a field K. Let $f_1, \cdots, f_n \in K[V]^G$ be primary invariants of degrees $d_1, \cdots, d_n$, and let $g_1, \cdots, g_m$ be a minimal system of secondary invariants. Then $K[V]^G$ is Cohen-Macaulay if and only if $m|G| = \prod_{i=1}^n d_i$.*

The following theorem shows that the Cohen-Macaulay property of $K[V]^G$ in characteristic p is controlled by a p-Sylow subgroup of G.

Theorem 2.6.5. *([109, pg. 257]). Let V be a representation of a finite group G over a field K of characteristic p. If $K[V]^{Syl_p(G)}$ is Cohen-Macaulay then so is $K[V]^G$, where $Syl_p(G)$ denote a Sylow p-subgroup of G.*

The following theorem can be found in [109, pg. 260].

Theorem 2.6.6. *Let $\rho : G \to GL(V)$ be a finite dimensional representation of a finite group G over F_p. If $\rho(Syl_p(G)) = Uni(m, F_p)$ for some integer $m \leq n$ then $F_p[V]^G$ is Cohen-Macaulay, where $Uni(m, F_p)$ denote the group of unipotent matrices.*

Kemper has classified all groups, whose modular regular representation has a Cohen-Macaulay ring of invariants.

Theorem 2.6.7. *(Kemper [64]) For a finite group G with group algebra FG, $F[FG]^G$ is Cohen-Macaulay if and only if $char(F) \nmid |G|$ or $G \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_3\}$.*

The following theorem is due to Hochster and Roberts for a linearly reductive group over a field K that is not necessarily algebraically closed.

Theorem 2.6.8. *(Hochster and Roberts [42]). If V is a representation of a linearly reductive group over a field K, then $K[V]^G$ is Cohen-Macaulay.*

If X is a smooth affine G-variety, Hochster and Huneke [43] proved that the ring of invariants is Cohen-Macaulay. Kemper proved a partial converse of the above theorem.

Theorem 2.6.9. *(Kemper [65]). Suppose that G is a reductive group and that for every rational representation V of G the invariant ring $K[V]^G$ is Cohen-Macaulay. Then G is linearly reductive.*

Since in characteristic 0 the notion of reductivity and linear reductivity are same, for a reductive group the ring of invariant is Cohen-Macaulay. But, the above theorem shows that the classical groups in positive characteristic have rational representation with non-Cohen-Macaulay invariant rings. The following example shows that for a rational representation of a non-reductive group, the ring of invariants can be Cohen-Macaulay.

Example: Let V be a rational representation of G_a over $\mathbb{C}$. The action of G_a on V can be extended to an action of $SL_2(\mathbb{C})$, and since G_a is a maximal unipotent subgroup of $SL_2(\mathbb{C})$, we have an isomorphism $\mathbb{C}[V]^{G_a(\mathbb{C})} \simeq \mathbb{C}[V \oplus \mathbb{C}^2]^{SL_2(\mathbb{C})}$ ([114, pg. 69]), where $SL_2(\mathbb{C})$ acts naturally on $\mathbb{C}^2$. Now since $SL_2(\mathbb{C})$ is linearly reductive, $\mathbb{C}[V]^{G_a}$ is Cohen-Macaulay, although G_a is not reductive.

2.7 Depth of an Invariant Ring

In the last section we saw that the ring of invariants may not be always Cohen-Macaulay. If it is not Cohen-Macaulay, then the question is, how close is a $K[V]^G$ to being a Cohen-Macaulay ring ? That is measured by $depth(K[V]^G)$:= maximal regular sequence in $K[V]^G$. From Auslander-Buchsbaum-formula it follows that $K[V]^G$ is Cohen-Macaulay if and only if $depth(K[V]^G) = dim(K[V]^G)$.

The first formula for the depth of a cyclic p group was given by Ellingsrud and Skjelbred [30].

Theorem 2.7.1. *Let V be a finite dimensional representation of a cyclic p group (with $p = char(K)$). Then $depth(K[V]^G) = min\{dim_K(V^G) + 2, dim_K(V)\}$.*

The following theorem is due to Kemper.

Theorem 2.7.2. *(Kemper [66]). Let V be a finite dimensional representation of a finite group G over a field K. Suppose that $|G|$ is divisible by $p = char(K)$ but not by p^2. Let r be the smallest positive number such that $H^r(G, K[V]) \neq 0$. Then*

$$depth(K[V]^G) = min\{dim_K(V^P) + r + 1, dim_K(V)\},$$

where $P \leq G$ is a Sylow p-subgroup.

A group G is called p-nilpotent if it has a normal p-complement, i.e. a normal subgroup N of order co-prime to p, such that G/N is a p-group (which then has to be isomorphic to a Sylow p-group of G) (see page 68 of [10]). Then recently (2005) Fleischmann, Kemper and Shank [34] proved the following:

Theorem 2.7.3. *If G is p-nilpotent with cyclic Sylow p-subgroup $P \leq G$, then $depth(K[V]^G) = min\{dim_K(V^P) + 2, dim_K(V)\}$.*

Let K be an algebraically closed field. For a finitely generated graded commutative K-algebra R, let $cmdef(R) := dim(R) - depth(R)$ denote the Cohen-Macaulay defect of R. Then the following result can be found in [68].

Theorem 2.7.4. *Let G be a linear algebraic group over K that is reductive but not linearly reductive. Then there exists a faithful rational representation V of G such that $cmdef(K[V^{\oplus k}]^G) \geq k - 2$ for all $k \in \mathbb{N}$.*

2.8 Noether's Degree Bound

Let V be a finite dimensional representation of a group G. We define the *Noether Number* of V, $\beta(K[V]^G) := min\{d : K[V]^G$ is generated by homogeneous invariants of degree $\leq d\}$ and the *Noether Number* of G, $\beta(G) := max\{\beta(K[V]^G) : V$ a representation of $G\}$.

In the second proof (1926) of the finite generation of the invariant ring of a finite group Noether also proved that the invariant ring can be generated by homogeneous invariants of degree at most the order of G, provided the characteristic of K is 0 or bigger than $|G|$. For the case that $p = char(K)$ is smaller than $|G|$ but $p \nmid |G|$, the question whether Noether's bound holds was open for quite a while. Recently (2001) Fleischmann and Fogarty independently found proofs for the general non-modular case.

Theorem 2.8.1. *(Noether [89], Fleischmann [33], Fogarty [37]). For a finite group G, if $char(K) \nmid |G|$, then $\beta(G) \leq |G|$.*

Remark: Noether bound is sharp in the sense that no better bound can be given in terms of group order. Consider a finite cyclic group G of order n and let $char(K) = 0$ and containing a primitive n^{th} root of unity, ξ. Let g be a generator of G. There are exactly n inequivalent irreducible representations $W_0, W_1, \cdots, W_{n-1}$ of G, each of which is one dimensional. The action of G on W_i is given by $gv = \xi^i v$ for all $v \in W_i$. It is easy to see that $K[W_i]^G = F[x^{n/gcd(i,n)}]$ and thus if i is relatively prime to n then $\beta(K[W_i]^G) = n$. Therefore we see that Noethers bound is sharp for cyclic groups. However Schmid [96] proved that if G is non-cyclic and $char(K) = 0$, then $\beta(G) < |G|$. Domokos and Hegedüs [26] examined Schmid's proof by induction and sharpened the bound. Sezer extended the results to non-modular case.

Theorem 2.8.2. *(Sezer [103]). Let G be a finite, non-cyclic group and $char(K) \nmid |G|$. Then*

$$\beta(G) \leq \begin{cases} \frac{3}{4}|G| & \text{if } |G| \text{ is even} \\ \frac{5}{8}|G| & \text{if } |G| \text{ is odd} \end{cases}$$

The following theorem says, in order to compute Noether number of a group in characteristic 0, it is sufficient to consider only the regular representation of that group.

Theorem 2.8.3. *(Schmid [96]). Let G be a finite group and let K be a field of characteristic 0. Let V_{reg} denote the regular representation of G. Then $\beta(G) = \beta(K[V_{reg}]^G)$.*

In the modular case the Noether bound does not hold and the behaviour of $\beta(K[V]^G)$ is in sharp contrast to the non-modular situation.

Example: Let $G = C_2$ acts on $F_2[x_1, \cdots, x_k, y_1, \cdots, y_k]$ as algebra automorphism by $g : x_i \rightarrow y_i, \ \forall\ i = 1, \cdots, k$. For $k \geq 3$, the element $f := x_1 \cdots x_k + y_1 \cdots y_k$ is indecomposable in $F_2[x_1, \cdots, x_k, y_1, \cdots, y_k]^G$, i.e., can not be written as a polynomial in lower degree invariants. So the Noether bound does not hold in the modular case.

Theorem 2.8.4. *(Richman [93], (1996)). Let K be a field of characteristic p, G be a finite group whose order is divisible by p and let V be a faithful representation of G. Then*

$$\beta(K[V^{\oplus m}]^G) \geq \frac{m(p-1)}{p^{|G|-1} - 1}$$

The above theorem shows that there is a positive number α depending only on $|G|$ and p, such that every set of K-algebra generators of $K[V^{\oplus m}]^G$ contain a generator of degree $\geq \alpha m$. So, $\beta(G)$ may be infinite when G is a finite modular group. Indeed the next theorem due to Bryant and Kemper shows this is always the case.

Theorem 2.8.5. *(Bryant-Kemper [7]). Let G be any linear algebraic group. If $\beta(G)$ is finite then G is a finite group with $|G|$ is invertible in K.*

However, if we content ourselves with finding a so-called separating subalgebra of invariants rather than the entire ring of invariants then, the invariants of degree at most $|G|$ will always suffice (see [22, pg. 54]).

Derksen and Kemper (see [22, pg. 117]) proved the following bound for any modular representation of a finite group.

Theorem 2.8.6. *Let V be an n-dimensional modular representation of a finite group G. Then $\beta(K[V]^G) \leq n(|G|-1) + (|G|^{(2^{n-1})n+1)}(n^{2^{n-1}+1})$.*

Campbell, Geramita, Hughes, Shank and Wehlau [8] showed that if $K[V]^G$ is Gorenstein then, $\beta(K[V]^G)) \leq max\{|G|, dim(V)(|G|-1)\}$. Then Broer [4] showed that for any representation V of a finite group G if $K[V]^G$ is Cohen-Macaulay then $\beta(K[V]^G)) \leq max\{|G|, dim(V)(|G|-1)\}$. Then many people conjectured that the hypothesis of Cohen-Macaulayness is not required. Recently (2009) Symonds proved the conjecture.

Theorem 2.8.7. *(Symonds [117]). For any finite dimensional representation V of a finite group G we have $\beta(K[V]^G) \leq max\{|G|, dim(V)(|G|-1)\}$.*

Kemper has conjectured that if V is a representation of a finite group G such that $K[V]^G$ is Cohen-Macaulay then $\beta(K[V]^G] \leq |G|$.

The $2p-1$ Conjecture: It has been conjectured that if V_{reg} is the regular representation of C_p over F_p then $\beta(V_{reg}) = 2p-3$. Then recently (2006) Fleischmann, Sezer, Shank, Woodcock [35] proved the conjecture with a more general result.

Theorem 2.8.8. *For any representation V of C_p we have $\beta(F_p[V]^{C_p}) = (p-1)dim(V^{C_p}) + p-2$.*

Kemper [62] found the following lower bound for a permutation representation. Let V be a faithful modular permutation representation of the finite group G over a field K of characteristic p and suppose G contains an element of order p^k for some $k \in \mathbb{N}$. Then $\beta(K[mV]^G) \geq m(p^k-1)$. About the same time Fleischmann obtained the following exact result.

Theorem 2.8.9. *(Fleischmann [32]). Let $G = S_n$ be the symmetric group on $n = p^k$ letters acting naturally by permuting a basis of the n dimensional representation V over the field F_q of order $q = p^r$. Then $\beta(K[mV]^{S_n}) = max\{n, m(n-1)\}$.*

For a graded K-algebra $R = \oplus_{d \in \mathbb{Z}_{\geq 0}} R_d$, define the constant $\gamma(R)$ as the smallest integer d such that there exist homogeneous $f_1, f_2, \cdots, f_r \in R$ with $deg(f_i) \leq d$ for all i and R is finite over $K[f_1, f_2, \cdots, f_r]$.

For a connected semi-simple algebraic group Popov gave an explicit upper bound for $\beta(K[V]^G)$ in terms of $\gamma(K[V]^G)$.

Theorem 2.8.10. *(Popov [91, 92]). Suppose V is an almost faithful representation (0-dimensional kernel) of a connected semi-simple group G defined over a field K of characteristic zero. Then $\beta(K[V]^G) \leq dim(V)lcm\{1, 2, \cdots, \gamma(K[V]^G)\}$.*

In 1989, Knop gave the following improvement of the above theorem.

Theorem 2.8.11. *(Knop [67]). Let V be a representation of a connected semi-simple algebraic group G defined over a field K of characteristic 0. Suppose $a_1, a_2, \cdots, a_r$ is a homogeneous system of parameters for $K[V]^G$. Then $\beta(K[V]^G) \leq max\{a_1+a_2+\cdots+a_r-r, a_1, a_2, \cdots, a_r\}$.*

In 1999, Derksen gave a better upper bound for $\beta(K[V]^G)$ in terms of $\gamma(K[V]^G)$ for a reductive group.

Theorem 2.8.12. *(Derksen [21]). Let G be a reductive group defined over an algebraically closed field K of characteristic zero. Then $\beta(K[V]^G) \leq max\{2, \frac{3}{8}r(\gamma(K[V]^G))^2\}$, where r is the Krull dimension of $K[V]^G$.*

In the case of $G = T$ a torus Wehlau gave a better upper bound for $\beta(K[V]^G)$. Let V be an n-dimensional faithful representation of an r-dimensional torus with weights $\varpi_1, \varpi_2, \cdots, \varpi_n$. The character group $X(T)$ of T is isomorphic to $\mathbb{Z}^r$ and has a natural embedding into $X(T) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^r$. We have the usual volume form $d\nu$ on $\mathbb{R}^r$ which is independent of the chosen basis of $X(T)$.

Theorem 2.8.13. *(Wehlau [120]). In the situation above we have $\beta(K[V]^T) \leq max(n - r - 1, 1)r!vol(C_V)$, where C_V is the convex hull of $\varpi_1, \varpi_2, \cdots, \varpi_n$ in $\mathbb{R}^r$.*

2.9 Vector Invariants

Let V be an n dimensional representation of G over a field K. For $m \in \mathbb{N}$, we denote by mV the nm dimensional representation $mV := \underbrace{V \oplus V \oplus \cdots \oplus V}_{m\ copies}$ on which G acts diagonally via $g.(v_1, v_2, \cdots, v_m) = (gv_1, gv_2, \cdots, gv_m)$. Invariants lying in $K[mV]^G$ are called vector invariants of V. The first fundamental theorem of invariant theory of G refers to a description of a minimal system of homogeneous generators of $K[mV]^G$ and the second fundamental theorem of invariant theory describes the relations among these generators. In this section we talk about only the first fundamental theorem for Weyl groups and classical groups. If $m = 1$ then a generating set for the algebra $K[mV]^G$ of invariants in one vector variable is called a system of basic invariants. The classical procedure, known as polarization constructs invariants of mV from invariants of V as follows. Let $f \in \mathbb{K}[V]^G$ be a homogeneous polynomial of degree d. For $v_1, v_2, \cdots, v_m \in V$ and $t_1, t_2, \cdots, t_m$ are indeterminates, we consider the function $f(\sum_i t_i v_i)$. Then

$$f(\sum_i t_i v_i) = \bigoplus_{\alpha \in (\mathbb{Z}^+)^m, |\alpha| = d} f_\alpha(v_1, \cdots, v_m) t^\alpha, \tag{2.1}$$

where the $f_\alpha \in \mathbb{K}[mV]^G$ are multihomogeneous of the indicated degree α. Here for $\alpha = (a_1, a_2, \cdots, a_m) \in (\mathbb{Z}^+)^m$, we have $t^\alpha = t^{a_1} \ldots t^{a_m}$ and $|\alpha| = a_1 + \ldots + a_m$. We call the polynomials f_α, the *polarizations* of f.

Polarizations of a polynomial can also be defined in terms of some linear differential operators called the polarization operators. Choosing a basis for V and writing $v_i = (x_{i1}, \cdots, x_{in})$ we define

$$D_{ij} = \sum_{k=1}^{n} x_{ik} \frac{\partial}{\partial x_{jk}}.$$

The operators D_{ij}'s are called polarization operators. They commute with the action of G on $\mathbb{K}[mV]$ and applying successively operators D_{ij} $(i > j)$ to $f \in \mathbb{K}[V]^G$ we obtain precisely (up to a constant) the polarizations of f in any number of variables.

The following theorem is due to H. Weyl (see [123, Ch. II]).

Theorem 2.9.1. *Suppose V is a representation of a finite group G over a field K of characteristic zero. If $m \in \mathbb{N}$ is an integer with $m > n$ then a complete fundamental system of invariants for $K[mV]^G$ is obtained by polarizing a complete fundamental system of invariants for $K[nV]^G$.*

Let V be an n-dimensional vector space over $K = \mathbb{C}$ and let G be a finite subgroup of $GL(V)$ generated by reflections. If $m = 1$ then the algebra $K[V]^G$ of invariants in one vector variable is very nice. As discussed earlier a celebrated theorem of Chevalley implies $K[V]^G$ is a polynomial ring. Let W be a Weyl group and V be the natural representation of W. A set of basic invariants for each type of Weyl groups is listed in appendix-B.

The first fundamental theorem for $W = S_n$ is proved by Weyl long ago. Then Wallach [119] and Hunziker [48] independently proved the same for Weyl group of type $B_n = C_n$ and G_2.

Theorem 2.9.2. *Let W be a Weyl group of type $A_n, B_n = C_n$ or G_2, then the polarization of a set of basic invariants generate $K[mV]^W$.*

However for Weyl group of type D_n Wallach observed that the polarizations of a set of basic invariants do not generate the algebra $K[mV]^W$ for $m \geq 2$. Explicitly, Wallach [119] used bidegree considerations to show that the invariant

$$f := \sum_{i=1}^{n} x_1 x_2 \cdots \hat{x}_i \cdots x_n y_i^3$$

can not be expressed in terms of polarizations of a set of basic invariants (Here $x_i = x_{1i}$ and $y_i = x_{2i}$). Then Wallach introduced the notion of a generalized polarization operator as follows. Assume that we have chosen a orthonormal basis for V. For $k = 1, 2, \cdots, n$ define

$$D_{ij}^k := \sum_{l=1}^{n} \frac{\partial f_k}{\partial x_{il}}(x_{i1}, \cdots, x_{in}) \frac{\partial}{\partial x_{jl}}$$

where $f_1, f_2, \cdots, f_n$ are basic invariants. The operators D_{ij}^k's are called generalized polarization operators. For type D_n the polynomials

$$f_k = \sum_{i=1}^{n} x_i^{2k} \;\; 1 \leq k \leq n-1, \;\; f_n = x_1 x_2 \cdots x_n$$

can be taken as a system of basic invariants. In [119], Wallach proved that the generalized polarizations of $f_1, \cdots, f_n$ generate $K[mV]^W$. Then Hunziker [48] sharpened his result in the following way: For odd $r \geq 1$ define

$$P_r = \sum_{i=1}^{n} y_i^r \frac{\partial}{\partial x_i}$$

where $x_i = x_{1i}$ and $y_i = x_{2i}$. Then;

Theorem 2.9.3. *The algebra $K[2V]^W$ is generated by the polarizations of the basic invariants and the generalized polarizations*

$$P_{r_1} \cdots P_{r_k}(f_n) \ (r_i \geq 1 \textit{ odd } \sum_{i=1}^{k} r_i \leq n-k)$$

of the basic invariant f_n.

Theorem 2.9.4. *For $m \geq 2$, the algebra $K[mV]^W$ is generated by the polarizations of $K[2V]^W$.*

For the other types of Weyl group the problem of finding a generating set for $K[mV]^W$ is still open. Recently Domokos [27] found a generating set for a class of pseudo-reflection groups.

Let V be a finite dimensional K-vector space. Consider the representation of $GL(V)$ on the vector space $W := V^{\oplus p} \oplus V^{*\oplus q}$ consisting of p copies of V and q copies of its dual space V^*, given by

$$g(v_1, \cdots, v_p, \phi_1, \cdots, \phi_q) := (gv_1, \cdots, gv_p, g\phi_1, \cdots, g\phi_q)$$

where $g\phi_i$ is defined by $(g\phi_i)(v) := \phi_i(g^{-1}v)$. The elements of V are classically called vectors, those of the dual space V^* covectors. For every pair $(i,j),\ i = 1, \cdots, p,\ j = 1, \cdots, q$, we define the bilinear function $(i|j)$ on $W := V^{\oplus p} \oplus V^{*\oplus q}$ by

$$(i|j) : (v_1, \cdots, v_p, \phi_1, \cdots, \phi_q) \mapsto (v_i|\phi_j) := \phi_j(v_i).$$

These functions are usually called contractions. They are clearly invariant:

$$(i|j)(g(v,\phi)) = (g\phi_j)(gv_i) = \phi_j(g^{-1}gv_i) = (i|j)(v,\phi).$$

Now the first fundamental theorem (shortly FFT) states that these functions generate the ring of invariants.

Theorem 2.9.5. *(FFT for $GL(V)$, [70, Th. 2.1]).*
The ring of invariants for the action of $GL(V)$ on $V^{\oplus p} \oplus V^{*\oplus q}$ is generated by the invariants $(i|j)$:

$$K[V^{\oplus p} \oplus V^{*\oplus q}]^{GL(V)} = K[(i|j) : i = 1, \cdots, p, j = 1, \cdots, q].$$

Let $n := dimV$. Fix a basis $\{v_1, \cdots, v_n\}$ of V and choose in V^* the dual basis $\{\phi_1, \cdots, \phi_n\}$. Then the determinant $det(v_1, \cdots, v_n)$ is defined for every n-tuple of vectors $v_i \in V = K^n$ as the determinant of the $n \times n$ matrix consisting of the column vectors $v_1, \cdots, v_n$. This allows to define, for every sequence $1 \leq i_1 < i_2 < \cdots < i_n \leq p$, an $SL(V)$-invariant function

$$[i_1, \cdots, i_n] : V^p \oplus V^{*q} \to K,\ (v,\phi) \mapsto det(v_{i_1}, \cdots, v_{i_n}).$$

Similarly the determinants

$$[j_1, \cdots, j_n]_* : V^p \oplus V^{*q} \to K,\ (v,\phi) \mapsto det(\phi_{j_1}, \cdots, \phi_{j_n}).$$

are $SL(V)$ invariants.

Theorem 2.9.6. *(FFT for $SL(V)$, [70, Th. 8.4]).*
The ring of invariants for the action of $SL(V)$ on $V^{\oplus p} \oplus V^{*\oplus q}$ is generated by the scalar products $\langle j|i\rangle$ and the determinants $[i_1, \cdots, i_n]$ and $[j_1, \cdots, j_n]_*$.

For the action of O_n and SO_n on $V^{\oplus p} = (K^n)^{\oplus p}$, clearly the functions $(i|j)$ defined by $(i|j)(v_1, \cdots, v_p) := (v_i|v_j)$ are invariants.

Theorem 2.9.7. *(FFT for O_n and SO_n, [70, Th. 10.2]).*
(a) The invariant algebra $K[V^{\oplus p}]^{O_n}$ is generated by the invariants $(i|j),\ 1 \leq i \leq j \leq p$.
(b) The invariant algebra $K[V^{\oplus p}]^{SO_n}$ is generated by the invariants $(i|j),\ 1 \leq i \leq j \leq p$ together with the determinants $[i_1, \cdots, i_n], 1 \leq i_1 < i_2 < \cdots < i_n \leq p$.

For every pair $1 \leq i, j \leq p$ the following functions on V^p are invariant by Sp_{2m}:

$$\langle i|j\rangle(v_1, \cdots, v_p) := \langle v_i|v_j\rangle.$$

Theorem 2.9.8. *(FFT for Sp_{2m}, [70, Th. 10.3]).*
The algebra $K[V^{\oplus p}]^{Sp_{2m}}$ is generated by the invariants $\langle i|j\rangle,\ 1 \leq i \leq j \leq p$.

2.10 Geometric Invariant Theory

Though there are several good introductory books on this subject, in this section we will try to explain the most important concepts in Geometric Invariant theory. Through out this section we assume G is an affine algebraic group over an algebraic closed field K.

2.10.1 Group Actions on Algebraic Varieties

The main purpose of Geometric Invariant Theory is as follows. Given a variety (or scheme) X and a group G, acting on X, one wants to construct a quotient of this action. In the category of sets one just takes X/G to be the set of orbits. In the category of varieties (or schemes) it is far more difficult. In general there will be no quotient which is an orbit space. This is easy to see. Suppose that X/G has the structure of a variety, such that the map $\pi : X \to X/G$ is a morphism. Then in particular each orbit has to be closed, because π is continuous. But this need not always be the case as the following example shows.

Example: The action of $GL_n(K)$ on K^n has two orbits: $\{0\}$ and $K^n \setminus \{0\}$. The orbit $K^n \setminus \{0\}$ is not closed.

This trivial example is typical: the fact is almost never all orbits of the actions considered are closed. However, we will see that "in most cases" there exists an open set $U \subset X$ such that U/G has the structure of algebraic variety and $U \to U/G$ is a morphism (so in particular U is a union of closed orbits).

The following definition gives us the minimal requirements for what we should call a quotient of an algebraic group action.

Definition: Let $\sigma : G \times X \to X$ be an algebraic group action. A categorical quotient of X by G is a variety Y with a morphism $\Phi : X \to Y$ such that

1. the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \arrow[d, "\Phi"] \\ X \arrow[r, "\Phi"] & Y \end{tikzcd}$$

commutes, i.e., Φ is constant on orbits, and

2. if $\Psi : X \to Z$ is any morphism that is constant on orbits, then there exists a unique morphism $\eta : Y \to Z$ with $\Psi = \eta \circ \Phi$.

Note that if a categorical quotient exists, it is unique and has good functorial properties, but not necessarily good geometric ones.

Example: G_m acts on K^2 by $\lambda.(x, y) = (\lambda x, \lambda^{-1} y)$. The orbits are (1) for each $\alpha \in K^*$, the conic $\{(x, y) : xy = \alpha\}$, (2) the punctured x-axis $\{(x, 0) : x \in K^*\}$, (3) the punctured y-axis $\{(0, y) : y \in K^*\}$, (4) the origin $\{(0, 0)\}$. In order to get a separated quotient, one has to combine the last three orbits listed and indeed one then gets a categorical quotient isomorphic to K, the quotient morphism being given by $(x, y) \to xy$. One cannot obtain a separated orbit space (that is, as a variety), even if one deletes the orbit $\{(0, 0)\}$, which has lower dimension than the others, since both punctured axes are limits of the other orbits as $\alpha \to 0$.

Example: G_m acts on K^n $(n \geq 2)$ by $\lambda.(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n)$. The origin lies in the closure of every orbit, so any morphism which is constant on orbits is constant. Thus there is no orbit space but there is a categorical quotient consisting of a single point.

Example: G_m acts on $K^n \setminus \{(0, 0)\}$ by the same formula. This time the projective space P^{n-1} is a categorical quotient and also an orbit space.

In order to construct categorical quotients in general we first restrict our attention to a special case. Suppose X is an affine variety on which G acts rationally and, let $K[X]$ denote the algebra of morphisms $X \to K$. Then we have an rational action of G on $K[X]$. Now, we can ask for a candidate for a categorical quotient Y. Suppose it exists and is affine, write $Y = Spec(B)$. The definition of categorical quotient tells us that such a morphism factors through Y if and only if it is constant on orbits. Algebraically this means that $B = K[X]^G$.

So, if Y is to be affine, $K[X]^G$ has to be finitely generated. In general, given a rational action of an algebraic group G on a finitely generated K-algebra R, the subalgebra of invariants R^G is not finitely generated as we have already discussed in §-(2.2). This is the famous counterexample of Nagata against Hilbert's fourteenth problem. However, we have seen already that by Nagata's theorem (Th. 2.2.2) when G is geometrically reductive $K[X]^G$ is finitely generated. We recall the notion of linear reductivity and geometrically reductivity from §-(2.2).

A linear algebraic group G is called linearly reductive (resp. geometrically reductive) if for any rational representation V and any nonzero invariant vector $v \in V$ there exists a homogeneous G-invariant polynomial f on V with $deg(f) = 1$ (resp. $deg(f) \geq 1$) such that $f(v) \neq 0$. In characteristic 0 an algebraic group is reductive $\Longleftrightarrow$ linear reductive $\Longleftrightarrow$ geometrically reductive. For any characteristic reductivity $\Longleftrightarrow$ geometric reductivity and also clearly linear reductivity $\Longrightarrow$ geometric reductivity. The converse is not true though in positive characteristic, as discussed in §-(2.2).

The aim of the rest of this sub-section is to give a theorem about the categorical quotient of an affine variety for the action of a reductive group. Before we give this theorem, we give some more definitions. There are more properties one would like to have when one has constructed a categorical quotient.

Definition: Let $\sigma : G \times X \to X$ be an algebraic group action. A good quotient of X by G is a variety Y and a morphism $\Phi : X \to Y$ such that

(1) Y (together with Φ) is a categorical quotient,

(2) for any subset $U \subset Y$, the inverse image $\Phi^{-1}(U)$ is open if and only if U is open,

(3) for any open subset $U \subset Y$, the homomorphism $\Phi^* : K[U] \to K[\Phi^{-1}(U)]$ is an isomorphism onto $K[\Phi^{-1}(U)]^G$, and

(4) Φ is surjective.

If we are in the situation of the definition above, it follows that

(1) if W is a closed G-invariant subset of X, then $\Phi(W)$ is closed in Y, and

(2) if W_1 and W_2 are closed G-invariant subsets of X with $W_1 \cap W_2 = \emptyset$, then $\Phi(W_1) \cap \Phi(W_2) = \emptyset$.

In fact one can rephrase the definition above using these properties of closed invariant subsets.

Definition: Let $\sigma : G \times X \to X$ be an algebraic group action. A geometric quotient of X by G is a variety Y and a morphism $\Phi : X \to Y$ such that

(1) Y (together with Φ) is a good quotient, and

(2) the image of the map $\Psi : G \times X \to X \times X$ given by $(g, x) \mapsto (\sigma(g, x), x)$ is $X \times_Y X$.

A geometric quotient is the best we can hope for, because it is even an orbit space for the action of G on X.

Recall we were looking for a categorical quotient for the action of an algebraic group on an affine variety. The following theorem says that, if the group is reductive, our candidate for an affine categorical quotient is in fact a very good candidate.

Theorem 2.10.1. *Let G be a reductive group acting on an affine variety X. Then $Y = Spec(K[X]^G)$, together with the map $\phi : X \to Y$ is a good quotient of X by G.*

Usually we will work in this thesis with projective varieties, and not with affine varieties. In the next paragraph we will look more closely at reductive group actions on projective varieties.

2.10.2 G.I.T. Quotients

Until now we required our actions on affine varieties to be linear, i.e. the groups act via a rational representation. The analogue of this, when we consider actions on quasi-projective varieties, is the concept of linearization with reference to a line bundle.

Definition: Let $\sigma : G \times X \to X$ be an algebraic group action on a quasi-projective variety X, and let $\mathcal{L}$ be a line bundle on X, with projection map: $\pi : \mathcal{L} \to X$. A linearization of this action is an action $\beta : G \times \mathcal{L} \to \mathcal{L}$ such that

1. the diagram

$$\begin{tikzcd} G \times \mathcal{L} \arrow[r, "\beta"] \arrow[d, "id\times\pi"] & \mathcal{L} \arrow[d, "\pi"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd}$$

commutes, and

2. for all $x \in X$ and all $g \in G$, the map $\mathcal{L}_x \to \mathcal{L}_{\sigma(g,x)}$ given by $y \mapsto \beta(g, y)$ is linear.

We call a G-linearized line bundle over X a pair of a line bundle $\mathcal{L}$ and its linearization β. A morphism of G-linearized line bundles is a G-equivariant morphism of line bundles. Thus we can speak of isomorphism classes of G-linearized line bundles on X and one can show (see [25, Ch. 7]) that the set of isomorphism classes of G-linearized line bundles on X has an abelian group structure. We denote this group by $Pic^G(X)$, and we have a natural homomorphism

$$\theta : Pic^G(X) \to Pic(X)$$

which is forgetting the linearization. This homomorphism is not necessarily surjective.

Definition: Let X be a quasi-projective variety with an action of a reductive algebraic group G. Let $\mathcal{L}$ be a G-linearized line bundle on X. Let $x \in X$.

1. x is called semi-stable with respect to $\mathcal{L}$ if there exists $m \geq 0$ and $s \in H^0(X, L^{\otimes m})^G$ such that $X_s = \{y \in X | s(y) \neq 0\}$ is affine and contains x,

2. x is called stable with respect to $\mathcal{L}$ if there exists $m \geq 0$ and $s \in H^0(X, L^{\otimes m})^G$ such that $X_s = \{y \in X | s(y) \neq 0\}$ is affine and contains x, G_x is finite and all orbits of G in X_s are closed, and

3. x is called unstable with respect to $\mathcal{L}$ if x is not semi- stable.

Here are some notations we use. $X^{ss}(\mathcal{L})$: locus of semi-stable points, $X^s(\mathcal{L})$: locus of stable points, $X^{us}(\mathcal{L})$: locus of unstable points, $X^{sss}(\mathcal{L}) := X^{ss}(\mathcal{L}) \setminus X^s(\mathcal{L})$. Often we will omit the line bundle in question and write X^s, etc.. Elements of X^{sss} are called strictly semi-stable points which are not stable.

Remark: In the definition of semi-stable points above, if the line bundle $\mathcal{L}$ is ample, the set X_s is affine automatically. So in that case we only have to find an invariant section (of a power) of $\mathcal{L}$ which is non-zero on x. This is important for us because all line bundle we will consider, are ample.

As shows the following theorem, the semi-stable locus is an open subset of X over which a good quotient exists.

Theorem 2.10.2. *(Mumford) Let G be a reductive group acting on a quasi-projective variety X. Let $\mathcal{L}$ be a G-linearized line bundle on X. Then there exists a good quotient*

$$\pi : X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L})//G.$$

There exists an open set $U \subset X^{ss}(\mathcal{L})//G$ such that $X^s(\mathcal{L}) = \pi^{-1}(U)$ and the restriction of π to $X^s(\mathcal{L})$ is a geometric quotient of $X^s(\mathcal{L})$ by G. Moreover: $X^{ss}(\mathcal{L})//G$ is a quasi-projective variety.

The following corollary we will use in practice.

Corollary 2.10.3. *If in the theorem above, we further assume that X is projective, and $\mathcal{L}$ is very ample, we have*

$$X^{ss}(\mathcal{L})//G \simeq Proj(R^G)$$

where

$$R = \oplus_{k \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{L}^{\otimes k}).$$

Thus $X^{ss}(\mathcal{L})//G$ is a projective variety.

Remark: If $X \subset \mathbb{P}^n$, $\mathcal{L} := O_X(1)$ in corollary (2.10.3) and the action is by the group $G = SL_{n+1}$ and, it is linearized with respect to this $\mathcal{L}$, the unstable points are precisely the points on which all invariant functions vanish. We define:

$$\mathcal{N} := X^{us}(\mathcal{L}) = \{x \in X : s(x) = 0 \text{ for all } s \in R^G\}.$$

The idea to consider this set $\mathcal{N}$ goes back to Hilbert. It is called the *nullcone* and its elements are called *nullforms*. Nullforms can't be distinguished by invariant functions. In fact if we consider the special case that R^G is generated by generators $s_0, \cdots, s_k$ of the same degree, then the rational map $X \dashrightarrow \mathbb{P}^k$ given by

$$x \mapsto (s_0(x), \cdots, s_k(x))$$

is the quotient map (when restricted to the semi-stable locus). The nullcone is the locus where this map isn't defined.

In most cases it turns out to be very difficult to find explicit invariants. Nevertheless we have a useful tool to determine which points are (semi-)stable. This is the so called one-parameter criterion or numerical criterion for stability. The idea is as follows. Let G be a reductive algebraic group acting on a projective variety $X \subset \mathbb{P}^n$ via a homomorphism $G \to GL_{n+1}$. In other words: the action is linearized with respect to the line bundle $\mathcal{L}$ corresponding to the embedding $X \subset \mathbb{P}^n$. We can consider the induced action of G on the affine cone $\hat{X} \subset K^{n+1}$. Let $\hat{x} \in \hat{X}$ be a point whose class is $x \in X$. Then another way to say whether or not x is unstable is given by this equivalence:

$$x \in \mathcal{N} \Longleftrightarrow 0 \in \overline{G\hat{x}}.$$

We could also check this for subgroups of G. If $0 \in \overline{H\hat{x}}$ for some subgroup $H \subset G$, x is unstable, because $\overline{H\hat{x}} \subset \overline{G\hat{x}}$. In fact the numerical criterion will say it is sufficient to check this only for the one-parameter subgroups of G.

Recall from the last chapter that a one-parameter subgroup of G is a non- trivial homomorphism of algebraic groups $G_m \to G$ and we denote the set of one-parameter subgroups of G by $Y(G)$.

A one-parameter subgroup λ of G can be viewed as an action of G_m on X, or as an action on $\hat{X}$. It is a fact (see [87]) that we can choose coordinates such that the action on $\hat{X}$ is given by

$$\lambda(t)\hat{x} = (t^{a_0}x_0, \cdots, t^{a_m}x_m)$$

for certain $a_0, \cdots, a_m \in \mathbb{Z}$. Now consider the map

$$\phi_x^* : \mathbb{A}^1 \setminus \{0\} \to K^{n+1},\ t \mapsto \lambda(t)\hat{x}.$$

If this map can be extended to a map $\mathbb{A}^1 \to K^{n+1}$ by sending the origin to the origin then it is clear that 0 is in the closure of the orbit of $\hat{x}$ of the one-parameter subgroup of G, so that x is unstable. Using the diagonal form of the action we see that 0 is in this closure if and only if all a_i for which $x_i \neq 0$ are strictly positive. This observation leads to the following definition.

Definition:

$$\mu^{\mathcal{L}}(x, \lambda) := -min\{a_i : x_i \neq 0\}.$$

One can show that the function $\mu^{\mathcal{L}}$ doesn't depend on the diagonalization of the one-parameter action. We can use this function to check unstability. If a one-parameter subgroup λ of G and a point $x \in X$ satisfy $\mu^{\mathcal{L}}(x, \lambda) < 0$, then x is unstable. Now we are in a position to state the numerical criterion.

Theorem 2.10.4. *(Hilbert-Mumford) Let G be a reductive group acting on a projective variety X. Let the action be linearized with respect to an ample line bundle $\mathcal{L}$. Let $x \in X$. Then:*

$$x \in X^{ss}(\mathcal{L}) \Longleftrightarrow \mu^{\mathcal{L}}(x, \lambda) \geq 0 \text{ for all } \lambda \in Y(G)$$

$$x \in X^{s}(\mathcal{L}) \Longleftrightarrow \mu^{\mathcal{L}}(x, \lambda) > 0 \text{ for all } \lambda \in Y(G).$$

2.10.3 Linearization of the Action

We probably hoped, in the previous section, that an action of an algebraic group G on X can always be linearized with reference to some line bundle $\mathcal{L}$. This is certainly true in the example we consider in this thesis. In fact we will see in a moment that often more than one linearization is possible, with fixed line bundle $\mathcal{L}$. The quotient can change together with a change of linearization. Whenever no confusion arises as to which linearization we choose, we just write $X^{ss}(\mathcal{L})//G$ for the GIT quotient. It will happen, however, that we do consider our quotients with respect to different linearization. When this is the case, we denote by $X_\beta(\mathcal{L})$ the semi-stable locus for the linearization $\beta : G \times \mathcal{L} \to \mathcal{L}$, and by $X_\beta(\mathcal{L})//G$ it's GIT quotient. We give some theorems saying how many linearizations are possible for a given action and a line bundle.

Let $\chi(G)$ denote the group of rational characters of G. Recall that $\phi : Pic^G(X) \to Pic(X)$ is the homomorphism that forgets the linearization. The dimension of the kernel of ϕ is a measure of the amount of linearizations a given line bundle on X allows.

Theorem 2.10.5. *If $K[G \times X]^* = p_1^{-1}(K[G]^*)$ then*

$$ker(\phi) \simeq \chi(G).$$

The condition is fulfilled for example if X is just affine space or if X is connected and proper over K, because in those cases $K[X]^* = K^*$.

The following is the main theorem ([25, §. 7.2]) about the existence and the amount of linearizations.

Theorem 2.10.6. *Let G be a connected, affine algebraic group acting on a normal variety X then, we have an exact sequence of groups*

$$0 \to ker(\phi) \to Pic^G(X) \xrightarrow{\theta} Pic(X) \to Pic(G).$$

Remark: Since it can happen that there are many possible linearizations, a priori there are many different quotients for the same action. It turns out however, that in the case of line bundles giving projective geometric quotients, there are only finitely many quotients and all these are birational to one another. See an article of Dolgachev and Hu [26]. We will not focus in this thesis on all possible linearizations and their resulting quotients, but almost always take a standard linearization.

Chapter 3

Torus Quotients of Homogeneous Spaces

This chapter reports the work done in [57]. Our main aim in this chapter is to describe all the minimal Schubert varieties admitting semi-stable points for the action of a maximal torus on G/P, where G is a semi-simple simply connected algebraic group and P is a maximal parabolic subgroup of G. We also describe for any semi-simple simply connected algebraic group G and for any Borel subgroup B of G, all Coxeter elements τ for which the Schubert variety $X(\tau)$ admits a semi-stable point for the action of the maximal torus T with respect to a non-trivial line bundle on G/B. In this chapter the author also gives a C-program that describes all the minimal Schubert varieties admitting semi-stable points for the exceptional algebraic groups.

3.1 Introduction

Let G be a simply connected semi-simple algebraic group over an algebraically closed field K. Let T be a maximal torus of G and let B be a Borel subgroup of G containing T. In [52, 53], Kannan described all parabolic subgroups P of G containing B for which there exists an ample line bundle $\mathcal{L}$ on G/P such that the semi-stable points $(G/P)^{ss}_T(\mathcal{L})$ are the same as the stable points $(G/P)^{s}_T(\mathcal{L})$. In [116], Strickland gives a shorter proof of Kannan's result. In [126] and [127], Zhgun studied how the quotients vary as the line bundle varies. In [107], Skorobogatov described the automorphism group of $T\backslash\backslash(G/P)$.

Let $\mathcal{L}$ be an ample line bundle on G/P. It is an interesting question to study the minimal Schubert varieties in G/P admitting semi-stable points with respect to $\mathcal{L}$ for the action of a maximal torus T. In [56], when Q is a maximal parabolic subgroup of G and $\mathcal{L} = \mathcal{L}_{\varpi}$, where ϖ is a minuscule dominant weight, it is shown that there exists unique minimal Schubert variety $X(w)$ admitting semi-stable points with respect to $\mathcal{L}$. Note that this includes type A.

Let G be a simple algebraic group of type B, C or D and P is a maximal parabolic subgroup of G. Let $\mathcal{L}$ be an ample line bundle on G/P. In §-3.3, we describe all minimal Schubert varieties in G/P admitting semi-stable points with respect to $\mathcal{L}$.

In the case of G/Q, where G is of exceptional type (E_6, E_7, E_8, F_4 and G_2), Q is a maximal parabolic and $\mathcal{L}$ is an ample line bundle, the combinatorics of minimal elements $w \in W/W_Q$ for which $X(w)^{ss}_T(\mathcal{L}) \neq \emptyset$ is complicated. In §-3.3, we give a C-program that describes all such $w \in W/W_Q$.

Now, let G be a semi-simple simply connected algebraically group over an algebraic closed field K. Let T be a maximal torus of G and let B be a Borel subgroup of G containing T. A Schubert variety $X(w)$ in G/B contains a (rank G)-dimensional T-orbit if and only if $w \geq \tau$ for some Coxeter element τ. So, it is a natural question to ask if for every Coxeter element τ, there is a non-trivial line bundle $\mathcal{L}$ on G/B such that $X(\tau)^{ss}_T(\mathcal{L}) \neq \emptyset$. In §-3.4 we describe all such Coxeter elements τ.

3.2 Preliminary Notations and Combinatorial Lemmas

In this section we recall some notations from chapter-1 and prove some combinatorial lemmas.

Let G be a semi-simple algebraic group over an algebraically closed field K. Let T be a maximal torus of G, B a Borel subgroup of G containing T and let U be the unipotent radical of B. Let $N_G(T)$ be the normalizer of T in G. Let $W = N_G(T)/T$ be Weyl group of G with respect to T and Φ denote the set of roots with respect to T, Φ^+ positive roots with respect to B. As in chapter-1, for the enumeration of roots we refer to [3]. Let U_α denote the one dimensional T-stable subgroup of G corresponding to the root α and let $\Delta = \{\alpha_1, \cdots, \alpha_l\} \subseteq \Phi^+$ denote the set of simple roots. For a subset $I \subseteq \Delta$ denote $W^I = \{w \in W | w(\alpha) > 0,\, \alpha \in I\}$ and W_I is the subgroup of W generated by the simple reflections $s_\alpha, \alpha \in I$. Then every $w \in W$ can be uniquely expressed as $w = w^I.w_I$, with $w^I \in W^I$ and $w_I \in W_I$. Denote $\Phi(w) = \{\alpha \in \Phi^+ : w(\alpha) < 0\}$ and w_0 is the longest element of W with respect to Δ. Let $X(T)$ (resp. $Y(T)$) denote the set of characters of T (resp. one parameter subgroups of T). Let $E_1 := X(T) \otimes \mathbb{R}$, $E_2 = Y(T) \otimes \mathbb{R}$. Let $\langle ., . \rangle : E_1 \times E_2 \longrightarrow \mathbb{R}$ be the canonical non-degenerate bilinear form. Choose λ_j's in E_2 such that $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ for all i. Let $\overline{C} := \{\lambda \in E_2 | \langle \alpha, \lambda \rangle \geq 0 \;\forall\, \alpha \in \Phi^+\}$ and for all $\alpha \in \Phi$, there is a homomorphism $SL_2 \xrightarrow{\phi_\alpha} G$ (see [12, pg. 19]). We have $\check{\alpha} : G_m \longrightarrow G$ defined by $\check{\alpha}(t) = \phi_\alpha(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})$. We also have $s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$ for all $\alpha \in \Phi$ and $\chi \in E_1$. Set $s_i = s_{\alpha_i}\;\; \forall\; i = 1, 2, \cdots, l$. Let $\{\omega_i : i = 1, 2, \cdots, l\} \subset E_1$ be the fundamental weights; i.e. $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$ for all $i, j = 1, 2, \cdots, l$.

We recall the definition of the Hilbert-Mumford numerical function and definition of the semi-stable points from chapter-2. We also refer to the same chapter for notations in geometric invariant theory.

Let X be a projective variety with an action of reductive group G. Let λ be a one-parameter subgroup of G. Let $\mathcal{L}$ be a G-linearized very ample line bundle on X. Let $x \in \mathbb{P}(H^0(X, \mathcal{L})^*)$ and $\hat{x} = \sum_{i=1}^{r} v_i$, where each v_i is a weight vector of λ of weight m_i. Then we have

$$\mu^{\mathcal{L}}(x, \lambda) = -min\{m_i : i = 1, \cdots, r\}$$

A point $x \in X$ is said to be semi-stable with respect to a G-linearized line bundle $\mathcal{L}$ if there is a positive integer $m \in \mathbb{N}$, and a G-invariant section $s \in H^0(X, \mathcal{L}^m)$ with $s(x) \neq 0$.

For any character χ of B, we denote by $\mathcal{L}_\chi$, the line bundle on G/B given by the character χ. We denote by $X(w)^{ss}_T(\mathcal{L}_\chi)$ the semi-stable points of $X(w)$ for the action of T with respect to the line bundle $\mathcal{L}_\chi$. For the simplicity of notation we will denote by $X(w)$ for a Schubert variety in both G/B and G/P, where P is a maximal parabolic subgroup of G.

We now recall the following definition from [54, pg. 90]. Let $w, \phi \in W$. Define $W^-(w,\phi) = \{\tau \leq w : \Phi^+(\tau) \cap \Phi^+(\phi) = \emptyset\}$. By [54, lemma. 5.4(1)] $W^-(w,\phi)$ has a unique maximal element in the Bruhat order and is denoted by $\tau^-(w,\phi)$.

The following proposition describes a criterion for a Schubert variety to admit semi-stable points.

Proposition 3.2.1. *Let $\chi = \sum_{\alpha \in \Delta} a_\alpha \varpi_\alpha$ be a dominant character of T which is in the root lattice. Let $I = Supp(\chi) = \{\alpha \in \Delta : a_\alpha \neq 0\}$ and let $w \in W^{I^c}$, where $I^c = \Delta \setminus I$. Then $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ if and only if $w\chi \leq 0$.*

Proof. Let $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$. Since $X(w)^{ss}_T(\mathcal{L}_\chi)$ is an open subset of the irreducible variety $X(w)$, we have $X(w)^{ss}_T(\mathcal{L}_\chi) \cap BwP_I/P_I \neq \emptyset$ where P_I is the parabolic corresponding to I. Let $x \in X(w)^{ss}_T(\mathcal{L}_\chi) \cap BwP_I/P_I$. Then by Hilbert-Mumford criterion ([Ch. 2, Th. 2.10.4], we have $\mu^{\mathcal{L}_\chi}(x,\lambda) \geq 0$ for all one parameter subgroup λ of T.

On the other hand by [101, lemma. 5.1] we have $\mu^{\mathcal{L}_\chi}(x,\lambda) = -\langle w\chi, \lambda\rangle$ for all one parameter subgroup λ of T lying in the dominant chamber. So $-\langle w\chi, \lambda\rangle \geq 0$ for all one parameter subgroup λ of T lying in the dominant chamber. Hence, $w\chi \leq 0$.

Conversely, let $w\chi \leq 0$.

<u>**Step 1**</u> - We prove that if $w, \tau \in W^{I^c}$ are such that $X(w) \subseteq \bigcup_{\phi \in W} \phi X(\tau)$, then, $w \leq \tau$. Now, suppose that $X(w) \subseteq \bigcup_{\phi \in W} \phi X(\tau)$. Then, since $X(w)$ is irreducible and W is finite, we must have

$$X(w) \subseteq \phi X(\tau), \text{ for some } \phi \in W.$$

Hence, $\phi^{-1}X(w) \subseteq X(\tau)$. Now, let $P_I = BW_IB$ and consider the projection

$$\pi : G/B \longrightarrow G/P_I$$

Then, $\pi^{-1}(\phi^{-1}X(w)) \subseteq \pi^{-1}(X(\tau))$. Let w^{max}(resp. τ^{max}) be the unique maximal element in wW_{I^c} (resp. τW_{I^c}). Then we have $\phi^{-1}X(w^{max}) \subseteq X(\tau^{max})$. So, we may assume that $I = \Delta$.

Now, since $\phi^{-1}X(w) \subseteq X(\tau)$, we have $\phi^{-1}w_1 \leq \tau$, $\forall\, w_1 \leq w$. Therefore $w_1\phi \leq \tau^{-1}$ $\forall\, w_1 \leq w^{-1}$. Hence, by the definition of τ^-, we have $\tau^-(w^{-1}, \phi^{-1})\phi \leq \tau^{-1}$.

Now we claim that $w^{-1} \leq \tau^-(w^{-1}, \phi^{-1})\phi$ for all $w, \phi \in W$, i.e., $w \leq \tau^-(w,\phi)\phi^{-1}$ for all $w, \phi \in W$.

We will prove this by induction on $l(\phi)$. Let $\phi = s_\alpha$. Now we consider two cases.

Case-1: $ws_\alpha < w$.

By [54, lemma. 5.6] we have $ws_\alpha \leq \tau^-(w, s_\alpha)s_\alpha$. Since $\tau^-(w, s_\alpha)s_\alpha(\alpha) < 0$ and $ws_\alpha \leq \tau^-(w, s_\alpha)s_\alpha$, we have $w = max\{w, ws_\alpha\} \leq \tau^-(w, s_\alpha)s_\alpha$.

Case-2: $ws_\alpha > w$

Again by [54, lemma. 5.6] we have $ws_\alpha \leq \tau^-(w, s_\alpha)s_\alpha$. Hence $w \leq ws_\alpha \leq \tau^-(w, s_\alpha)s_\alpha$.

Assume by induction that $w \leq \tau^-(w, \eta)\eta^{-1}$ for all $\eta \leq \phi$ and let $\phi = \eta s_\alpha$ for some $\alpha \in \Delta$ such that $l(\phi) = 1 + l(\eta)$.

Case-1: $w(\alpha) > 0$

In this case $w < ws_\alpha$.

Now we have $\quad ws_\alpha \leq \tau^-(ws_\alpha, \eta)\eta^{-1} \quad$ by induction, since $l(\eta) = l(\phi) - 1$.

$$\begin{aligned} &= \tau^-(ws_\alpha, \phi s_\alpha)\eta^{-1} \\ &= \tau^-(w, \phi)s_\alpha\eta^{-1} \quad \text{by [54, lemma. 5.4, 4(b)]} \\ &= \tau^-(w, \phi)\phi^{-1} \end{aligned}$$

Hence, $w \leq \tau^-(w, \phi)\phi^{-1}$.

Case-2: $w(\alpha) < 0$

Then $ws_\alpha(\alpha) > 0$.

Now we have $\quad w \leq \tau^-(w, \phi s_\alpha)s_\alpha\phi^{-1} \quad$ by induction, since $l(\phi s_\alpha) = l(\phi) - 1$.

$$\begin{aligned} &= \tau^-(ws_\alpha, \phi)s_\alpha s_\alpha\phi^{-1} \quad \text{by [54, lemma. 5.4, 4(b)]} \\ &= \tau^-(ws_\alpha, \phi)\phi^{-1} \\ &= \tau^-(w, \phi)\phi^{-1} \quad \text{by [54, lemma. 5.4, 4(a)]} \end{aligned}$$

Hence we have $w^{-1} \leq \tau^-(w^{-1}, \phi^{-1})\phi\tau^{-1}$. Thus $w \leq \tau$.

Now, let $w \in W^{I^c}$ be such that $w\chi \leq 0$. Then by step-1, there exist a point $x \in X(w) \setminus W$-translates of $X(\tau), \tau \in W^{I^c},\ \tau \not\geq w. \qquad \longrightarrow (1)$.

Step 2: We prove that x is semi-stable.

Let λ be an one parameter subgroup of T. Choose $\phi \in W$ such that $\phi\lambda \in \bar{C}$. Let $\tau \in W^{I^c}$ be such that $\phi x \in U_\tau \tau P_I$.

By (1) we have, $w \leq \tau$. Hence, $\tau\chi \leq w\chi \leq 0$.

Hence, by [101, lemma. 5.1], we have $\mu^{\mathcal{L}_\chi}(x, \lambda) = \mu^{\mathcal{L}_\chi}(\phi x, \phi\lambda) = \langle -\tau\chi, \phi\lambda \rangle \geq 0$.

Hence, by Hilbert-Mumford criterion ([Ch. 2, Th. 2.10.4], x is semi-stable. $\square$

We recall the following elementary properties of minuscule weights from [56], which were derived by Kannan and Sardar and they are be crucial in our description.

Lemma 3.2.2. *Let I be any nonempty subset of Δ, and let μ be a weight of the form $\sum_{\alpha_i \in I} m_i\alpha_i - \sum_{\alpha_i \notin I} m_i\alpha_i$, where $m_i \in \mathbb{Q}$ for all i, $1 \leq i \leq n-1$; $m_i > 0$ for all $\alpha_i \in I$ and $m_i \geq 0$ for all $\alpha_i \in \Delta \setminus I$. Then there is an $\alpha \in I$ such that $s_\alpha(\mu) < \mu$.*

Proof. Since $s_\alpha(\mu) = \mu - \langle \mu, \check{\alpha} \rangle \alpha$, we need to find an $\alpha \in I$ such that $\langle \mu, \check{\alpha} \rangle > 0$. Since $\langle \sum_{\alpha_i \in I} m_i\alpha_i, \sum_{\alpha_i \in I} m_i\check{\alpha_i} \rangle \geq 0$, we can find an $\alpha \in I$ such that $\langle \sum_{\alpha_i \in I} m_i\alpha_i, \check{\alpha} \rangle > 0$. Now we know that for any $\alpha_i, \alpha_j \in \Delta$, $i \neq j$, $\langle \alpha_i, \check{\alpha_j} \rangle \leq 0$. Hence, $\langle \sum_{\alpha_i \notin I} m_i\alpha_i, \check{\alpha} \rangle \leq 0$ for this $\alpha \in I$. Thus $\langle \mu, \check{\alpha} \rangle > 0$. This proves the lemma. $\square$

Lemma 3.2.3. *Let λ be any dominant weight and let $I = \{\alpha \in \Delta : \langle \lambda, \check{\alpha} \rangle = 0\}$. Let $w_1, w_2 \in W^I$ be such that $w_1(\lambda) = w_2(\lambda)$, then $w_1 = w_2$.*

Proof. See [12] and [45]. $\square$

In the rest of this section, ω will denote a minuscule weight and $I := \{\alpha \in \Delta : \langle \omega, \check{\alpha} \rangle = 0\}$

Lemma 3.2.4. *Let $\alpha \in \Delta$ and $\tau \in W$ such that $l(s_\alpha\tau) = l(\tau) + 1$ and $s_\alpha\tau \in W^I$, then $\tau \in W^I$; $s_\alpha\tau(\omega) = \tau(\omega) - \alpha$.*

Proof. The proof of the first part of the lemma is clear. Now $s_\alpha\tau(\omega) = \tau(\omega) - \langle \tau(\omega), \check{\alpha} \rangle \alpha$. Since the form $\langle ., . \rangle$ is W-invariant, $\langle \tau(\omega), \check{\alpha} \rangle = \langle \omega, \tau^{-1}\check{\alpha} \rangle$. Again since $l(s_\alpha\tau) = l(\tau) + 1$, we have $\tau^{-1}\alpha > 0$. Let $\tau^{-1}\alpha = \sum_{i=1}^{n-1} m_i\alpha_i$, $m_i \in \mathbb{Z}_{\geq 0}$. Now, if $\langle \omega, \tau^{-1}\check{\alpha} \rangle = 0$, then $m_i > 0 \Rightarrow \langle \omega, \tau^{-1}\check{\alpha_i} \rangle = 0$ for $1 \leq i \leq n-1$. This gives a contradiction, since $s_\alpha\tau \in W^I$ and $s_\alpha\tau(\tau^{-1}\alpha) = s_\alpha(\alpha) < 0$. Thus, $\langle \omega, \tau^{-1}\check{\alpha} \rangle = 1$. Hence the lemma is proved. $\square$

Corollary 3.2.5. 1. *For any $w \in W^I$, the number of times that s_i, $1 \leq i \leq n-1$ appears in a reduced expression of w = (coefficient of α_i in ω) $-$ (coefficient of α_i in $w(\omega)$) and hence it is independent of the reduced expression of w.*
2. *Let $w \in W^I$ and let $w = s_{i_1}.s_{i_2} \dots s_{i_k} \in W^I$ be a reduced expression. Then $w(\omega) = \omega - \sum_{j=1}^{k} \alpha_{i_j}$. and $l(w) = ht(\omega - w(\omega))$.*

Proof. Follows from lemma (3.2.4). $\square$

Lemma 3.2.6. *Let $w = s_{i_1}s_{i_2} \dots s_{i_k} \in W$ such that $ht(\omega - s_{i_1}s_{i_2} \dots s_{i_k}(\omega)) = k$ then $w \in W^I$ and $l(w) = k$.*

Proof. This follows from the corollary (3.2.5). $\square$

Lemma 3.2.7. *Let $\omega = \sum_{i=1}^{l} m_i\alpha_i$, $m_i \in \mathbb{Q}_{\geq 0}$ be a minuscule weight. Let $I = \{\alpha \in \Delta : \langle \omega, \check{\alpha} \rangle = 0\}$. Then, there exist a unique $w \in W^I$ such that $w(\omega) = \sum_{i=1}^{l}(m_i - \lceil m_i \rceil)\alpha_i$ where for any real number x,*

$$\lceil x \rceil := \begin{cases} x & \textit{if x is an integer} \\ [x]+1 & \textit{otherwise} \end{cases}$$

Proof. Using lemma (3.2.2) and the fact that ω is minuscule we can find a maximal sequence $s_{i_k}, s_{i_{k-1}}, \ldots, s_{i_1}$ of simple reflections in W such that for each j, $2 \leq j \leq k+1$, coefficient of α_{i_j} in $s_{i_{j-1}}.s_{i_{j-2}} \ldots s_{i_1}(\omega_r)$ is positive and $(s_{i_k}.s_{i_{k-1}} \ldots s_{i_1}(\omega_r)) = \omega_r - \sum_{j=1}^{k} \alpha_{i_j}$ for each j, $1 \leq j \leq k$. The existence part of the lemma follows from here. The uniqueness follows from lemma (3.2.3). $\square$

Now onwards, we say that for two elements w and τ in W, $w \preceq \tau$ if $l(\tau) = l(w) + l(\tau w^{-1})$. Note that this order if finer than the Bruhat order.

Lemma 3.2.8. *Let ω and I be as in the lemma (3.2.7) and $\tau, \sigma \in W^I$. Then $\tau(\omega) \leq \sigma(\omega) \Leftrightarrow \sigma \preceq \tau$.*

Proof. ($\Rightarrow$) The proof is by induction on $ht(\sigma(\omega) - \tau(\omega))$ which is a non-negative integer. $\underline{ht(w(\sigma\omega) - \tau(\omega)) = 1}$: This means $\sigma(\omega) = \tau(\omega) + \alpha$ for some $\alpha \in \Delta$. Applying s_α on both the sides of this equation, we have,

$$\begin{aligned} & s_\alpha\sigma(\omega) = -\alpha + s_\alpha\tau(\omega) \\ & \Longrightarrow \tau(\omega) - \langle \omega, \sigma^{-1}\check{\alpha} \rangle\alpha = -2\alpha + \tau(\omega) - \langle \omega, \tau^{-1}\check{\alpha} \rangle\alpha \\ & \Longrightarrow \langle \omega, \sigma^{-1}\check{\alpha} \rangle = 2 + \langle \omega, \tau^{-1}\check{\alpha} \rangle \end{aligned}$$

Since ω is minuscule, we get $\langle \omega, \sigma^{-1}\check{\alpha} \rangle = 1$ and $\langle \omega, \tau^{-1}\check{\alpha} \rangle = -1$. So, by lemma (3.2.6), $l(s_\alpha\sigma) = l(w) + 1$ and $s_\alpha w \in W^I$. Now, we have $s_\alpha\sigma(\omega) = \tau(\omega)$. Hence, by lemma (3.2.3), we get $\tau = s_\alpha\sigma$ with $l(\tau) = l(\sigma) + 1$. Thus the result follows in this case.

Let us assume that the result is true for $ht(\sigma(\omega) - \tau(\omega)) \leq m - 1$.

$\underline{ht(\sigma(\omega) - \tau(\omega)) = m}$: Let $\sigma(\omega) - \tau(\omega) = \sum_{\alpha_i \in J} m_i\alpha_i$ where $J \subseteq \Delta$ and m_i's are positive integers. Since $\langle \sum_{\alpha_i \in J} m_i\alpha_i, \sum_{\alpha_i \in J} m_i\check{\alpha_i} \rangle \geq 0$ there exist an $\alpha_j \in J$ such that $\langle \sigma(\omega) - \tau(\omega), \check{\alpha_j} \rangle > 0$. Hence either $\langle \sigma(\omega), \check{\alpha_j} \rangle > 0$ or $\langle \tau(\omega), \check{\alpha_j} \rangle < 0$.

<u>Case I</u>: Let us assume $\langle \sigma(\omega), \check{\alpha_j} \rangle > 0$. Then $l(s_{\alpha_j}\sigma) = l(\sigma) + 1$ and $s_{\alpha_j}\sigma \in W^I$. Now $ht(s_{\alpha_j}\sigma(\omega) - \tau(\omega)) = m - 1$. Hence, by induction $\tau = \phi_1 s_{\alpha_j}\sigma$ with $l(\tau) = l(\phi_1) + l(s_{\alpha_j}\sigma)$. Thus taking $\phi = \phi_1.s_{\alpha_j}$ we are done in this case.

<u>Case II</u>: Let us assume $\langle \tau(\omega), \check{\alpha_j} \rangle < 0$. Then $l(s_{\alpha_j}\tau) = l(\tau) - 1$ and $s_{\alpha_j}\tau \in W^I$. Since $\sigma(\omega) - s_{\alpha_j}\tau(\omega) = m - 1$ by induction $s_{\alpha_j}\tau = \phi_2\sigma$ with $l(s_{\alpha_j}\tau) = l(\phi_2) + l(\sigma)$. Thus taking $\phi = s_{\alpha_j}\phi_2$ we are done in this case also. This completes the proof.

($\Leftarrow$) $\sigma \preceq \tau \Rightarrow \sigma \leq \tau$. So $\tau(\omega) \leq \sigma(\omega)$ $\square$

Corollary 3.2.9. *Let ω, w and I be as in lemma (3.2.7). Let $\sigma \in W^I$ be such that $\sigma(n\omega) \leq 0$ for some positive integer n. Then, we have $w \preceq \sigma$.*

Proof. The proof follows from lemma (3.2.7), (3.2.8) and the fact that ω is minuscule. □

This corollary solves our problem in the case of minuscule weights since the order we have defined above is finer than the Bruhat order. Now we list all the minuscule findamental weights and the corresponding element w of the Weyl group for each type of simple algebraic group.

Type-A: All the fundamental weights are minuscule. For the description of the corresponding w such that $X(w)^{ss}_T \neq \emptyset$, see theorem (3.3.1) in the next section.

Type-B, C, D: The only minuscule fundamental weights in type B_n and C_n are ϖ_n and ϖ_1 respectively. For type D_n, the fundamental weights ϖ_1, ϖ_{n-1} and ϖ_n are minuscule. For each of the fundamental weight the corresponding Weyl group element is written explicitly in theorem (3.3.3).

Type-E_6: The minuscule fundamental weights are ϖ_1 and ϖ_6. The corresponding Weyl group elements are $s_5s_6s_1s_3s_4s_5s_2s_4s_3s_1$ and $s_6s_5s_3s_4s_1s_3s_2s_4s_5s_6$ respectively.

Type-E_7: The only minuscule fundamental weight is w_7 and the corresponding Weyl group element is $s_7s_5s_6s_2s_4s_5s_3s_4s_1s_3s_2s_4s_5s_6s_7$.

Type-E_8, F_4, G_2: There is no minuscule fundamental weight in these cases.

3.3 Minimal Schubert Varieties in G/P admitting Semi-stable Points

In this section, we describe all minimal Schubert varieties $X(w)$ in G/P, where G is a simple algebraic group and P is a maximal parabolic subgroup of G, for which $X(w)$ admits a semi-stable point for the action of a maximal torus of G with respect to an ample line bundle on G/P. Let $\mathcal{L}_{\varpi_r}$ denote the line bundle corresponding to the fundamental weight ϖ_r.

3.3.1 Classical Types

For type A, the following theorem is due to Kannan and Sardar (see [56, lemma. 2.7]).

Theorem 3.3.1. *Let $rank(G) = n = qr + t$, with $1 \leq t \leq r$ and let $w \in W^{I_r}$. Then $X(w)^{ss}_T(\mathcal{L}_{\varpi_r}) \neq \emptyset$ if and only if $w = (s_{a_1} \cdots s_1) \cdots (s_{a_r} \cdots s_r)$, where $\{a_i : i = 1, 2 \cdots r\}$ is an increasing sequence of positive integers such that $a_i \geq i(q+1)\ \forall\ i \leq t-1$ and $a_i = iq + (t+1)\ \forall\ t \leq i \leq r$.*

Now assume G is of type B, C or D, and P is a maximal parabolic subgroup of G for which $X(w)$ admits a semi-stable point for the action of a maximal torus of G with respect to an ample line bundle on G/P.

Let $I_r = \Delta \setminus \{\alpha_r\}$ and let $P_{I_r} = BW_{I_r}B$ be the maximal parabolic corresponding to the simple root α_r. Let $\mathcal{L}_r$ denote the line bundle associated to the weight ϖ_r. In this section we will describe all minimal elements of W^{I_r} for which $X(w)^{ss}_T(\mathcal{L}_r) \neq \emptyset$.

At this point, we recall a standard property of the fundamental weights of type A, B, C and D which will be used in the proof of our next proposition.

In types A_n, B_n, C_n and D_n, we have $|\langle \varpi_r, \check{\alpha} \rangle| \leq 2$ for any fundamental weight ϖ_r and any root α.

Proof. Now $\langle \varpi_r, \check{\alpha} \rangle \leq \langle \varpi_r, \check{\eta} \rangle$, where η is a highest root for the corresponding root system.

The highest root for type A_n is $\alpha_1 + \alpha_2 + \ldots + \alpha_n$, the highest roots for type B_n are $\alpha_1 + 2(\alpha_2 + \ldots + \alpha_n)$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_n$, the highest roots for type C_n are $2(\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$ and $\alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$ and the unique highest root for type D_n is $\alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$.

In all these cases, we have $\langle \varpi_r, \check{\eta} \rangle \leq 2$. So $|\langle \varpi_r, \check{\alpha} \rangle| \leq 2$, for any root α. ☐

Let G be a simple simply-connected algebraic group of type B, C or D. Let T be a maximal torus of G and let Δ be the set of simple roots with respect to a Borel subgroup B of G containing T.

Proposition 3.3.2. *Let $r \in \{1, 2, \cdots, n\}$ such that ϖ_r is non-miniscule. Let $I_r = \Delta \setminus \{\alpha_r\}$ and let $w \in W^{I_r}$ be of maximal length such that $w(\varpi_r) \in \mathbb{Q}_{\geq 0}\Delta$. Write $w(\varpi_r) = \sum_{i=1}^{n} a_i \alpha_i$ and let $a = max\{a_i : i = 1, 2, \cdots, n\}$. Then $a \in \{1, \frac{3}{2}\}$. Further, if $a = \frac{3}{2}$, then r must be odd and G must be of type D_n with $a = a_{n-1}$ or $a = a_n$.*

Proof. Let $r \in \{1, 2, \cdots, n\}$ such that ϖ_r is non-miniscule. Then $r \neq n$ in type B_n, $r \neq 1$ in type C_n, $r \neq 1, n-1, n$ in type D_n.

Assume that $a \notin \{1, \frac{3}{2}\}$, then $a \geq 2$ or $a = \frac{1}{2}$. We first show that $a \ngeq 2$.

Assume $a \geq 2$. Let i_0 be the least integer such that $a_{i_0} = a$. Using the tables in appendix-B we see that $i_0 \neq 1$. We first observe that, $s_{i_0} w(\varpi_r) = w(\varpi_r) - \langle w(\varpi_r), \check{\alpha_{i_0}} \rangle \alpha_{i_0} = \sum_{i \neq i_0} a_i \alpha_i + (a - \langle w(\varpi_r), \check{\alpha_{i_0}} \rangle) \alpha_{i_0} \in \mathbb{Q}_{\geq 0}\Delta$, since $\langle w(\varpi_r), \check{\alpha_{i_0}} \rangle \leq 2 \leq a = a_{i_0}$.

For all the cases except $i_0 = n$ in type B_n, $i_0 = n-1, n$ in type C_n and $i_0 = n-2, n-1, n$ in type D_n, we have $\langle w(\varpi_r), \check{\alpha_{i_0}} \rangle = 2a - (a_{i_0 - 1} + a_{i_0 + 1}) > 0$. So, $s_{\alpha_{i_0}} w > w$, a contradiction to the maximality of w.

Now, we treat the special cases explicitly.

$i_0 = n$ *in type* B_n: In this case, $\langle w(\varpi_r), \check{\alpha_n}\rangle = -2a_{n-1} + 2a_n > 0$, since $a_n = a > a_{n-1}$. So, $s_n w > w$, a contradiction to the maximality of w.

$i_0 = n-1$ *in type* C_n: In this case $\langle w(\varpi_r), \check{\alpha_{n-1}}\rangle = -a_{n-2} + 2a_{n-1} - 2a_n$ and $\langle w(\varpi_r), \check{\alpha_n}\rangle = 2a_n - a_{n-1}$. So we have

$$\langle w(\varpi_r), \check{\alpha_{n-1}}\rangle + \langle w(\varpi_r), \check{\alpha_n}\rangle > 0.$$

So we have either $\langle w(\varpi_r), \check{\alpha_{n-1}}\rangle > 0$ or $\langle w(\varpi_r), \check{\alpha_n}\rangle > 0$. If $\langle w(\varpi_r), \check{\alpha_{n-1}}\rangle > 0$, then $s_{n-1}w > w$, a contradiction to the maximality of w. Otherwise, $\langle w(\varpi_r), \check{\alpha_n}\rangle > 0$. Then, $s_n w(\varpi_r) = \sum_{i \neq n-1} a_i\alpha_i + (a_{n-1} - a_n)\alpha_n \in \mathbb{Q}_{\geq 0}\Delta$. Hence, $s_n w > w$, a contradiction to the maximality of w.

$i_0 = n$ *in type* C_n: In this case $\langle w(\varpi_r), \check{\alpha_n}\rangle = 2a_n - a_{n-1} > 2$ as $a_n = a \geq 2$. This gives a contradiction to the fact that $|\langle \varpi_r, \check{\alpha}\rangle| \leq 2$, for any root α.

$i_0 = n$ *in type* D_n: Here, we have $\langle w(\varpi_r), \check{\alpha_n}\rangle = 2a_n - a_{n-2} > 2$ as $a_n = a > a_{n-2}$, a contradiction to the fact that $|\langle \varpi_r, \check{\alpha}\rangle| \leq 2$, for any root α.

$i_0 = n-1$ *in type* D_n: This case is similar to the previous case.

$i_0 = n-2$ *in type* D_n: We have $\langle w(\varpi_r), \check{\alpha_{n-2}}\rangle = -a_{n-3} + 2a_{n-2} - a_{n-1} - a_n$, $\langle w(\varpi_r), \check{\alpha_{n-1}}\rangle = 2a_{n-1} - a_{n-2}$ and $\langle w(\varpi_r), \check{\alpha_n}\rangle = 2a_n - a_{n-2}$. Now if $\langle w(\varpi_r), \check{\alpha_{n-2}}\rangle > 0$ then $s_{n-2}w > w$, a contradiction to the maximality of w. Otherwise we claim that, either $\langle w(\varpi_r), \check{\alpha_{n-1}}\rangle > 0$ or $\langle w(\varpi_r), \check{\alpha_n}\rangle > 0$.

Now assume $\langle w(\varpi_r), \check{\alpha_{n-2}}\rangle \leq 0$. Then $a_{n-1} + a_n > a_{n-2}$, since $a_{n-3} < a_{n-2}$. Then we have either $2a_{n-1} > a_{n-2}$ or $2a_n > a_{n-2}$. Then, either $\langle w(\varpi_r), \check{\alpha_{n-1}}\rangle > 0$ or $\langle w(\varpi_r), \check{\alpha_n}\rangle > 0$. If $\langle w(\varpi_r), \check{\alpha_{n-1}}\rangle > 0$, then $s_{n-1}w > w$, a contradiction to the maximality of w, since $s_{n-1}w(\varpi_r) = \sum_{i \neq n-1} a_i\alpha_i + (a_{n-2} - a_{n-1}) \in \mathbb{Q}_{\geq 0}\Delta$. Similarly, if $\langle w(\varpi_r), \check{\alpha_n}\rangle > 0$, we get a contradiction.

Thus, we conclude that $a \leq 2$.

Now we show that $a \neq \frac{1}{2}$. Assume in contrary that $a = \frac{1}{2}$. Then using the tables in appendix-B we see that r is odd with the following four possibilities;

$a_n = \frac{1}{2}$ in type C_n
$a_{n-1} = a_n = \frac{1}{2}$ in type D_n
$a_{n-1} = 0$, $a_n = \frac{1}{2}$ in type D_n
$a_{n-1} = \frac{1}{2}$, $a_n = 0$ in type D_n.

In first two cases $w(\varpi_r)$ is conjugate to ϖ_1, a contradiction. In third and fourth case $w(\varpi_r) = \frac{1}{2}a$. So $2\varpi_r$ is conjugate to the unique highest root $\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n = \varpi_2$, a contradiction.

Thus, we conclude that $a \in \{1, \frac{3}{2}\}$.

Now, if $a = \frac{3}{2}$, then clearly r is odd and G is not of type B_n (see Appendix-B). We now prove that G can not be of type C_n.

Suppose on the contrary let G be of type C_n. We first note that $\langle w(\varpi_r), \check{\alpha_n}\rangle = 3-a_{n-1} \leq 2$. So, we have $a_{n-1} = 1$. Let t be the least positive integer such that $\sum_{i=t}^{n-1} \alpha_i + \frac{3}{2}\alpha_n \leq w(\varpi_r)$. If $t \leq n-2$, then $0 \leq s_t w(\varpi_r) = \sum_{i\neq t} a_i\alpha_i < w(\varpi_r)$. So, $s_t w > w$, a contradiction to the maximality of w. Hence, $a_{n-2} = 0$.

We now claim that $a_i = 0\ \forall\ i \leq n-3$. For otherwise, let $m \leq n-3$ be the largest integer such that $a_m = 1$. Then, $\langle w(\varpi_r), \alpha_{m+1} + \alpha_{m+2} \check{+} \ldots \alpha_{n-1}\rangle = -3$, a contradiction to the fact that $|\langle w(\varpi_r), \check{\beta}\rangle| \leq 2$ for all root β. So, $a_i = 0\ \forall\ i \leq n-2$. Hence, $w(\varpi_r) = \alpha_{n-1} + \frac{3}{2}\alpha_n$. But then, $w(\varpi_r)$ is conjugate to ϖ_1, a contradiction.

Thus, G can not be of type C_n.

Now let G be of type D_n. We have already proved that $a_i \leq \frac{3}{2}\ \forall\ i = 1, 2, \cdots, n$. We now claim that, $a_{n-1} + a_n \leq 2$. Suppose on the contrary, let $a_{n-1} = a_n = \frac{3}{2}$. We claim that $a_m = 0\ \forall\ m \leq n-3$. Otherwise, let t be the least positive integer such that $\sum_{i=t}^{n-2} \alpha_i + \frac{3}{2}\alpha_{n-1} + \frac{3}{2}\alpha_n \leq w(\varpi_r)$. Then, $a_{t-1} = 0$ and $t \leq n-3$.

Hence, $\langle w(\varpi_r), \alpha_t + \alpha_{t+1} + \check{\ldots} \alpha_{n-1} + \alpha_n\rangle = 3$, a contradiction to the fact that $|\langle w(\varpi_r), \check{\beta}\rangle| \leq 2$ for all root β. Thus, $a_m = 0\ \forall\ m \leq n-3$. So, $w(\varpi_r) = \alpha_{n-2} + \frac{3}{2}(\alpha_{n-1} + \alpha_n)$.

Then, $\langle w(\varpi_r), \alpha_{n-2} + \check{\alpha}_{n-1} + \alpha_n\rangle = 3$, a contradiction to the fact that $|\langle w(\varpi_r), \check{\beta}\rangle| \leq 2$ for all root β.

Thus, in type D_n not both a_{n-1} and a_n can be $\frac{3}{2}$. $\square$

Notation: $J_{p,q} = \{(i_1, i_2, \cdots, i_p) : i_k \in \{1, 2, \cdots, q\}\ \forall\ k$ and $i_{k+1} - i_k \geq 2\}$.

Now, we describe the set of all elements $w \in W^{I_r}$ of minimal length such that $w\varpi_r \leq 0$ for types B_n, C_n and D_n. Note that for $w \in W$ we have $w(\varpi_r) = \varpi_r$ if and only if $w \in W_{I_r}$, and that if $w \in W^{I_r}$ then w is the unique minimal element of the coset wW_{I_r}.

Theorem 3.3.3. *Let $W^{I_r}_{min}$ = Minimal elements of the set of all $\tau \in W^{I_r}$ such that $X(\tau)^{ss}_T(\mathcal{L}_{\varpi_r}) \neq \emptyset$.*

(1)Type B_n: (i) Let $r = 1$. Then $w = s_n s_{n-1} \ldots s_1$. Further, $W^{I_1}_{min} = \{w\}$.

(ii) Let r be an even integer in $\{2, 3, \cdots, n-1\}$. For any $\underline{i} = (i_1, i_2, \cdots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n-1}$, there exists unique $w_{\underline{i}} \in W^{I_r}_{min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$. Further, $W^{I_r}_{min} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2}, n-1}\}$.

(iii) Let r be an odd integer in $\{2, 3, \cdots, n-1\}$. For any $\underline{i} = (i_1, i_2, \cdots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2}, n-2}$, there exists unique $w_{\underline{i}} \in W^{I_r}_{min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \alpha_n)$. Further, $W^{I_r}_{min} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r-1}{2}, n-2}\}$.

(iv) Let $r = n$. If n is even, then, $w = w_{\frac{n}{2}} \cdots w_1$, where, $w_i = s_{2i-1} \dots s_n,\ i = 1, 2, \cdots \frac{n}{2}$ and if n is odd, then, $w = w_{[\frac{n}{2}]+1} \cdots w_1$, where, $w_i = s_{2i-1} \dots s_n,\ i = 1, 2, \cdots [\frac{n}{2}]+1$. Further, $W^{I_n}_{min} = \{w\}$.

(2) <u>Type C_n</u>: (i) Let $r = 1$. Then $w = s_n s_{n-1} \dots s_1$. Further, $W^{I_1}_{min} = \{w\}$.

(ii) Let r be an even integer in $\{2, 3, \cdots, n\}$. For any $\underline{i} = (i_1, i_2, \cdots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n-1}$, there exists unique $w_{\underline{i}} \in W^{I_r}_{min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$. Further, $W^{I_r}_{min} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2},n-1}\}$.

(iii) Let r be an odd integer in $\{2, 3, \cdots, n\}$. For any $\underline{i} = (i_1, i_2, \cdots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2},n-2}$, there exists unique $w_{\underline{i}} \in W^{I_r}_{min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_n)$. Further, $W^{I_r}_{min} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r-1}{2},n-2}\}$.

(3)<u>Type D_n</u>: (i) Let $r = 1$. Then $w = s_n s_{n-1} \dots s_1$. Further, $W^{I_1}_{min} = \{w\}$

(ii) Let r be an even integer in $\{2, 3, \cdots, n-2\}$. For any $\underline{i} = (i_1, i_2, \cdots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n} \backslash Z$, there exists unique $w_{\underline{i}} \in W^{I_r}_{min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$, where $Z = \{(i_1, i_2, \cdots, i_{\frac{r}{2}-2}, n-2, n) : i_k \in \{1, 2, \cdots, n-4\}$ and $i_{k+1} - i_k \geq 2\ \forall\ k\}$. Further, $W^{I_r}_{min} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2},n} \setminus Z\}$.

(iii) Let r be an odd integer in $\{2, 3, \cdots, n-2\}$. For any $\underline{i} = (i_1, i_2, \cdots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2},n-3}$, there exists unique $w_{\underline{i}} \in W^{I_r}_{min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n)$. Also, for any $\underline{i} = (i_1, i_2, \cdots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2},n-2}$, there exists unique $w_{\underline{i},1} \in W^{I_r}_{min}$ such that $w_{\underline{i},1}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_{n-1} + \frac{3}{2}\alpha_n)$ and there exists unique $w_{\underline{i},2} \in W^{I_r}_{min}$ such that $w_{\underline{i},2}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{3}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n)$. Further, $W^{I_r}_{min} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r-1}{2},n-3}\} \bigcup \{w_{\underline{i},j} : \underline{i} \in J_{\frac{r-1}{2},n-2}$ and $j = 1, 2\}$.

(iv) Let $r = n-1$ or n. Then, $w = \prod_{i=1}^{[\frac{n-1}{2}]} w_i$, where,

$$w_i = \begin{cases} \tau_i s_n & \text{if } i \text{ is odd.} \\ \tau_i s_{n-1} & \text{if } i \text{ is even.} \end{cases}$$

with, $\tau_i = s_{2i-1} \dots s_{n-2},\ i = 1, 2, \cdots [\frac{n-1}{2}]$. Further, $W^{I_r}_{min} = \{w\}$.

Proof. <u>*Proof of 1*</u>:

(i) $\varpi_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

Take $w = s_n s_{n-1} \dots s_1$. Then $w(\varpi_1) = -\alpha_n \leq 0$. Clearly, $W^{I_1}_{min} = \{w\}$.

(ii) Let r be an even integer in $\{2, 3, \cdots, n-1\}$.

We have, $\varpi_r = \sum_{i=1}^{r-1} i\alpha_i + r(\alpha_r + \ldots + \alpha_n),\ 2 \leq r \leq (n-1)$.

Now, $J_{\frac{r}{2},n-1} = \{(i_1, i_2, \cdots, i_{\frac{r}{2}}) : i_k \in \{1, 2, \cdots, n-1\}$ and $i_{k+1} - i_k \geq 2 \ \forall \ k\}$. Consider the partial order on $J_{\frac{r}{2},n-1}$, given by $(i_1, i_2, \cdots, i_{\frac{r}{2}}) \leq (j_1, j_2, \cdots, j_{\frac{r}{2}})$ if $i_k \leq j_k \ \forall \ k$ and $(i_1, i_2, \cdots, i_{\frac{r}{2}}) < (j_1, j_2, \cdots, j_{\frac{r}{2}})$ if $i_k < j_k$ for some k. We will prove by descending induction on this order that there exists a $w \in W^{I_r}$ such that $w(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

For $(j_1, j_2, \cdots, j_{\frac{r}{2}}) = (n-r+1, n-r+3, \cdots, n-1)$, we have $(s_{n-r+1} \ldots s_1)(s_{n-r+3} \ldots s_2) \ldots (s_{n-1} \ldots s_{\frac{r}{2}})(s_n s_{n-1} \ldots s_{\frac{r}{2}+1})(s_n s_{n-1} \ldots s_{\frac{r}{2}+2}) \ldots (s_n s_{n-1} \ldots s_r)(\varpi_r) = -(\sum_{t=1}^{\frac{r}{2}} \alpha_{n-r+2t-1})$.

Now, if $(i_1, i_2, \cdots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n-1}$ is not maximal, then, there exists t maximal such that $i_t < n - r + 2t - 1$.

Now, $(i_1, i_2, \cdots, i_{t-1}, 1+i_t, i_{t+1}, \cdots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n-1}$ and $(i_1, i_2, \cdots, i_{t-1}, 1+i_t, i_{t+1}, \cdots, i_{\frac{r}{2}}) > (i_1, i_2, \cdots, i_{\frac{r}{2}})$. So by induction, there exists $w_1 \in W^{I_r}$ such that $w_1\varpi_r = -(\sum_{k\neq t} \alpha_{i_k} + \alpha_{1+i_t})$. Taking $w = s_{1+i_t} s_{i_t} w_1$ we have $w\varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

Hence, for any $(i_1, i_2, \cdots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n-1}$, there exists $w \in W^{I_r}$ such that $w\varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

Now, we prove that the w's in W^{I_r} having this property are minimal in the set of τ with $\tau(\varpi_r) \leq 0$.

Let $w \in W^{I_r}$ such that $w\varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

Suppose w is not minimal. Then there exists $\beta \in \Phi^+$ such that $s_\beta w(\varpi_r) \leq 0$ and $l(s_\beta w) = l(w) - 1$. So we have $-(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k}) = w(\varpi_r) < s_\beta w(\varpi_r) \leq 0$. Again since $i_{k+1} - i_k \geq 2 \ \forall \ k$, $\beta = \alpha_{i_t}$ for some $t = 1, 2, \cdots \frac{r}{2}$. Hence, $s_\beta w(\varpi_r) = -(\sum_{k\neq t} \alpha_{i_k}) + \alpha_{i_t} \not\leq 0$, a contradiction. Thus, all the w's are minimal.

Now, it remains to prove that for all elements of two types, (i) $-(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ and (ii) $-(\sum_{k=1}^{s} \alpha_{i_k}), \ s > \frac{r}{2}$ in the weight lattice such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$, for some k, there does not exist $w \in W^{I_r}$ *minimal* in the set of τ with $\tau(\varpi_r) \leq 0$ such that $w\varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

We first consider the first case. Let $\mu = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ be such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$ for some k. Choose k minimal such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$.

If $i_k = n - 1$, then $i_{k+1} = 1$ and $s_n w(\varpi_n) = -(\sum_{i_j \neq n} \alpha_{i_j}) > -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$. Hence, $s_n w < w$, a contradiction to the minimality of w.

Otherwise, $s_{i_k} w(\varpi_r) = -(\sum_{j\neq k} \alpha_{i_j}) > -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$. Hence, $s_{i_k} w < w$, a contradiction to the minimality of w.

We now consider the second case. Let $\mu = -(\sum_{k=1}^{s} \alpha_{i_k}), \ s > \frac{r}{2}$. Using the same argument as above we see that there does not exist $w \in W^{I_r}$ *minimal* in the set of τ with $\tau(\varpi_r) \leq 0$ such that $w\varpi_r = -(\sum_{k=1}^{s} \alpha_{i_k})$.

Hence, $W_{min}^{I_r} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2},n-1}\}$ follows from proposition (3.3.2).

(iii) Let r be an odd integer in $\{2, 3, \cdots, n-1\}$.

The proof is similar to the case when r is even.

(iv) We have, $\varpi_n = \frac{1}{2}\sum_{i=1}^{n} i\alpha_i$.

Then, $2\varpi_n = \sum_{i=1}^{n} i\alpha_i$.

<u>$Case\ 1:$</u> n is even.

Take $w_i = s_{2i-1}\dots s_n,\ i = 1, 2, \cdots \frac{n}{2}$.

Let $w = w_{\frac{n}{2}} \cdots w_1$. Then $w(2\varpi_n) = -\sum_{i=1}^{\frac{n}{2}} \alpha_{2i-1} \leq 0$.

<u>$Case\ 2:$</u> n is odd.

Take $w_i = s_{2i-1}\dots s_n,\ i = 1, 2, \cdots, \frac{n+1}{2}$.

Let $w = w_{\frac{n+1}{2}} \cdots w_1$. Then $w(2\varpi_n) = -\sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} \leq 0$. Note that $W_{min}^{I_n} = \{w\}$, since ϖ_n is miniscule.

<u>*Proof of 2*</u>:

(i) We have, $\varpi_1 = \alpha_1 + \alpha_2 + \dots + \frac{1}{2}\alpha_n$.

Then, $2\varpi_1 = 2(\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}) + \alpha_n$.

Take $w = s_n s_{n-1}\dots s_1$. Then $w(2\varpi_1) = -\alpha_n \leq 0$. Note that $W_{min}^{I_1} = \{w\}$, since ϖ_1 is miniscule.

Proof of (ii) and (iii) are similar to Cases (ii) and (iii) of type B_n.

<u>*Proof of 3*</u>:

(i) We have, $\varpi_1 = \sum_{i=1}^{n-2} \alpha_i + \frac{1}{2}(\alpha_{n-1} + \alpha_n)$. Then, $2\varpi_1 = 2(\sum_{i=1}^{n-2} \alpha_i) + \alpha_{n-1} + \alpha_n$.

Take $w = s_n s_{n-1}\dots s_1$. Then $w(2\varpi_1) = -(\alpha_{n-1} + \alpha_n) \leq 0$. Note that $W_{min}^{I_1} = \{w\}$, since ϖ_1 is miniscule.

Proof of (ii) and (iii) are similar to Cases (ii) and (iii) of type B_n.

(iv) We have, $\varpi_{n-1} = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2}) + \frac{1}{4}(n\alpha_{n-1} + (n-2)\alpha_n)$.

Then, $4\varpi_{n-1} = 2(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2}) + n\alpha_{n-1} + (n-2)\alpha_n$

Take

$$w_i = \begin{cases} \tau_i s_{n-1} & \text{if } i \text{ is odd.} \\ \tau_i s_n & \text{if } i \text{ is even.} \end{cases}$$

where, $\tau_i = s_{2i-1} \dots s_{n-2}, \ i = 1, 2, \cdots [\frac{n-1}{2}]$.

Let $w = \prod_{i=1}^{[\frac{n-1}{2}]} w_i$. Then,

$$w(4\varpi_{n-1}) = \begin{cases} \mu - 2\alpha_n & \text{if } n \equiv 0 \pmod 4, \\ \mu - 2\alpha_{n-1} & \text{if } n \equiv 2 \pmod 4, \\ \mu - 2\alpha_{n-2} - 3\alpha_{n-1} - \alpha_n & \text{if } n \equiv 1 \pmod 4, \\ \mu - 2\alpha_{n-2} - \alpha_{n-1} - 3\alpha_n & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where, $\mu = -2(\sum_{i=1}^{[\frac{n-1}{2}]} \alpha_{2i-1})$.

We have, $\varpi_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \ldots + (n-2)\alpha_{n-2}) + \frac{1}{4}((n-2)\alpha_{n-1} + n\alpha_n)$.

Then, $4\varpi_n = 2(\alpha_1 + 2\alpha_2 + \ldots + (n-2)\alpha_{n-2}) + (n-2)\alpha_{n-1} + n\alpha_n$.

Take

$$w_i = \begin{cases} \tau_i s_n & \text{if } i \text{ is odd.} \\ \tau_i s_{n-1} & \text{if } i \text{ is even.} \end{cases}$$

where, $\tau_i = s_{2i-1} \dots s_{n-2}, \ i = 1, 2, \cdots [\frac{n-1}{2}]$.

Let $w = \prod_{i=1}^{[\frac{n-1}{2}]} w_i$. Then,

$$w(4\varpi_n) = \begin{cases} \mu - 2\alpha_{n-1} & \text{if } n \equiv 0 \pmod 4, \\ \mu - 2\alpha_n & \text{if } n \equiv 2 \pmod 4, \\ \mu - 2\alpha_{n-2} - \alpha_{n-1} - 3\alpha_n & \text{if } n \equiv 1 \pmod 4, \\ \mu - 2\alpha_{n-2} - 3\alpha_{n-1} - \alpha_n & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where, $\mu = -2(\sum_{i=1}^{[\frac{n-1}{2}]} \alpha_{2i-1})$. Note that $W_{min}^{I_i} = \{w\}$ for $i = n-1, n$, since ϖ_{n-1} and ϖ_n are miniscule. □

3.3.2 Exceptional Types

In this section, we describe all minimal Schubert varieties $X(w)$ in G/P (where G is a simple algebraic group of type E_6, E_7, E_8, F_4 or G_2, and P is a maximal parabolic subgroup of G) for which $X(w)$ admits a semi-stable point for the action of a maximal torus of G with respect to an ample line bundle on G/P.

Let $I_r = \Delta \setminus \{\alpha_r\}$ and let $P_{I_r} = BW_{I_r}B$ be the maximal parabolic corresponding to the simple root α_r. Let $\mathcal{L}_r$ denote the line bundle associated to the weight ϖ_r. In this section we describe all minimal elements of W^{I_r} for which $X(w)^{ss}_T(\mathcal{L}_r) \neq \emptyset$.

Now, we describe the set of all elements $w \in W^{I_r}$ of minimal length such that $w\varpi_r \leq 0$ for type E_6, E_7, E_8, F_4 or G_2. For the Dynkin diagrams and labelling of simple roots, we refer to chapter-1 and for fundamental weights we refer to appendix-B.

Type $\mathbf{F_4}$:

(1) $\varpi_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$

$(s_2s_1)(s_3s_2s_4s_3s_2s_1)(\varpi_1) = -\alpha_2$

$(s_1s_2)(s_3s_2s_4s_3s_2s_1)(\varpi_1) = -\alpha_1$

(2) $\varpi_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4$

$(s_3s_4)(s_1s_2s_1s_3s_2s_1s_4s_3s_2)(\varpi_2) = -\alpha_1 - 2\alpha_3$

$(s_4s_3)(s_1s_2s_1s_3s_2s_1s_4s_3s_2)(\varpi_2) = -\alpha_1 - 2\alpha_4$

$(s_1s_2s_3s_2s_1s_4s_3s_2s_3s_1s_2)(\varpi_2) = -2\alpha_1 - \alpha_2$

$(s_2s_1s_3s_2s_3s_4s_3s_2s_3s_1s_2)(\varpi_2) = -\alpha_1 - 2\alpha_2$

$(s_4s_2s_3s_1s_2s_3s_1s_2s_4s_3s_2)(\varpi_2) = -\alpha_2 - 2\alpha_4$

(3) $\varpi_3 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4$

$(s_1s_2)(s_3s_4s_3s_2s_3s_4s_1s_2s_3)(\varpi_3) = -\alpha_1 - \alpha_4$

$(s_2s_1)(s_3s_4s_3s_2s_3s_4s_1s_2s_3)(\varpi_3) = -\alpha_2 - \alpha_4$

$(s_1s_3)(s_2s_3s_4s_3s_2s_3s_1s_2s_3)(\varpi_3) = -\alpha_1 - \alpha_3$

$(s_3s_4)(s_2s_1s_3s_2s_3s_4s_1s_2s_3)(\varpi_3) = -2\alpha_3 - \alpha_4$

$(s_4s_3)(s_2s_3s_4s_1s_2s_3s_4s_2s_3)(\varpi_3) = -\alpha_3 - 2\alpha_4$

(4) $\varpi_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$

$(s_3s_4)(s_2s_3s_1s_2s_3s_4)(\varpi_4) = -\alpha_3$

$(s_4s_3)(s_2s_3s_1s_2s_3s_4)(\varpi_4) = -\alpha_4$

Type $\mathbf{G_2}$:

(1) $\varpi_1 = 2\alpha_1 + \alpha_2$

$s_2 s_1(\varpi_1) = -\alpha_1 - 4\alpha_2$

$s_1 s_2 s_1 s_2 s_1 s_2(\varpi_1) = -2\alpha_1 - \alpha_2$

(2) $\varpi_2 = 3\alpha_1 + 2\alpha_2$

$s_2 s_1(\varpi_2) = -\alpha_1 - 5\alpha_2$

$s_1 s_2 s_1 s_2 s_1 s_2(\varpi_2) = -3\alpha_1 - 2\alpha_2$

Type $\mathbf{E_6, E_7, E_8}$:

Since the combinatorics in this case is very complicated, for given a fundamental weight ϖ we give a C-program that will generate all the $w \in W^{I_r}$ such that $w\varpi \leq 0$. For the program please refer to appendix-A.

Program Description:

Given a fundamental weight ϖ we can write it as a tuple of rational numbers $(a_1, a_2 \dots a_l)$. Since $w(\varpi) \leq 0$ if and only if $w(k\varpi) \leq 0$, by multiplying a suitable integer we can assume that all the co-ordinates of the tuple are positive integers. Since we are interested in minimal Schubert varieties admitting semi-stable points, so given a fundamental weight ϖ we need to compute all the minimal length Weyl group elements having the property that $w(\varpi) \leq 0$, i.e., all the entries of the tuple is non-positive. Let s_i denote the reflection corresponding to the simple root α_i. We keep applying the simple reflections to the tuple. We say the operation s_i is valid for this tuple if $\varpi^r > 0$ and $s_i(\varpi^r) < \varpi^r$, where ϖ^r is the resultant tuple after r^{th} operation. A sequence of s_i's is valid if at each stage the operation applied is valid. Our goal is to find all valid sequence of operations which when applied consecutively, takes each element of the tuple to a non-positive integer. Also, if two or more valid sequences map the tuple to the same tuple of non-positive numbers, we want to retain the one which is lexicographically smallest and this is possible by lemma (3.2.3). Note that if two valid sequences map to the same value, they must be of the same length. This follows from the definition of valid.

The algorithm does an exhaustive search through the set of all possible function sequences i.e. sequence of s_i's, with a little pruning to cut down on the running time. Note that this set is infinite, but in our case, we know a bound on the length of such a sequence, thus restricting the set to only finitely many sequences. This is based on the following observation: Given a fundamental weights $\varpi = (a_1, a_2, \dots a_l)$, the length of the sequence is bounded by $\sum_i a_i$. For the cases we are interested in (Type E_6, E_7, E_8), the sum of the weights does not exceed 200 for any tuple. Hence, we restrict our search space to all sequences of s_i's of length at most 200. Let us call this set M.

It is clear that we can define a lexicographical ordering on M. The algorithm goes through the elements in M in a lexicographical manner. However, the algorithm does not consider every element in order to cut down on the running time. Certain relations listed below are used to prune equivalent sequences:

1. The commuting relations: $s_i s_j = s_j s_i$ if i, j are not neighbouring vertices.
2. The Braid relations: $s_i s_j s_i = s_j s_i s_j$ if i, j are neighbouring vertices.

Moreover, once a valid sequence s has been found to map to a tuple of non- positive integers, we do not consider sequences which have s as a prefix, since they clearly would not be valid. Apart from this, a sequence which contains an s_i followed by another s_i is not considered, since $s_i^2 = 1$. Algorithmically speaking, we build sequences in lexicographical order, in increasing order of length i.e. starting with some s_i as the first operation, we recursively keep appending operations (in lexicographically order i.e. s_j would be tried before s_{j+1}) to the sequence, whilst preserving validity and applying the above heuristics for commutativity etc.

Even after employing these conditions we could not handle the block commutation of s_i's, for example in type E_8, the program was not able to detect the equivalence of $s_7 s_6 s_5 s_4 s_2 s_3 s_1 s_4 s_2$ and $s_5 s_7 s_6 s_5 s_4 s_2 s_3 s_1 s_4$ i.e. that the two sequences map to the same tuple. We made certain modifications to the program to get around these duplications. Since the output is integer valued and $w_1(\lambda) = w_2(\lambda)$ if and only if $w_1 = w_2$, we could use hashing map to filter out the repeated elements.

We represent the fundamental weight as an l-tuple $(a_1, a_2 \ldots, a_l)$. We keep on applying reflections till each of the component entries in the tuple become less than or equal to 0. It can be checked easily that none of the entries of the final non-positive tuple go beyond -8. We now define the hashing function $h : \{0, -1, -2 \ldots, -7\}^l \to \mathbb{N}$, which maps a reflected tuple to a unique value in $\mathbb{N}$. Let the tuple after reflections be $(a'_1, a'_2 \ldots, a'_l)$. Let us define $x_i = a'_i + 7, 1 \leq i \leq l$. Then, $0 \leq x_i \leq 7$, and the tuple $(x_1, x_2, \ldots, x_l)$ can be interpreted as an octal number, read from left to right (x_1 is the unit position). We can then map the octal number to its decimal value. Thus, the hash function is defined as follows:

$$h(a'_1, a'_2 \ldots, a'_l) = \sum_{i=1}^{l} 8^{i-1}(a'_i + 7)$$

Here, $(a'_1, a'_2 \ldots a'_l)$ is a reflected tuple. It is easy to see that the hashing function is injective. We know that two octal numbers are equal if and only if there corresponding digits are equal, and thus, two tuples map to the same value if and only if there component entries are same.

Input and Output formats:

Create an input file called "data" in the same directory as the program. The top-most line of the input file contains the length of the tuple ($6, 7$ or 8) . The rest of the lines contain a tuple of the given length. For the output, the file generated by the program is "output". For each line of input (except the first line, which is used to give the length), the program generates pairs of lines. The first line in each pair corresponds to the sequence of operations applied, while the second line corresponds to the final tuple obtained after applying the sequence to the tuple in the corresponding input line. For example, in type E_6, ϖ_2 = (1 2 2 3 2 1), so the input and output are given below:

Input: 6 1 2 2 3 2 1
Output:
0 0 0 0 0 -1 0 0
$\mathbf{s_2s_4s_3s_1s_5s_4s_2s_3s_4s_5s_6}$
0 0 0 0 -1 0 0 0
$\mathbf{s_2s_4s_3s_1s_5s_4s_2s_3s_4s_6s_5}$
0 0 0 -1 0 0 0 0
$\mathbf{s_2s_4s_3s_1s_5s_4s_2s_3s_6s_5s_4}$
0 0 -1 0 0 0 0 0
$\mathbf{s_2s_4s_3s_1s_5s_4s_2s_6s_5s_4s_3}$
0 -1 0 0 0 0 0 0
$\mathbf{s_2s_4s_3s_1s_5s_4s_3s_6s_5s_4s_2}$
-1 0 0 0 0 0 0 0
$\mathbf{s_2s_4s_3s_5s_4s_2s_6s_5s_4s_3s_1}$

The last two zeros in the output are filling up the dummy places which are reserved for type E_7 and E_8. Since the output is huge we list here only the number of minimal Schubert varieties admitting semi-stable points in each of the three cases.

Let n denote the number of minimal Schubert varieties admitting semi-stable points.

Type $\mathbf{E_6}$:

Fundamental weights	n
$3\varpi_1$ =(4 3 5 6 4 2)	1
ϖ_2 =(1 2 2 3 2 1)	6
$3\varpi_3$ =(5 6 10 12 8 4)	6
ϖ_4 =(2 3 4 6 4 2)	30
$3\varpi_5$ =(4 6 8 12 10 5)	6
$3\varpi_6$ =(2 3 4 6 5 4)	1

Type $\mathbf{E_7}$:

Fundamental weights	n
ϖ_1 =(2 2 3 4 3 2 1)	7
$2\varpi_2$ =(4 7 8 12 9 6 3)	10
ϖ_3 =(3 4 6 8 6 4 2)	51
ϖ_4 =(4 6 8 12 9 6 3)	186
$2\varpi_5$ =(6 9 12 18 15 10 5)	52
ϖ_6 =(2 3 4 6 5 4 2)	15
$2\varpi_7$ =(2 3 4 6 5 4 3)	1

<u>**Type $\mathbf{E_8}$:**</u>

Fundamental weights	n
ϖ_1 =(4 5 7 10 8 6 4 2)	21
ϖ_2 =(5 8 10 15 12 9 6 3)	192
ϖ_3 =(7 10 14 20 16 12 8 4)	623
ϖ_4 =(10 15 20 30 24 18 12 6)	4014
ϖ_5 =(8 12 16 24 20 15 10 5)	2115
ϖ_6 =(6 9 12 18 15 12 8 4)	589
ϖ_7 =(4 6 8 12 10 8 6 3)	94
ϖ_8 =(2 3 4 6 5 4 3 2)	8

3.4 Coxeter Elements admitting Semi-stable Points

In this section, we describe all Coxeter elements $w \in W$ for which the corresponding Schubert variety $X(w)$ admits a semi-stable point for the action of a maximal torus with respect to a non-trivial line bundle on G/B.

We now assume that the root system Φ is irreducible.

<u>Coxeter elements of Weyl group:</u>

An element $w \in W$ is said to be a Coxeter element if it is of the form $w = s_{i_1}s_{i_2}\dots s_{i_n}$, with $s_{i_j} \neq s_{i_k}$ unless $j = k$, see [47, pg. 74].

Let $\chi = \sum_{\alpha\in\Delta} a_\alpha\alpha$ be a non-zero dominant weight and let w be a Coxeter element of W.

Lemma 3.4.1. *If $w\chi \leq 0$ and $\alpha \in \Delta$ is such that $l(ws_\alpha) = l(w) - 1$, then,*

(1) $|\{\beta \in \Delta \setminus \{\alpha\} : \langle\beta, \check{\alpha}\rangle \neq 0\}| = 1$ *or* 2.

(2) *Further if $|\{\beta \in \Delta \setminus \{\alpha\} : \langle\beta, \check{\alpha}\rangle \neq 0\}| = 2$, then Φ must be of type A_3 and χ is of the form $a(2\alpha + \beta + \gamma)$ for some $a \in \mathbb{Z}_{\geq 0}$, where α, β and γ are labelled as*

$$\circ_\beta\text{———}\circ_\alpha\text{———}\circ_\gamma$$

Proof. Since Φ is irreducible and χ is non zero dominant weight, a_β is a positive rational number for each $\beta \in \Delta$. Further since $w\chi \leq 0$, χ must be in the root lattice and so a_β is a positive integer for every β in Δ.

Since w is a Coxeter element and $l(ws_\alpha) = l(w) - 1$, the coefficient of α in $w\chi$ = coefficient of α in $s_\alpha\chi$. $\longrightarrow (1)$

We have $s_\alpha\chi = \chi - \langle\chi,\check{\alpha}\rangle\alpha$

$$= \chi - \langle\sum_{\beta\in\Delta} a_\beta\beta, \check{\alpha}\rangle\alpha$$

$$= \sum_{\beta\in\Delta} a_\beta\beta - \sum_{\beta\in\Delta} a_\beta\langle\beta,\check{\alpha}\rangle\alpha.$$

The coefficient of α in $s_\alpha\chi$ is $-(\sum_{\beta\in\Delta\setminus\{\alpha\}}\langle\beta,\check{\alpha}\rangle a_\beta + a_\alpha)$. $\longrightarrow (2)$

Since $w\chi \leq 0$, from (1) and (2) we have

$$-(\sum_{\beta\in\Delta\setminus\{\alpha\}}\langle\beta,\check{\alpha}\rangle a_\beta + a_\alpha) \leq 0.$$

Hence, $-(\sum_{\beta\in\Delta\setminus\{\alpha\}}\langle\beta,\check{\alpha}\rangle a_\beta) \leq a_\alpha$

Thus, we have $-2(\sum_{\beta\in\Delta\setminus\{\alpha\}}\langle\beta,\check{\alpha}\rangle a_\beta) \leq 2a_\alpha$. $\longrightarrow (3)$

Since χ is dominant, we have,

$$\langle\chi,\check{\beta}\rangle \geq 0,\ \forall\ \beta\in\Delta$$

$$\Rightarrow \langle\sum_{\gamma\in\Delta} a_\gamma\gamma, \check{\beta}\rangle \geq 0$$

$$\Rightarrow \sum_{\gamma\in\Delta} a_\gamma\langle\gamma,\check{\beta}\rangle \geq 0$$

Now if $\langle\beta,\check{\alpha}\rangle \neq 0$, the left hand side of the inequality is $2a_\beta - a_\alpha-$(a non-negative integer).

Thus, we have, $2a_\beta \geq a_\alpha$ if $\langle\beta,\check{\alpha}\rangle \neq 0$ $\longrightarrow (4)$.

Now if $|\{\beta\in\Delta\setminus\{\alpha\} : \langle\beta,\check{\alpha}\rangle \neq 0\}| \geq 3$, from (3) and (4) we have,

$$3a_\alpha \leq -(2\sum_{\beta\in\Delta\setminus\{\alpha\}}\langle\beta,\check{\alpha}\rangle a_\beta) \leq 2a_\alpha.$$

This is a contradiction to the fact that a_α is a positive integer.

So $|\{\beta\in\Delta\setminus\{\alpha\} : \langle\beta,\check{\alpha}\rangle \neq 0\}| \leq 2$.

Proof of (2):

Suppose $|\{\beta\in\Delta\setminus\{\alpha\} : \langle\beta,\check{\alpha}\rangle \neq 0\}| = 2$. Let β,γ be the two distinct elements of this set. Using (3) and the facts that $\langle\beta,\check{\alpha}\rangle \leq -1$, $\langle\gamma,\check{\alpha}\rangle \leq -1$, we have

$$2(a_\beta + a_\gamma) \leq -2(\langle\beta,\check{\alpha}\rangle a_\beta + \langle\gamma,\check{\alpha}\rangle a_\gamma) \leq 2a_\alpha \qquad \longrightarrow (5)$$

Since $\langle\chi,\check{\beta}\rangle \geq 0$ and $\langle\chi,\check{\gamma}\rangle \geq 0$ we have

$$2a_\beta \geq -\sum_{\delta\neq\beta,\alpha}\langle\delta,\check{\beta}\rangle a_\delta + a_\alpha \text{ and}$$

$$2a_\gamma \geq -\sum_{\delta\neq\gamma,\alpha} \langle \delta, \check{\gamma}\rangle a_\delta + a_\alpha.$$

Hence, $-\sum_{\delta\neq\beta,\alpha}\langle\delta,\check{\beta}\rangle a_\delta - \sum_{\delta\neq\gamma,\alpha}\langle\delta,\check{\gamma}\rangle a_\delta + 2a_\alpha \leq 2(a_\beta + a_\gamma)$.

Using (5), we get

$$-\sum_{\delta\neq\beta,\alpha}\langle\delta,\check{\beta}\rangle a_\delta - \sum_{\delta\neq\gamma,\alpha}\langle\delta,\check{\gamma}\rangle a_\delta + 2a_\alpha \leq 2a_\alpha.$$

$$\Rightarrow \sum_{\delta\neq\gamma,\beta,\alpha}\langle -\delta,\ check\beta\rangle a_\delta + \sum_{\delta\neq\gamma,\beta,\alpha}\langle -\delta,\check{\gamma}\rangle a_\delta \leq 0,\ \text{ since } \langle\beta,\check{\gamma}\rangle = \langle\gamma,\check{\beta}\rangle = 0$$

Since each a_δ is positive and $\langle -\delta,\check{\beta}\rangle, \langle -\delta,\check{\gamma}\rangle$ are non-negative integers, we have

$\langle -\delta,\check{\beta}\rangle = 0$ and $\langle -\delta,\check{\gamma}\rangle = 0,\ \forall\ \delta\neq\alpha,\beta,\gamma.$

Since Φ is irreducible, we have $\Delta = \{\alpha,\beta,\gamma\}$. So, from the classification theorem (theorem (1.7.1)) of irreducible root systems, we have $\langle\beta,\check{\alpha}\rangle \in \{-1,-2\}$.

If $\langle\beta,\check{\alpha}\rangle = -2$, then $\langle\gamma,\check{\alpha}\rangle = -1$.

Hence, from (3) we get $\quad 4a_\beta + 2a_\gamma \leq 2a_\alpha \qquad \longrightarrow (6)$

Again, from (4) we have $2a_\beta \geq a_\alpha$ and $2a_\gamma \geq a_\alpha$. So using (6), we get $3a_\alpha \leq 4a_\beta + 2a_\alpha \leq 2a_\alpha$, a contradiction to the fact that a_α is a positive integer. Thus $\langle\beta,\check{\alpha}\rangle = -1$.

Using a similar argument, we see that $\langle\gamma,\check{\alpha}\rangle = -1$.

Now, let us assume that $\langle\alpha,\check{\beta}\rangle = -2$.

Then,

$$\begin{aligned} 0 \leq \langle\chi,\check{\beta}\rangle &= a_\gamma\langle\gamma,\check{\beta}\rangle - 2a_\alpha + 2a_\beta \\ &= -2a_\alpha + 2a_\beta,\ \text{ since } \langle\gamma,\check{\beta}\rangle = 0 \\ \Rightarrow 2a_\alpha \leq 2a_\beta. & \end{aligned}$$

From (3), we have $\quad 2a_\beta + 2a_\gamma \leq 2a_\alpha \leq 2a_\beta$.
Hence, $2a_\gamma \leq 0$, a contradiction. So $\langle\alpha,\check{\beta}\rangle = -1$. Similarly $\langle\alpha,\check{\gamma}\rangle = -1$.

Hence Φ is of the type A_3.

$$\circ_\beta\text{———}\circ_\alpha\text{———}\circ_\gamma$$

We now show that $\chi = a(\beta + 2\alpha + \gamma)$, for some $a \in \mathbb{Z}_{\geq 0}$.

Let $\chi = a_\alpha\alpha + a_\beta\beta + a_\gamma\gamma$. By assumption, we have $s_\gamma s_\beta s_\alpha(\chi) \leq 0$.

So $(a_\beta + a_\gamma - a_\alpha)\alpha + (a_\beta - a_\alpha)\gamma + (a_\gamma - a_\alpha)\beta \leq 0$.

Hence, we have $a_\beta + a_\gamma \leq a_\alpha \qquad \longrightarrow (7)$

Since χ is dominant, we have $\langle \chi, \check{\beta} \rangle \geq 0$ and $\langle \chi, \check{\gamma} \rangle \geq 0$.

So we have, $a_\alpha \leq 2a_\beta$ and $a_\alpha \leq 2a_\gamma$ $\longrightarrow (8)$.

Using (7) and (8), $2a_\alpha \geq 2(a_\beta + a_\gamma) \geq 2a_\alpha$. This is possible only if $2a_\beta = a_\alpha = 2a_\gamma$.

Then, χ must be of the form $a(\beta + 2\alpha + \gamma)$, for some $a \in \mathbb{Z}_{\geq 0}$. $\square$

Let G be a simple simply connected algebraic group. We now describe all the Coxeter elements $w \in W$ for which $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.

Theorem 3.4.2. *(A) Type A_n: (1) A_3: For any Coxeter element w, $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ.*

(2) $A_n, n \geq 4$: If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ and w is a Coxeter element, then w must be either $s_n s_{n-1} \ldots s_1$ or $s_i \ldots s_1 s_{i+1} \ldots s_n$ for some $1 \leq i \leq n-1$.

(B) Type B_n: (1) B_2: For any Coxeter element w, $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ.

(2) $B_n, n \geq 3$: If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ and w is a Coxeter element, then $w = s_n s_{n-1} \ldots s_1$.

(C) Type C_n: If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ and w is a Coxeter element, then $w = s_n s_{n-1} \ldots s_1$.

(D) Type D_n: (1) D_4: If w is a Coxeter element, then $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ if and only if $l(ws_2) = l(w) + 1$ and $l(ws_i) = l(w) - 1$ for exactly one $i \neq 2$.

(2) $D_n, n \geq 5$: If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ and w is a Coxeter element, then $w = s_n s_{n-1} \ldots s_1$.

(E) E_6, E_7, E_8: There is no Coxeter element w for which there exist a non-zero dominant weight χ such that $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.

(F) F_4: There is no Coxeter element w for which there exist a non-zero dominant weight χ such that $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.

(G) G_2: There is no Coxeter element w for which there exist a non-zero dominant weight χ such that $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.

Proof. By proposition (3.2.1), $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for a non-zero dominant weight χ if and only if $w\chi \leq 0$. So, using this lemma we investigate all the cases.

Proof of (A):

(1) The Coxeter elements of A_3 are precisely $s_1s_2s_3, s_1s_3s_2, s_2s_1s_3, s_3s_2s_1$. For $w = s_1s_3s_2$, take $\chi = \alpha_1 + 2\alpha_2 + \alpha_3$. Otherwise take $\chi = \alpha_1 + \alpha_2 + \alpha_3$. Then $w\chi \leq 0$.

(2) Let $n \geq 4$, and let $w\chi \leq 0$ for some dominant weight χ. By lemma (3.4.1), if $l(ws_i) = l(w) - 1$, then $i = 1$ or $i = n$.

If $l(ws_n) \neq l(w) - 1$, then using the fact that s_i commute with s_j for $j \neq i-1, i+1$, it is easy to see that $w = s_ns_{n-1}\dots s_2s_1$.

If $l(ws_n) = l(w) - 1$, then, let i be the least integer in $\{1, 2, \cdots, n-1\}$ such that $w = \phi s_{i+1}\dots s_n$, for some $\phi \in W$ with $l(w) = l(\phi) + (n-i)$. Then, we have to show that $\phi = s_is_{i-1}\dots s_1$.

If $\phi = \phi_1 s_j$ for some $j \in \{2, 3, \cdots, i-1\}$, then w is of the form

$$\begin{aligned} w &= \phi_1 s_j(s_{i+1}\dots s_{n-1}s_n) \\ &= \phi_1(s_{i+1}\dots s_{n-1}s_ns_j). \end{aligned}$$

This contradicts lemma (3.4.1). So $j \in \{1, i\}$. Again $j = i$ is not possible unless $i = 1$ by the minimality of i. Thus, we have $\phi = s_i\dots s_1$.

<u>Proof of (B):</u>

(1) For $w = s_1s_2$, take $\chi = \alpha_1 + 2\alpha_2$.

For $w = s_2s_1$, take $\chi = \alpha_1 + \alpha_2$.

(2) For $w = s_ns_{n-1}\dots s_1$, take $\chi = \alpha_1 + \alpha_2 + \dots \alpha_n$. Then $w\chi = -\alpha_n \leq 0$.

Conversely, let w be a Coxeter element and let χ be a non-zero dominant weight such that $w\chi \leq 0$. By lemma (3.4.1), if $l(ws_i) = l(w) - 1$ then either $i = 1$ or $i = n$.

If $l(ws_n) \neq l(w) - 1$, then using the fact that s_i commute with s_j for $j \neq i-1, i+1$, it is easy to see that $w = s_ns_{n-1}\dots s_2s_1$.

We now claim that $l(ws_n) = l(w) + 1$. If not, then, the coefficient of α_n in $w\chi =$ coefficient of α_n in $s_n\chi$.

Now, the coefficient of α_n in $s_n\chi$ is $2a_{n-1} - a_n$. Since $w\chi \leq 0$, we have $2a_{n-1} - a_n \leq 0$.

$$\Rightarrow 2a_{n-1} \leq a_n. \qquad \longrightarrow (1)$$

Since χ is dominant, we have $\langle \chi, \check{\alpha_{n-1}} \rangle \geq 0$. Thus, we get

$$-a_{n-2} + 2a_{n-1} - a_n \geq 0.$$

$$\Rightarrow a_{n-2} \leq 2a_{n-1} - a_n \leq 0, \text{ by (1)}.$$

So $a_{n-2} = 0$, a contradiction to the assumption that $n \geq 3$ and χ is a non-zero dominant weight. Thus $l(ws_n) = l(w) + 1$.

So the only possibility for w is $s_n s_{n-1} \ldots s_1$.

Proof of (C):

For $w = s_n s_{n-1} \ldots s_1$, take $\chi = 2(\sum_{i \neq n} \alpha_i) + \alpha_n$. Then, χ is dominant and $w\chi = -\alpha_n$.

Conversely, let w be a Coxeter element and let χ be a non-zero dominant weight such that $w\chi \leq 0$. By lemma (3.4.1), if $l(ws_i) = l(w) - 1$ then $i \in \{1, n\}$.

If $l(ws_n) \neq l(w) - 1$, then using the fact s_i commute with s_j for $j \neq i-1, i+1$, it is easy to see that $w = s_n s_{n-1} \ldots s_2 s_1$.

Claim: $l(ws_n) = l(w) + 1$.

If not, then, the coefficient of α_n in $w\chi =$ coefficient of α_n in $s_n\chi$.

Now, the coefficient of α_n in $s_n\chi$ is $a_{n-1} - a_n$. Since $w\chi \leq 0$, we have $a_{n-1} - a_n \leq 0$.

Hence, we have $a_{n-1} \leq a_n$. $\longrightarrow (2)$

Since χ is dominant, we have $\langle \chi, \check{\alpha_{n-1}} \rangle \geq 0$. Thus, we get

$$-a_{n-2} + 2a_{n-1} - 2a_n \geq 0.$$

$$\Rightarrow a_{n-2} \leq 2a_{n-1} - 2a_n \leq 0, \text{ by (2).}$$

So $a_{n-2} = 0$, a contradiction to the assumption that χ is a non-zero dominant weight. Thus $l(ws_n) = l(w) + 1$.

So the only possibility for w is $s_n s_{n-1} \ldots s_1$.

Proof of (D):

(1) The Coxeter elements in this case are precisely $s_4s_3s_2s_1$, $s_4s_1s_2s_3$, $s_3s_1s_2s_4$, $s_4s_2s_3s_1$, $s_2s_4s_3s_1$, $s_3s_2s_4s_1$, $s_4s_3s_1s_2$, $s_1s_2s_3s_4$.

For $w = s_4s_3s_2s_1$, take $\chi = 2(\alpha_1 + \alpha_2) + \alpha_3 + \alpha_4$, for $w = s_4s_1s_2s_3$, take $\chi = 2(\alpha_3 + \alpha_2) + \alpha_1 + \alpha_4$ and for $w = s_3s_1s_2s_4$, take $\chi = 2(\alpha_4 + \alpha_2) + \alpha_1 + \alpha_3$. Then $w(\chi) \leq 0$ in each of these cases. For other Coxeter elements we have either $l(ws_2) \neq l(w) + 1$ or $l(ws_i) = l(w) - 1$ for more than one $i \neq 2$. In these cases we show that there is no dominant weight χ such that $w(\chi) \leq 0$.

Assume that there exists a dominant weight of the form $\chi = \sum_{k=1}^{4} a_k\alpha_k$ and there exist $i, j \in \{1, 3, 4\}$ such that $l(ws_i) = l(w) - 1$ or $l(ws_j) = l(w) - 1$ with $w(\chi) \leq 0$. Since

$w(\chi) \leq 0$, we have $a_2 \leq a_i$ and $a_2 \leq a_j$. Then $\langle \chi, \alpha_2 \rangle < 0$, a contradiction to the fact that χ is dominant.

Now assume $l(ws_2) \neq l(w) + 1$, then $w = s_4 s_3 s_1 s_2$. Then by lemma (3.4.1)(1), the proof follows.

The converse follows from lemma (3.4.1).

(2) For $w = s_n s_{n-1} \ldots s_1$, take $\chi = 2(\sum_{i=1}^{n-2} \alpha_i) + \alpha_{n-1} + \alpha_n$. Then $w\chi \leq 0$.

Conversely, let w be a Coxeter element and let χ be a non-zero dominant weight such that $w\chi \leq 0$. By lemma (3.4.1), if $l(ws_i) = l(w) - 1$ then $i \in \{1, n-1, n\}$.

Now, if $l(ws_1) = l(w) - 1$, then, it is easy to see that $w = s_n s_{n-1} \ldots s_2 s_1$.

So, it is sufficient to prove that $l(ws_n) = l(w) + 1$ and $l(ws_{n-1}) = l(w) + 1$.

If $l(ws_n) = l(w) - 1$, then, the coefficient of α_n in $w\chi$ = coefficient of α_n in $s_n\chi = a_{n-2} - a_n$.

Since $w\chi \leq 0$, we have $a_{n-2} - a_n \leq 0$. $\longrightarrow (4)$

Since χ is dominant we have $\langle \chi, \check{\alpha_{n-2}} \rangle \geq 0$. Therefore, we have

$$2a_{n-2} \geq a_{n-1} + a_{n-3} + a_n. \qquad \longrightarrow (5)$$

Also, since $\langle \chi, \check{\alpha_{n-1}} \rangle \geq 0$ and $\langle \chi, \check{\alpha_{n-3}} \rangle \geq 0$, we have

$$2a_{n-1} - a_{n-2} \geq 0 \qquad \longrightarrow (6)$$

and

$$2a_{n-3} - a_{n-4} - a_{n-2} \geq 0. \qquad \longrightarrow (7)$$

From (5), we get

$$\begin{aligned} 4a_{n-2} &\geq 2a_{n-1} + 2a_{n-3} + 2a_n \\ &\geq a_{n-2} + (a_{n-4} + a_{n-2}) + 2a_n, \text{ from (6) and (7)} \\ &\geq 2a_{n-2} + 2a_{n-2} + a_{n-4}, \text{ by (4)} \\ &= 4a_{n-2} + a_{n-4}. \end{aligned}$$

So $a_{n-4} = 0$, a contradiction to the assumption that χ is a non-zero dominant weight. So $l(ws_n) = l(w) + 1$.

Using a similar argument, we can show that $l(ws_{n-1}) = l(w) + 1$.

Proof of (E):

Type $\mathbf{E_8}$:

Let w be a Coxeter element and let χ be a non-zero dominant weight χ such that $w\chi \leq 0$. Further, if $l(ws_i) = l(w) - 1$, then by lemma (3.4.1), $i \in \{1, 2, 8\}$.

Case 1 : $i = 8$

Co-efficient of α_8 in $w\chi =$ Co-efficient of α_8 in $s_8(\chi) = a_7 - a_8 \leq 0$.

Since χ is dominant, $\langle \chi, \check{\alpha_i} \rangle \geq 0 \ \forall \ i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$\langle \chi, \check{\alpha_7} \rangle \geq 0 \Rightarrow 2a_7 \geq a_6 + a_8 \geq a_6 + a_7$.

Hence, we have $a_7 \geq a_6$.

$$\langle \chi, \check{\alpha_6} \rangle \geq 0 \Rightarrow 2a_6 \geq a_5 + a_7 \geq a_5 + a_6$$

$$\Rightarrow a_6 \geq a_5.$$

$$\langle \chi, \check{\alpha_5} \rangle \geq 0 \Rightarrow 2a_5 \geq a_4 + a_6 \geq a_4 + a_5.$$

$$\Rightarrow a_5 \geq a_4$$

$\langle \chi, \check{\alpha_3} \rangle \geq 0 \Rightarrow 2a_3 \geq a_1 + a_4$.

$\langle \chi, \check{\alpha_2} \rangle \geq 0 \Rightarrow 2a_2 \geq a_4$.

Now, $\langle \chi, \check{\alpha_4} \rangle \geq 0 \Rightarrow 2a_4 \geq a_2 + a_3 + a_5$

$$\Rightarrow 4a_4 \geq 2a_2 + 2a_3 + 2a_5.$$

$$\geq a_4 + a_1 + a_4 + 2a_4, \text{ since } a_5 \geq a_4.$$

So, $a_1 = 0$. Thus in this case, there is no Coxeter element w for which there is a non-zero dominant weight such that $w\chi \leq 0$.

Case 2 : $i = 1$

Co-efficient of α_1 in $w\chi =$ Co-efficient of α_1 in $s_1\chi = a_3 - a_1 \leq 0$.

Since χ is dominant, we have $\langle \chi, \check{\alpha_3} \rangle \geq 0$. Therefore, $2a_3 \geq a_1 + a_4 \geq a_3 + a_4$

Hence, we have $a_3 \geq a_4$.

Since, $\langle \chi, \check{\alpha_4} \rangle \geq 0$, we have $2a_4 \geq a_3 + a_2 + a_5$.

Since, $\langle \chi, \check{\alpha_2} \rangle \geq 0$ and $\langle \chi, \check{\alpha_5} \rangle \geq 0$ we have $2a_2 \geq a_4$ and $2a_5 \geq a_4 + a_6$.

Then, $4a_4 \geq 2a_3 + 2a_2 + 2a_5 \geq 2a_4 + a_4 + a_4 + a_6$, from the above inequalities.

So, $a_6 = 0$. Hence we have $\chi = 0$. Thus, in this case also, there is no Coxeter element w for which there exist a non-zero dominant weight χ such that $w\chi \leq 0$.

$Case\ 3 : i = 2$

Co-efficient of α_2 in $w\chi$ = Co-efficient of α_2 in $s_2\chi = a_4 - a_2 \leq 0$.

Since χ is dominant, $\langle \chi, \check{\alpha_i} \rangle \geq 0 \ \forall \ i \in \{1, 2, 3, 4, 5, 6\}$.

$\langle \chi, \check{\alpha_5} \rangle \geq 0 \Rightarrow 2a_5 \geq a_4 + a_6$.

$\langle \chi, \check{\alpha_3} \rangle \geq 0 \Rightarrow 2a_3 \geq a_1 + a_4$.

$\langle \chi, \check{\alpha_4} \rangle \geq 0 \Rightarrow 2a_4 \geq a_3 + a_2 + a_5$.

So, we have $4a_4 \geq 2a_3 + 2a_2 + 2a_5$.

$$\geq (a_1 + a_4) + 2a_4 + (a_4 + a_6) = a_1 + a_6 + 4a_4.$$

$\Rightarrow a_1 + a_6 = 0$. So, $a_1 = a_6 = 0$.

Hence, we have $\chi = 0$. Thus, in this case also, there is no Coxeter element w for which there exist a non-zero dominant weight χ such that $w\chi \leq 0$.

Type $\mathbf{E_6}, \mathbf{E_7}$:

Proof is similar to the case of E_8.

Proof of F:

Let w be a Coxeter element. Let χ be a non-zero dominant weight such that $w\chi \leq 0$. If $l(ws_i) = l(w) - 1$, then $i \in \{1, 4\}$, by lemma (3.4.1).

$Case\ 1 : i = 1$

Co-efficient of α_1 in $w\chi$ = Co-efficient of α_1 in $s_1\chi = a_2 - a_1 \leq 0$.

Since χ is dominant, we have $\langle \chi, \check{\alpha_3} \rangle \geq 0$ and $\langle \chi, \check{\alpha_2} \rangle \geq 0$.

$\langle \chi, \check{\alpha_2} \rangle \geq 0 \Rightarrow 2a_2 \geq a_1 + a_3 \geq a_2 + a_3$, since $a_2 \leq a_1$.

Hence, we have $a_2 \geq a_3$.

$\langle \chi, \check{\alpha_3} \rangle \geq 0 \Rightarrow 2a_3 \geq 2a_2 + a_4 \geq 2a_3 + a_4$.

So, we have $a_4 = 0$. Hence, $\chi = 0$. Thus, in this case there is no Coxeter element w for which there exist a non-zero dominant weight χ such that $w\chi \leq 0$.

Case 2 : $i = 4$

Co-efficient of α_4 in $w\chi$ = Co-efficient of α_4 in $s_4\chi = a_3 - a_4 \leq 0$.

Since χ is dominant, we have $\langle \chi, \check{\alpha_3} \rangle \geq 0$ and $\langle \chi, \check{\alpha_2} \rangle \geq 0$.

$\langle \chi, \check{\alpha_3} \rangle \geq 0 \Rightarrow 2a_3 \geq 2a_2 + a_4 \geq 2a_2 + a_3$, since $a_3 \leq a_4$.

Hence, we have $a_3 \geq 2a_2$.

$\langle \chi, \check{\alpha_2} \rangle \geq 0 \Rightarrow 2a_2 \geq a_1 + a_3 \geq a_1 + 2a_2$.

So, we have $a_1 = 0$. Hence, $\chi = 0$. Thus, in this case also, there is no Coxeter element w for which there exist a non-zero dominant weight χ such that $w\chi \leq 0$.

<u>Proof of G</u>:

Let w be a Coxeter element and $\chi = a_1\alpha_1 + a_2\alpha_2$, be a dominant weight such that $w\chi \leq 0$

Case 1 : $l(ws_1) = l(w) - 1$.

Co-efficient of α_1 in $w\chi$ = Co-efficient of α_1 in $s_1\chi = a_2 - a_1 \leq 0$.

Since χ is dominant, we have $\langle \chi, \check{\alpha_2} \rangle \geq 0$.

$$\Rightarrow 2a_2 \geq 3a_1 \geq 3a_2.$$

So, we have $a_2 = 0$. Hence, $\chi = 0$. Thus, in this case, there is no Coxeter element w for which there exist a non-zero dominant weight χ such that $w\chi \leq 0$.

Case 2 : $l(ws_2) = l(w) - 1$.

Co-efficient of α_2 in $w\chi$ = Co-efficient of α_2 in $s_2\chi = 3a_1 - a_2 \leq 0$.

Since χ is dominant, we have $\langle \chi, \check{\alpha_1} \rangle \geq 0$.

$$\Rightarrow 2a_1 \geq a_2 \geq 3a_1.$$

So, we have $a_1 = 0$. Hence, $\chi = 0$. Thus, in this case also, there is no Coxeter element w for which there exist a non-zero dominant weight χ such that $w\chi \leq 0$. □

We now turn to the general case. Let G be a semi-simple simply connected algebraic group. Then G is of the form $G = \prod_{i=1}^{r} G_i$, for some simple simply connected algebraic groups $G_1, \cdots, G_r$. So, a maximal torus T (resp. a Borel subgroup B containing T) is of the form $\prod_{i=1}^{r} T_i$ (resp. $\prod_{i=1}^{r} B_i$), where each T_i is a maximal torus of G_i, and each B_i is a Borel subgroup of G_i containing T_i. Also the Weyl group of G with respect to T is of the form $\prod_{i=1}^{r} W_i$, where each W_i is the Weyl group of G_i with respect to T_i.

Now, let $\chi = (\chi_1, \cdots \chi_r) \in \oplus_{i=1}^{r} X(T_i)$ be a dominant weight, where $X(T_i)$ denote the group of characters of T_i. Then, clearly each χ_i is dominant. Let $w = (w_1, w_2, \cdots, w_r) \in \prod_{i=1}^{r} W_i$ be a Coxeter element of W. Then, each w_i is a Coxeter element. Then, we have;

Theorem 3.4.3. *$X(w)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$ if and only if w_i must be as in theorem (3.4.2) for all i such that χ_i is nonzero.*

Proof. Follows from theorem (3.4.2) and the fact that $w\chi \leq 0$ if and only if $w_i\chi_i \leq 0$ for all $i = 1, 2, \cdots, r$. ∎

Chapter 4

Projective Normality of GIT Quotient Varieties

This chapter reports the work done in [55, 58, 59]. In this chapter we investigate projective normality of quotient varieties modulo finite groups. In sections (4.2) and (4.3) we prove that for any finite dimensional vector space V over an algebraically closed field K, and for any finite subgroup G of $GL(V)$ which is either solvable or is generated by pseudo reflections such that $|G|$ is a unit in K, the projective variety $\mathbb{P}(V)/G$ is projectively normal with respect to the descent of $\mathcal{O}(1)^{\otimes |G|}$. In section (4.4) we prove the projective normality of $\mathbb{P}(V^m)/W$, where V^m denote m-copies of the standard representation V of the Weyl group W of a semi-simple algebraic group of type A_n, B_n, C_n, D_n, F_4 and G_2 over $\mathbb{C}$. In section (4.5) we prove a result connecting normality of an affine semigroup and the EGZ-theorem.

4.1 Introduction

Projective Normality: A projective variety X is said to be projectively normal if the affine cone $\hat{X}$ over X is normal at its vertex, i.e., the stalk at the vertex is a normal domain. Projective normality depend on the particular projective embedding of the variety (unlike the affine varieties) as the following example shows.

Example: The projective line $\mathbb{P}^1$ is obviously projectively normal since its cone is the affine plane K^2 (which is non-singular). However, it can be also embedded in $\mathbb{P}^3$ as the quartic curve, namely,

$$V_+ = \{(a^4, a^3b, ab^3, b^4) \in \mathbb{P}^3 : (a, b) \in \mathbb{P}^1\},$$

i.e., $V_+ = V_+(XT - YZ, TY^2 - XZ^2)$, but the coordinate ring of its cone V which is $K[X, Y, Z, T]/(XT - YZ, TY^2 - XZ^2)$ is not normal.

Remark: Let $\mathcal{L}$ be a very ample line bundle on a projective variety X. Then the polarized variety $(X, \mathcal{L})$ is projectively normal if the natural map $Sym^m H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L}^m)$ is surjective for all $m \geq 0$ (see [39, Ch. II, Ex. 5.14]).

Let G be a finite group. Let V be a finite dimensional representation of G over a field K. In 1916, E. Noether proved that if characteristic of K does not divide $|G|$, then the K-algebra of invariants $K[V]^G$ is finitely generated. In 1926, she proved that the same result holds in all characteristics (Th. 2.2.1). So, when K is algebraically closed, it is an interesting problem to study quotient varieties $V/G = Spec(K[V]^G)$ and $\mathbb{P}(V)/G$. Also, $\forall x \in \mathbb{P}(V)$, the isotropy G_x acts trivially on the fiber of the line bundle $\mathcal{O}(1)^{\otimes|G|}$ at x. Hence, by a descent lemma of Kempf (see [29]), when $|G|$ is a unit in K, the line bundle $\mathcal{O}(1)^{\otimes|G|}$ descends to the quotient $\mathbb{P}(V)/G$, where $\mathcal{O}(1)$ denotes the ample generator of the Picard group of $\mathbb{P}(V)$. Let us denote it by $\mathcal{L}$. On the other hand, V/G is normal. So, it is a natural question to ask if $\mathbb{P}(V)/G$ is projectively normal with respect to the line bundle $\mathcal{L}$. More generally, an interesting question is the following:-

Question: Let G be a finite group acting on a projectively normal polarized variety $(X, \mathcal{O}(1))$, where $\mathcal{O}(1)$ is G-linearized very ample line bundle making $X \subset \mathbb{P}(V)$ projectively normal and $\mathcal{L} \in Pic(X/G)$ is the descent of $\mathcal{O}(1)^{\otimes|G|}$ on X. Is the polarized variety $(X/G, \mathcal{L})$ projectively normal ?

Here we give an affirmative answer to this question in many cases.

4.2 Solvable Case

In this section, we prove projective normality of the quotient variety $\mathbb{P}(V)/G$ when the group G is solvable. We begin the section with the celebrated theorem in additive number theory due to Erdös, Ginzburg and Ziv.

Erdös-Ginzburg-Ziv Theorem [31]: Let $n \geq 1$ and $a_1, \ldots, a_{2n-1} \in \mathbb{Z}$. Then there exist $i_1, \ldots, i_n$ such that: $a_{i_1} + \cdots + a_{i_n} \equiv 0 mod n$.

Proposition 4.2.1. *Let G be a finite solvable group, and let V be a finite dimensional faithful representation of G over a field K of characteristic not dividing $|G|$. Let $m = |G|$, $R := \oplus_{d\geq 0} R_d$; $R_d := (Sym^{dm}V^*)^G$. Then R is generated as a K-algebra by R_1.*

Proof. Step 1: We first prove the statement when G is cyclic of order m. Let ξ be a primitive m^{th} root of unity in an algebraic closure $\bar{K}$ of K. Let $F = K(\xi)$. Since F is a free K module, we have $V^G \otimes_K F = (V \otimes_K F)^G$. Hence, we may assume that $\xi \in K$.

Let $G =< g >$. Write $V = \oplus_{i=0}^{m-1} V_i$ where $V_i := \{v \in V : g.v = \xi^i.v\}$, $0 \leq i \leq m-1$. Now, let $f \in R_d$ be of the form $f = X_0 \cdot X_1 \cdots X_{m-1}$ with $X_i = X_{i,1}.X_{i,2} \cdots X_{i,a_i} \in Sym^{a_i} V_i$, where $X_{i,j} \in V_i$ such that $\sum_{i=0}^{m-1} a_i = dm$. Since f is G-invariant we have

$$\sum_{i=0}^{m-1} i.a_i \equiv 0 \mod m$$

If $d = 1$, $f \in R_1$; so we may assume that $d \geq 2$. Now, consider the sequence of integers

$$\underbrace{0, \ldots, 0}_{a_0 \text{ times}}, \ \underbrace{1, \ldots, 1}_{a_1 \text{ times}}, \ \cdots, \ \underbrace{m-1, \ldots, m-1}_{a_{m-1} \text{ times}}$$

Since the sequence has dm terms and $d \geq 2$, by a theorem of Erdös-Ginzburg-Ziv (see [31]), there is a subsequence with exactly m-terms whose terms add up to a multiple of m. Thus there exist $f_1 \in R_1$ and $f_2 \in R_{d-1}$ such that $f = f_1.f_2$. Hence the proof follows by induction on $deg(f)$.

Step 2: Now we assume that G is any finite solvable group of order m. We use induction on m to prove the statement. We may assume that m is not a prime number. Since G is solvable it has a normal subgroup H such that G/H is a cyclic group of prime order.

Let $W := (Sym^{|H|}V)^H$. Since H is a normal subgroup of G, both $\underbrace{W \otimes \ldots \otimes W}_{d|G/H| \text{ copies}}$ and $(Sym^{d|H|}V)^H$ have natural G/H-module structures. Let $G_1 = G/H$. Since $|H| < |G|$, by induction, the homomorphism $\underbrace{W \otimes \ldots \otimes W}_{d|G_1| \text{ copies}} \longrightarrow (Sym^{d|G|}V)^H$ is surjective. $\cdots (1)$

Claim: The natural map $(Sym^{d|G_1|}W)^{G_1} \longrightarrow (Sym^{d|G|}V)^G$ is surjective. $\ldots (2)$

The surjectivity of the natural map $Sym^{d.|G_1|}W \longrightarrow (Sym^{d|G|}V)^H$ of G_1-modules follows from (1) and the following commutative diagram

$$\begin{tikzcd}
\otimes^{d|G_1|}W \arrow[r] \arrow[d] & (Sym^{d|G|}V)^H \\
Sym^{d.|G_1|}W \arrow[ur] &
\end{tikzcd}$$

Hence applying Reynold's operator we have the claim.

Now, consider the commutative diagram:

$$\begin{tikzcd}
\otimes^d (Sym^{|G_1|}W)^{G_1} \arrow[r] \arrow[d] & (Sym^{d.|G_1|}W)^{G_1} \arrow[d] \\
\otimes^d (Sym^{|G|}V)^G \arrow[r] & (Sym^{d.|G|}V)^G
\end{tikzcd}$$

The first horizontal map is surjective by step (1) and the second vertical map is surjective by (2). Thus the second horizontal map is surjective. Thus the proposition follows. $\square$

Theorem 4.2.2. *Let G be a finite solvable group, and let V be a finite dimensional faithful representation of G over a field K of characteristic not dividing $|G|$. Then, the polarized variety $(\mathbb{P}(V)/G, \mathcal{L})$ is projectively normal.*

Proof. The polarized variety $(\mathbb{P}(V)/G, \mathcal{L})$ is $Proj(\oplus_{d\in\mathbb{Z}_{\geq 0}}(H^0(\mathbb{P}(V), \mathcal{O}(1)^{\otimes d|G|})^G)$ which is same as $Proj(\oplus_{d\in\mathbb{Z}_{\geq 0}}(Sym^{d|G|}V^*)^G)$. Let $R := \oplus_{d\geq 0}R_d$; $R_d := (Sym^{dn}V^*)^G$. By proposition (4.2.1), the map $Sym^d R_1 \to R_d$ is surjective. So the result follows from the remark in the last section. $\square$

4.3 Group Generated by Pseudo Reflections

In this section, we prove the projective normality of the quotient variety $\mathbb{P}(V)/G$ when the group G is generated by pseudo reflections. First we prove a combinatorial lemma which will be used frequently in this chapter.

Let $\underline{a}$=$(a_1, a_2, \cdots a_r) \in \mathbb{N}^r$ and $N_{\underline{a}} = \prod_{i=1}^r a_i$. Consider the semigroup

$M_{\underline{a}} = \{(m_1, m_2, \cdots m_r) \in \mathbb{Z}^r_{\geq 0} : \sum_{i=1}^r m_i a_i \equiv 0 \bmod N_{\underline{a}}\}$ and the set

$S_{\underline{a}} = \{(m_1, m_2, \cdots m_r) \in \mathbb{Z}^r_{\geq 0} : \sum_{i=1}^r m_i a_i = N_{\underline{a}}\}$.

Lemma 4.3.1. *$M_{\underline{a}}$ is generated by $S_{\underline{a}}$ for $\underline{a} \in \mathbb{N}^r$.*

Proof. Suppose $(m_1, m_2, \cdots m_r) \in \mathbb{Z}^r_{\geq 0}$ such that:

$$\sum_{i=1}^r m_i.a_i = q.N_{\underline{a}}, \text{with } q \geq 2.$$

Let $n = q.N_{\underline{a}}$. Choose any $n \times r$ matrix
$A = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1r} \\ x_{21} & x_{22} & \cdots & x_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nr} \end{pmatrix}$ with entries $x_{i,j}$'s in $\{0, 1\}$ such that each row sum $\sum_{j=1}^r x_{i,j}$ is equal to 1 and for each $j = 1, 2, \cdots, r$, the j^{th} column sum $\sum_{i=1}^n x_{i,j}$ is $m_j a_j$.

Since $q \geq 2$ we have $n \geq 2.a_1$. Therefore, the sequence $\{x_{11}, x_{21}, \cdots, x_{n1}\}$ has atleast $2a_1$ number of terms. Hence, applying the theorem of Erdös-Ginzburg-Ziv repeatedly (see [31]) and rearranging the rows if necessary, we can assume that the n-terms of the sequence can be partitioned into $\frac{n}{a_1}$ number of subsequences $\{x_{11}, x_{21}, \cdots, x_{a_1 1}\}$, $\{x_{(a_1+1)1}, x_{(a_1+2)1}, \cdots, x_{2a_1 1}\}$, $\cdots$, $\{x_{(n-a_1+1)1}, x_{(n-a_1+2)1}, \cdots, x_{n1}\}$, each of length a_1 and sum of terms of each of these subsequences is a multiple of a_1.

Again, consider the sequence $\{\sum_{i=1}^{a_1} x_{i2}, \sum_{i=a_1+1}^{2a_1} x_{i2}, \cdots, \sum_{i=n-a_1+1}^{n} x_{i2}\}$. Using the same argument as above we can assume that this sequence can be partitioned into $\frac{n}{a_1 a_2}$ number of subsequences each of length a_2 and sum of terms of each subsequence is a multiple of a_2.

Proceeding in this way, we can see that for each $j = 1, 2, \cdots, r$, the sum of the first $N_{\underline{a}}$-terms in the jth column of the matrix A is a multiple of a_j.

Let $b_j = \sum_{i=1}^{N_{\underline{a}}} x_{ij}$. By construction of the x_{ij}'s , b_j is a multiple of a_j for every $j = 1, 2, \cdots, r$. So, for each $j = 1, 2, \cdots, r$, write $b_j = a_j c_j$, with $c_j \in Z_{\geq 0}$. Now, since $\sum_{j=1}^{r} x_{i,j} = 1, \ \forall \ i = 1, 2, \cdots, n$, we have $\sum_{j=1}^{r} b_j = N_{\underline{a}}$. Hence, $(c_1, c_2, \cdots, c_r) \in S_{\underline{a}}$. As $m_j a_j = \sum_{i=1}^{n} x_{ij}$, $b_j \leq m_j$ for every $j = 1, 2, \cdots, r$. Thus, we have $(m_1, m_2, \cdots, m_r) = (c_1, c_2, \cdots, c_r) + (m_1 - c_1, m_2 - c_2, \cdots, m_r - c_r)$, with $(c_1, c_2, \cdots, c_r) \in S_{\underline{a}}$. So, the lemma follows by induction on q. $\square$

Corollary 4.3.2. *Let V be a finite dimensional vector space over a field K. Let G be a finite subgroup of $GL(V)$ which is generated by pseudo reflections. Further assume that characteristic of K does not divide $|G|$. Let $R_d := (Sym^{d.|G|}(V^*))^G$. Then $R = \oplus_{d \in \mathbb{Z}_{\geq 0}} R_d$ is generated by R_1.*

Proof. By a theorem of Chevalley-Serre-Shephard-Todd (Th. 2.5.3), $(Sym(V^*))^G$ is a polynomial ring $K[f_1, f_2, \cdots, f_r]$ with each f_i is a homogeneous polynomial of degree a_i and $\prod_{i=1}^{r} a_i = |G|$. Thus, proof follows from lemma (4.3.1). $\square$

Theorem 4.3.3. *Let V be a finite dimensional vector space over a field K. Let G be a finite subgroup of GL(V) which is generated by pseudo reflections. Further assume that characteristic of K does not divide $|G|$, then the polarized variety $(\mathbb{P}(V)/G, \mathcal{L})$ is projectively normal.*

Proof. Proof follows from corollary (4.3.2). $\square$

4.4 Vector Invariants and Projective Normality

Let G be a semi-simple algebraic group of rank n over $\mathbb{C}$. Let T be a maximal torus of G. Let $N_G(T)$ be the normaliser of T in G and let $W = N_G(T)/T$ be the Weyl group of G with respect to T. Consider the standard representation $V = Lie(T)$ of W. For every integer $m \geq 1$, the group W acts on the algebra $\mathbb{C}[V^m]$ of polynomial functions on the direct sum $V^m := V \oplus \cdots \oplus V$ of m copies of V via the diagonal action

$$(wf)(v_1, \cdots, v_m) := f(w^{-1}v_1, \cdots, w^{-1}v_m), f \in \mathbb{C}[V^m], w \in W.$$

If $m = 1$ then the algebra $\mathbb{C}[V]^W$ of invariants in one vector variable is generated by n algebraically independent homogeneous invariants $f_1, f_2, \cdots, f_n$ of degrees $d_1, d_2, \cdots, d_n$ respectively such that $\prod_{i=1}^{n} d_i = |W|$ by a theorem of Chevalley-Serre-Shephard-Todd (Th. 2.5.3). We will refer to such a system of generators of $\mathbb{C}[V]^W$ as a system of basic invariants. Explicit systems of basic invariants are known for each type of irreducible Weyl groups W (see appendix-B).

Theorem 4.4.1. *Let G be a semi-simple algebraic group of type A_n, B_n, C_n, D_n, F_4 or G_2. Let W denote the corresponding Weyl group. Let V be the standard representation of W. Then $\mathbb{P}(V^m)/W$ is projectively normal with respect to the line bundle $\mathcal{O}(1)^{\otimes |W|}$.*

Proof. By a theorem of Chevalley-Serre-Shephard-Todd (Th. 2.5.3), the $\mathbb{C}$-algebra $\mathbb{C}[V]^W = (Sym(V^*))^W$ is a polynomial ring $\mathbb{C}[f_1, f_2, \cdots, f_n]$ with each f_i is a homogeneous polynomial of degree d_i and $\prod_{i=1}^n d_i = |W|$.

Let $R := \oplus_{q\geq 0} R_q$; where $R_q := (Sym^{q|W|}V^{*m})^W$. Since the $\mathbb{C}$-algebra R is integrally closed, so to prove our claim, it is enough to prove that it is generated by R_1. We prove this dealing case by case.

Type $\mathbf{A_n, B_n, C_n}$:

We first note that the Weyl groups of type B_n and C_n are same, and the root systems are dual to each other. Hence, the standard representation V for B_n and the standard representation V' for C_n are also the same. So, here we need to deal with cases A_n and B_n only.

For the diagonal action of the Weyl group on V^m, in type A_n by H. Weyl and in type B_n, by a theorem of Wallach, the algebra $\mathbb{C}[V^m]^W$ is generated by polarizations of the system of basic invariants $f_1, f_2, \cdots, f_n$ (see Th. 2.9.2).

For each $i \in \{1, 2, \cdots, n\}$, let $\{f_{ij} : j = 1, 2, \cdots a_i\}$ denote the polarizations of f_i where a_i is a positive integer. Since the polarization operators $D_{ij} = \sum_{k=1}^n x_{ik} \frac{\partial}{\partial x_{jk}}$ do not change the total degree of the original polynomial, we have

$$\text{degree of } f_{ij} = \text{degree of } f_i = d_i, \ \forall\ j = 1, 2, \cdots a_i. \tag{4.1}$$

Let us take an invariant polynomial $f \in (Sym^{q|W|}(V^m))^W$, where $q > 1$. Since f_{ij}'s generate $\mathbb{C}[V^m]^W$ with out loss of generality we can assume f is a monomial of the form $\prod_{i=1}^n \prod_{j=1}^{a_i} f_{ij}^{m_{ij}}$.

Since $f = \prod_{i=1}^n \prod_{j=1}^{a_i} f_{ij}^{m_{ij}} \in (Sym^{q|W|}(V^m))^W$, we have

$$\sum_{i=1}^n \sum_{j=1}^{a_i} m_{ij} d_i = q|W| = q(\prod_{i=1}^n d_i)$$

Let $m_i = \sum_{j=1}^{a_i} m_{ij}$ then we have $\sum_{i=1}^n m_i d_i = q(\prod_{i=1}^n d_i)$, and hence $(m_1, m_2, \cdots, m_n)$ is in the semigroup $M_{\underline{d}} = \{(m_1, m_2, \cdots m_n) \in \mathbb{Z}^n_{\geq 0} : \sum_{i=1}^r m_i d_i \equiv 0 \bmod N\}$.

By lemma (4.3.1), the semigroup $M_{\underline{d}}$ is generated by the set $S_{\underline{d}} = \{(m_1, m_2, \cdots m_n) \in \mathbb{Z}^n_{\geq 0} : \sum_{i=1}^n m_i d_i = \prod_{i=1}^n d_i\}$. So there exists $(m'_1, m'_2, \cdots m'_n) \in \mathbb{Z}^n_{\geq 0}$ such that for each i

$$m'_i < m_i \text{ and } \sum_{i=1}^n m'_i d_i = \prod_{i=1}^n d_i.$$

Again, since $m'_i < m_i = \sum_{j=1}^{a_i} m_{ij}$, for each i and j there exists $m'_{ij} \leq m_{ij}$ such that

$$m'_i = \sum_{j=1}^{a_i} m'_{ij}.$$

Then $g = \prod_{i=1}^{n} \prod_{j=1}^{a_i} f_{ij}^{m'_{ij}}$ is W-invariant and is in $(Sym^{|W|}(V^m))^W$.

Let $f' = \frac{f}{g}$. Then $f' \in (Sym^{(q-1)|W|}(V^m))^W$ and so by induction on q, f' is in the subalgebra generated by $(Sym^{|W|}(V^m))^W$.

Hence $f = g.f'$ is in the subalgebra generated by $(Sym^{|W|}(V^m))^W$.

Type $\mathbf{D_n}$:

Before proving the theorem for this case let us recall the action of the Weyl group of type B_n and D_n on the Euclidean space $\mathbb{R}^n$. Let W and W' denote the Weyl group of type D_n and B_n respectively. Then W' acts on $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ by permutation of $x_1, x_2, \cdots, x_n$ and the sign changes $x_i \to -x_i$ and the group W acts on x by permuting the coordinates and changes an even number of signs. Then it is clear that the group W' is generated by the group W and a reflection σ defined by

$$\sigma(x_1, x_2, \cdots, x_{n-1}, x_n) = (x_1, x_2, \cdots, x_{n-1}, -x_n).$$

We can take the polynomials

$$f_i = \sum_{k=1}^{n} x_k^{2i}, \ i = 1, 2, \cdots, n-1$$

and
$$f_n = x_1.x_2 \ldots x_n$$
for the basic invariants of $\mathbb{C}[V]^W$ (see appendix-B).

For $\mathbb{C}[V]^{W'}$ we can take the basic invariants the polynomials

$$f_i = \sum_{k=1}^{n} x_k^{2i}, \ i = 1, 2, \cdots, n-1$$

and
$$f'_n = \textstyle\sum_{k=1}^{n} x_k^{2n}.$$

For odd $r \geq 1$, define the operator

$$P_r := \sum_{k=1}^{n} x_{2k}^{r} \frac{\partial}{\partial x_{1k}},$$

where x_{1k}, x_{2k} are standard coordinates of $\mathbb{R}^{2n}$. The operator P_r commutes with the diagonal action of W and W' on $\mathbb{C}[V^2]$ and preserves $\mathbb{C}[V^2]^W$.

Now by theorem (2.9.3) the algebra $C[V^2]^W$ is generated by the polarizations of the basic invariants $f_1, f_2, \cdots, f_n$ and the polynomials

$$P_{r_1} \cdots P_{r_l}(f_n) \ \ (r_i \geq 1 \ \text{ odd}, \ \sum_{i=1}^{l} r_i \leq n-l).$$

Note that the degree of the polynomial $P_r(f_n)$ is $n+r-1$ and so the degrees of the polynomials $P_{r_1}\cdots P_{r_l}(f_n),\ (r_i \geq 1\ \text{ odd},\ \sum_{i=1}^{l} r_i \leq n-l)$ are

$$n + (r_1 + r_2 + \ldots + r_l) - l \leq 2n-2.$$

So $\mathbb{C}[V^2]^W$ is generated by homogeneous polynomials of degree $\leq 2n-2$.

Now we will prove the theorem for type D_n by dealing with two cases.

<u>Case -1:</u> n is even

In this case note that the degrees of the basic invariants $f_1, f_2, \cdots, f_n$ are all even. So the degrees of the polynomials $P_{r_1}\cdots P_{r_l}(f_n),\ (r_i \geq 1\ \text{ odd},\ \sum_{i=1}^{l} r_i \leq n-l)$ are all even. Since the polarizations of the basic invariants have the same degrees as the basic invariants, we conclude that in this case the algebra $\mathbb{C}[V^2]^W$ is generated by homogeneous polynomials of even degrees less than or equal to $2n-2$.

Now for $m > 2$, by theorem (2.9.4), the algebra $\mathbb{C}[V^m]^W$ is generated by the polarizations of the generators of $\mathbb{C}[V^2]^W$. Again since the polarization operators do not change the degree of the original polynomial we conclude that the algebra $\mathbb{C}[V^m]^W$ is generated by homogeneous polynomials of even degrees same as the degrees of the basic invariants. So in this case we can employ the same proof as in the case of type A_n, B_n and C_n.

<u>Case -2:</u> n is odd

In this case since the degree of the basic invariant f_n is odd and r_i's are all odd, we have degrees of all the polynomials $P_{r_1}\cdots P_{r_l}(f_n),\ (r_i \geq 1\ \text{ odd},\ \sum_{i=1}^{l} r_i \leq n-l)$ are odd.

Again, since for $m > 2$, the algebra $\mathbb{C}[V^m]^W$ is generated by the polarizations of the generators of $\mathbb{C}[V^2]^W$, among the generators of $\mathbb{C}[V^m]^W$ we have some odd degree invariants as well which are not necessarily having the same degrees as the degree of f_n.

Now, let us take one odd degree invariant $f \in \mathbb{C}[V^m]^W$ and write

$$f = \frac{f - \sigma(f)}{2} + \frac{f + \sigma(f)}{2}$$

where σ is the reflection $(x_1, x_2, \cdots, x_{n-1}, x_n) \to (x_1, x_2, \cdots, x_{n-1}, -x_n)$ defined before.

Since W is a normal subgroup of the Weyl group W' of type B_n and W' is generated by W and σ, we have

$$\frac{f + \sigma(f)}{2} \in \mathbb{C}[V^m]^{W'}$$

Again, since f is homogeneous of odd degree, the degree of $\frac{f+\sigma(f)}{2}$ is odd and hence $\frac{f+\sigma(f)}{2}$ is equal to 0 since $\mathbb{C}[V^m]^{W'}$ is generated by polarizations of the basic invariants $f_1, f_2, \cdots, f_{n-1}, f_n'$ which are all of even degrees. Hence, for an odd degree invariant $f \in \mathbb{C}[V^m]^W$, we have

$$\sigma(f) = -f.$$

So for any W-invariant polynomials f and g of odd degrees we have $\sigma(f.g) = fg$ and hence we conclude that f^2, g^2 and fg are in $\mathbb{C}[V^m]^{W'}$.

Now let us take a typical invariant monomial

$$f = (\prod_i \prod_{j=1}^{a_i} f_{ij}^{m_{ij}}) h_1^{l_1} h_2^{l_2} \dots h_p^{l_p} \in (Sym^{q|W|} V^m)^W$$

where f_{ij}'s $\in \mathbb{C}[V^m]^W$ are the even degree invariants of degrees $d_1, d_2, \cdots, d_{n-1}$ obtained by taking the polarizations of the even degree generators of $\mathbb{C}[V^2]^W$ and h_i's $\in \mathbb{C}[V^m]^W$ are the odd degree invariants obtained by taking the polarizations of the odd degree generators of $\mathbb{C}[V^2]^W$.

Again since h_i^2 and $h_i.h_j$ are in $\mathbb{C}[V^m]^{W'}$, they are polynomials in $f_{i,j}$'s and the polarizations of the even degree basic invariant f'_n. So we may assume that $\sum_{i=1}^p l_i = 0$ or 1.

Suppose $\sum_{i=1}^p l_i = 1$, then f is of the form

$$f = (\prod_i \prod_{j=1}^{a_i} f_{ij}^{m_{ij}}).h \in \mathbb{C}[V^m]^W,$$

where h is of odd degree, say t. So we have

$$\sum_i \sum_{j=1}^{a_i} m_{ij} d_i + t = q.|W|.$$

This is not possible since d_i's are all even and $|W|$ is even. So we conclude that $\sum_{i=1}^p l_i = 0$ and hence f is of the form $g_1^{m_1} g_2^{m_2} \dots g_r^{m_r}$ where g_i's are all of even degrees less than equal to $2n$. So in this case we can proceed with the proof as in the case of Type A_n, B_n and C_n.

Type$\mathbf{F_4}$ and $\mathbf{G_2}$:

Since the cardinality of the Weyl group of Type G_2 is $12 = 2^2.3$ and the cardinality of the Weyl group of Type F_4 is $1152 = 2^7.3^2$, by Burnside's $p^a q^b$ theorem (see [95, pg. 247]), they are solvable. Hence the result is true for each case by theorem (4.2.2). □

Remark: Although the reflection groups of type I are not Weyl group, they are solvable. So by theorem (4.2.2), the projective normality holds for these groups.

We deduce the following result of Chu-Hu-Kang (see [17]) as a consequence of the above theorem.

Corollary 4.4.2. *Let G be a finite group of order n and U be any finite dimensional representation of G over $\mathbb{C}$. Let $\mathcal{L}$ denote the descent of $\mathcal{O}(1)^{\otimes n!}$. Then $\mathbb{P}(U)/G$ is projectively normal with respect to $\mathcal{L}$.*

Proof. Let $G = \{g_1, g_2, \cdots, g_n\}$ and let $\{u_1, u_2, \cdots, u_k\}$ be a basis of U. Let V be the natural representation of the permutation group S_n. Let $\{x_1, x_2, \cdots, x_n\}$ be a basis of V; then the set $\{x_{11}, \cdots, x_{n1}, \cdots, x_{1k}, \cdots, x_{nk}\}$ is a basis of V^k.

Consider the Cayley embedding $G \hookrightarrow S_n$, $g \mapsto (g_j := gg_i)$. Then

$$\eta : Sym(V^k) \to Sym(U), \quad x_{il} \mapsto g_i(u_l)$$

is a G-equivariant and degree preserving algebra epimorphism.

Now we will use Noether's original argument (see [89]) to show that the restriction map

$$\tilde{\eta} : Sym(V^k)^{S_n} \to Sym(U)^G$$

is surjective. For any $f = f(u_1, \cdots, u_k) \in Sym(U)^G$, we define

$$f' := \frac{1}{n}(f(x_{11}, x_{12}, \cdots, x_{1k}) + \ldots + f((x_{n1}, x_{n2}, \cdots, x_{nk})) \in Sym(V^k)^{S_n}.$$

Then we have

$$\tilde{\eta}(f') = \frac{1}{n}(f(g_1(u_1), g_1(u_2), \cdots, g_1(u_k)) + \ldots + f(g_n(u_1), g_n(u_2), \cdots, g_n(u_k)))$$

$$= \frac{1}{n}(g_1 f(u_1, u_2, \cdots, u_k) + \ldots g_n f(u_1, u_2, \ldots, u_k)) = f$$

Hence, $\tilde{\eta}(f') = f$ and $\tilde{\eta}$ is surjective. So the corollary follows from theorem (4.4.1). $\square$

4.5 Normality, Projective Normality and EGZ Theorem

Let V be a finite dimensional representation of a finite cyclic group G over the field of complex numbers $\mathbb{C}$. Let $\mathcal{L}$ denote the descent of the line bundle $\mathcal{O}(1)^{\otimes |G|}$ to the GIT quotient $\mathbb{P}(V)/G$. By theorem (4.2.2), the polarized variety $(\mathbb{P}(V)/G, \mathcal{L})$ is projectively normal. Proof of this uses the well known arithmetic result due to Erdös-Ginzburg-Ziv (see [31]).

In this section, we prove that the projective normality of $(\mathbb{P}(V)/G, \mathcal{L})$, the Erdös-Ginzburg-Ziv theorem and normality of an affine semigroup are all equivalent.

4.5.1 Normality of a Semigroup

An affine semigroup M is a finitely generated sub-semigroup of $\mathbb{Z}^n$ containing 0 for some n. Let N be the subgroup of $\mathbb{Z}^n$ generated by M. Then, M is called normal if it satisfies the following condition: if $kx \in M$ for some $x \in N$ and $k \in \mathbb{N}$, then $x \in M$.

For an affine semigroup M and a field K we can form the affine semigroup algebra $K[M]$ in the following way: as a K vector space $K[M]$ has a basis consisting of the symbols X^a, $a \in M$, and the multiplication on $K[M]$ is defined by the K-bilinear extension of $X^a.X^b = X^{a+b}$.

We recall the following theorem from [6, Th. 4.40].

Theorem 4.5.1. *Let M be an affine semigroup, and K be a field. Then M is normal if and only if $K[M]$ is normal, i.e., it is integrally closed in its field of fractions.*

4.5.2 A Result connecting a Normal Semigroup and the EGZ Theorem

Theorem 4.5.2. *The following are equivalent*
1. *Erdös-Ginzburg-Ziv theorem: Let $(a_1, a_2, \cdots, a_m), m \geq 2n-1$ be a sequence of elements of $\mathbb{Z}/n\mathbb{Z}$. Then there exists a subsequence $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ of length n whose sum is zero.*
2. *Let G be a cyclic group of order n and V be any finite dimensional representation of G over $\mathbb{C}$. Let $\mathcal{L}$ be the descent of $\mathcal{O}(1)^{\otimes n}$. Then $(\mathbb{P}(V)/G, \mathcal{L})$ is projectively normal.*
$2'$. *Let G be a cyclic group of order n and V be the regular representation of G over $\mathbb{C}$. Let $\mathcal{L}$ be the descent of $\mathcal{O}(1)^{\otimes n}$. Then $(\mathbb{P}(V)/G, \mathcal{L})$ is projectively normal.*
3. *The sub-semigroup M of $\mathbb{Z}^n$ generated by the set $S = \{(m_0, m_1, \cdots, m_{n-1}) \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=0}^{n-1} m_i = n \text{ and } \sum_{i=0}^{n-1} i m_i \equiv 0 \text{ mod } n\}$ is normal.*

Proof. We first prove, (1), (2) and $(2')$ are equivalent.

$(1) \Rightarrow (2)$

Proof follows from the arguments given in proposition (4.2.1).

$(2) \Rightarrow (2')$

Proof is straightforward.

$(2') \Rightarrow (1)$

Let $G = \mathbb{Z}/n\mathbb{Z} =< g >$ and let V be the regular representation of G over $\mathbb{C}$. Let ξ be a primitive nth root of unity. Let $\{X_i : i = 0, 1, \cdots, n-1\}$ be a basis of V^* given by:

$$g.X_i = \xi^i X_i, \text{ for every } i = 0, 1, \cdots, n-1.$$

By assumption the algebra $\oplus_{d \in \mathbb{Z}_{\geq 0}} (Sym^{dn} V^*)^G$ is generated by $(Sym^n V^*)^G$ (*)

Let $(a_1, a_2, \cdots, a_m), m \geq 2n-1$ be a sequence of elements of G. Consider the subsequence $(a_1, a_2, \cdots, a_{2n-1})$ of length $2n-1$.

Take $a = -(\sum_{i=1}^{2n-1} a_i)$. Then $(\prod_{i=1}^{2n-1} X_{a_i}).X_a$ is a G-invariant monomial of degree $2n$, i.e., $(\prod_{i=1}^{2n-1} X_{a_i}).X_a \in (Sym^{2n} V^*)^G$.

By (*), there exists a subsequence $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ of $(a_1, a_2, \cdots, a_{2n-1}, a)$ of length n such that $\prod_{j=1}^{n} X_{a_{i_j}}$ is G-invariant. So, $\sum_{j=1}^{n} a_{i_j} = 0$. Thus, we have the implication.

We now prove $(1) \Rightarrow (3)$ and $(3) \Rightarrow (2')$, which completes the proof of the theorem.

$(1) \Rightarrow (3)$

Let N be the subgroup of $\mathbb{Z}^n$ generated by M. Suppose $q(m_0, m_1, \cdots, m_{n-1}) \in M$, $q \in \mathbb{N}$ and $(m_0, m_1, \cdots, m_{n-1}) \in N$. We need to prove that $(m_0, m_1, \cdots, m_{n-1}) \in M$.

Since $q(m_0, m_1, \cdots, m_{n-1}) \in M$ we have $q.m_i \geq 0 \; \forall \; i$. Hence, $m_i \geq 0 \; \forall \; i$. Since N is the subgroup of $\mathbb{Z}^n$ generated by M and M is the sub-semigroup of $\mathbb{Z}^n$ generated by S, N is generated by S as a subgroup of $\mathbb{Z}^n$. Therefore, the tuple $(m_0, m_1, \cdots, m_{n-1})$ is an integral (not necessarily non-negative) linear combination of elements of S, i.e.,

$$(m_0, m_1, \cdots, m_{n-1}) = \sum_{j=1}^{p} a_j (m_{0,j}, m_{1,j}, \cdots, m_{(n-1),j}),$$

where $a_j \in \mathbb{Z}$ for all $j = 1, 2, \cdots, p$ and $(m_{0,j}, m_{1,j}, \cdots, m_{(n-1),j}) \in S$. Therefore,

$$\sum_{i=0}^{n-1} m_i = \sum_{i=0}^{n-1} \sum_{j=1}^{p} a_j m_{ij} = (\sum_{j=1}^{p} a_j (\sum_{i=0}^{n-1} m_{i,j})) = (\sum_{j=1}^{p} a_j) n = kn$$

for some $k \in \mathbb{Z}$. Moreover $k \geq 0$, since $m_i \geq 0 \; \forall \; i$.

If $k = 1$ then $\sum_{i=0}^{n-1} m_i = n$ and hence, $(m_0, m_1, \cdots, m_{n-1}) \in M$. Otherwise $k \geq 2$ and consider the sequence of integers

$$\underbrace{0, \ldots, 0}_{m_0 \text{ times}}, \; \underbrace{1, \ldots, 1}_{m_1 \text{ times}}, \; \cdots, \; \underbrace{n-1, \ldots, n-1}_{m_{n-1} \text{ times}}$$

This sequence has atleast $2n$ terms, since $\sum_{i=0}^{n-1} m_i = kn, \; k \geq 2$ and the sum of it's terms is divisible by n by the assumption that $\sum_{i=0}^{n-1} i m_i \equiv 0 \; mod \; n$. So by (1) there exists a subsequence of exactly n terms whose sum is a multiple of n, i.e., there exists $(m'_0, m'_1, \cdots, m'_{n-1}) \in \mathbb{Z}^n_{\geq 0}$ with $m'_i \leq m_i, \; \forall \; i$ such that $\sum_{i=0}^{n-1} m'_i = n$ and $\sum_{i=0}^{n-1} i m'_i$ is a multiple of n. So $(m'_0, m'_1, \cdots, m'_{n-1}) \in M$. Then, by induction $(m_0, m_1, \cdots, m_{n-1}) - (m'_0, m'_1, \cdots, m'_{n-1}) \in M$ and, hence $(m_0, m_1, \cdots, m_{n-1}) \in M$ as required.

$(3) \Rightarrow (2')$

The polarized variety $(\mathbb{P}(V)/G, \mathcal{L})$ is $Proj(\oplus_{d \in \mathbb{Z}_{\geq 0}} (H^0(\mathbb{P}(V), \mathcal{O}(1)^{\otimes d|G|})^G)$ which is same as $Proj(\oplus_{d \in \mathbb{Z}_{\geq 0}} (Sym^{d|G|} V^*)^G)$. Let $R := \oplus_{d \geq 0} R_d$; $R_d := (Sym^{dn} V^*)^G$. Fix a generator g of G and let ξ be a primitive nth root of unity. Write $V^* = \oplus_{i=0}^{n-1} \mathbb{C} X_i$, where $\{X_i : i = 0, 1, \cdots, n-1\}$ is a basis of V^* given by: $g.X_i = \xi^i X_i$, for every $i = 0, 1, \cdots, n-1$.

Let R' be the $\mathbb{C}$-subalgebra of $\mathbb{C}[V]$ generated by $R_1 = (Sym^n V^*)^G$. We first note that $\{X_0^{m_0} . X_1^{m_1} \ldots X_{n-1}^{m_{n-1}} : (m_0, m_1, \cdots, m_{n-1}) \in M\}$ is a $\mathbb{C}$-vector space basis for R'. We now define the map

$$\Phi : \mathbb{C}[M] \to R' \text{ by extending linearly the map}$$

$$\Phi(X^{(m_0,m_1,\cdots,m_{n-1})}) = X_0^{m_0}.X_1^{m_1}\dots X_{n-1}^{m_{n-1}} \text{ for } (m_0,m_1,\cdots,m_{n-1}) \in M.$$

Clearly Φ is a homomorphism of $\mathbb{C}$-algebras. Since $\{X^{(m_0,m_1,\cdots,m_{n-1})} : (m_0,m_1,\cdots,m_{n-1}) \in M\}$ is a $\mathbb{C}$-vector space basis for $\mathbb{C}[M]$ and $\{X_0^{m_0}.X_1^{m_1}\dots X_{n-1}^{m_{n-1}} : (m_0,m_1,\cdots,m_{n-1}) \in M\}$ is a $\mathbb{C}$-vector space basis for R', Φ is an isomorphism of $\mathbb{C}$-algebras. Hence R' is the semigroup algebra corresponding to the affine semigroup M. Since by assumption M is a normal affine semigroup, by theorem (4.5.1) the algebra R' is normal. Thus, by [39, Ex. 5.14(a)], the implication $(3) \Rightarrow (2')$ follows. $\square$

4.6 A Counter Example

Let F be a field of characteristic $p \neq 2$ and V be the natural representation of the permutation group $G = S_{p^s}, s \geq 2$ over F. Consider $U = \underbrace{V \oplus V \oplus \ldots \oplus V}_{(p^s)! \text{ copies}}$. Then by a result of Fleischmann (Th. 2.8.9), we have

$$\beta(U,G) = max\{p^s, (p^s)!(p^s-1)\} = (p^s)!(p^s-1),$$

where $\beta(U,G)$ denote the Noether number, which can be defined as the minimal number t, such that the algebra $Sym(U^*)^G$ of invariants can be generated by finitely many elements of degree at most t.

So there exists a homogeneous polynomial $f \in (Sym^d U^*)^G;\ d = (p^s)!(p^s-1)$ which is not in the subalgebra generated by $(Sym^m U^*)^G;\ m \leq (d-1)$.

Hence, $f \in R_{(p^s-1)} = (Sym^{(p^s-1)|G|}U^*)^G$ but not in the subalgebra generated by $R_1 = (Sym^{|G|}U^*)^G$. Thus, projective normality does not hold in this case.

Remark 1: We couldn't find any reference for the generators of $\mathbb{C}[V^m]$ for type E_6, E_7, E_8. We are now working on it. Due to time constraint, we will write it in the future work.

Remark 2: We believe that from theorem (4.4.1), we will be able to prove the Projective normality result for any finite dimensional representation of any Weyl group. We are working on this problem.

Remark 3: It is an interesting and important problem to answer the following question:

Let G be a reductive group (not necessarily finite) acting morphically on a projective variety X. Let $\mathcal{M}$ be a G-linearized very ample line bundle on X such that $\mathcal{M}$ descends to the quotient $X//G$. Let $\mathcal{L}$ be the descent. Is the polarized variety $(X//G, \mathcal{L})$ projectively normal ?

Appendix-A

The C-Program:

```c
#include<stdio.h>
#include<stdlib.h>
FILE *fin,*fout;
int recStack[200][9],n,funcStack[200];
long cnt=0;
int adj[9][9];
int check[16777216];
long long oct[8]={1,8,64,512,4096,32768,262144L,2097152L};
long long hash,hash_max=-1;
int min=1;
int blkComm(int i,int level)
{
        int l=-1,j=level-1;
        while(j>=1 && (adj[i][funcStack[j]]==0))
        {
                if(l < funcStack[j])l=funcStack[j];
                j--;
        }
        if(l>i) return 1;
        else if(j>=2 && (i==funcStack[j-1]) && (i > funcStack[j]) && (adj[i][funcStack[j]]==1)) return 1;
        else return 0;
}
void DFS(int level)
{
        int flagt=0;
        for(int z=1;z<=n;z++)
                if(recStack[level-1][z]>0) flagt=1;
        if(flagt==0)
        {
                hash=0;
                for(int j=1;j<=8;j++)
                {
```

```
                hash=hash+(oct[j-1]*(recStack[level-1][j]+7));
            }
            if(check[hash]==0)
            {
                cnt++;
                check[hash]=1;
                for(int j=1;j<=8;j++) fprintf(fout,"%d ",recStack[level-1][j]);
                fprintf(fout,"\n");
                for(int k=1;k<level;k++) fprintf(fout,"s%d ",funcStack[k]);
                fprintf(fout,"\n");
            }
            return;
        }
        int flag=0;
        for(int i=1;i<=n;i++)
        {
            if(recStack[level-1][i]>0 && (i != funcStack[level-1]) && (blkComm(i,level)==0) && !(level>1 && (i==funcStack[level-2]) && (i>funcStack[level-1])))
            {
                if((i==5 || i==6 || i==7) && (recStack[level-1][i-1] + recStack[level-1][i+1] < 2*recStack[level-1][i]))
                {
                    funcStack[level]=i;
                    for(int j=1;j<=n;j++)
                    {
                        if(j!=i) recStack[level][j]=recStack[level-1][j];
                        else recStack[level][j]= recStack[level-1][j-1] + recStack[level-1][j+1]-recStack[level-1][j];
                    }
                    DFS(level+1);
                    flag=1;
                }
                else if(i==1 && (recStack[level-1][3]<2*recStack[level-1][1]))
```

```
{
        funcStack[level]=1;
        for(int j=1;j<=n;j++)
        {
                if(j!=1) recStack[level][j]=recStack[level-1][j];
                else recStack[level][j]= recStack[level-1][3] - recStack[level-1][j];
        }
        DFS(level+1);
        flag=1;
}
else if(i==8 && (recStack[level-1][7]<2*recStack[level-1][8]))
{
        funcStack[level]=8;
        for(int j=1;j<=n;j++)
        {
                if(j!=8) recStack[level][j]=recStack[level-1][j];
                else recStack[level][j]= recStack[level-1][7] - recStack[level-1][j];
        }
        DFS(level+1);
        flag=1;
}
else if(i==2 && (recStack[level-1][4]<2*recStack[level-1][2]))
{
        funcStack[level]=2;
        for(int j=1;j<=n;j++)
        {
                if(j!=2) recStack[level][j]=recStack[level-1][j];
                else recStack[level][j]= recStack[level-1][4] - recStack[level-1][j];
        }
```

```
                DFS(level+1);
                flag=1;
            }
            else if(i==4 && (recStack[level-1][3] + recStack[level-1][5] + recStack[level-1][2]<2*recStack[level-1][4]))
            {
                funcStack[level]=4;
                for(int j=1;j<=n;j++)
                {
                    if(j!=4) recStack[level][j]=recStack[level-1][j];
                    else recStack[level][j]= recStack[level-1][3] + recStack[level-1][5] + recStack[level-1][2] - recStack[level-1][j];
                }
                DFS(level+1);
                flag=1;
            }
            else if(i==3 && (recStack[level-1][1] + recStack[level-1][4]<2*recStack[level-1][3]))
            {
                funcStack[level]=3;
                for(int j=1;j<=n;j++)
                {
                    if(j!=3) recStack[level][j]=recStack[level-1][j];
                    else recStack[level][j]= recStack[level-1][1] + recStack[level-1][4]-recStack[level-1][j];
                }
                DFS(level+1);
                flag=1;
            }
        }
```

```
        }
}
int main()
{
        fin = fopen("data","r");
        fout = fopen("output","w");
        adj[1][3]=adj[3][1]=1;
        adj[2][4]=adj[4][2]=1;
        adj[3][4]=adj[4][3]=1;
        adj[4][5]=adj[5][4]=1;
        adj[6][5]=adj[5][6]=1;
        adj[6][7]=adj[7][6]=1;
        adj[8][7]=adj[7][8]=1;
        while(fscanf(fin,"%d",&n)!=EOF){
                int i;
                for(i=1;i<=n;i++) fscanf(fin,"%d",&
                   recStack[0][i]);
                DFS(1);
                fprintf(fout,"\n
                   ****************************************\
                   n");
        }
        printf("%d",cnt);
        fclose(fin);
        fclose(fout);
}
```

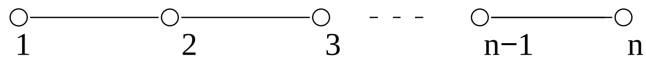
Appendix-B

Most of the information given in this appendix are collected from [13] and [45]. For the Weyl group invariants we refer to [73].

Type A_n:

Dynkin diagram:

An (n $\geq$ 1) :



Cartan matrix:

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ n-1 \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & . & . & . & n-1 & n \end{array} \\
\begin{pmatrix}
2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & \\
 & -1 & 2 & . & & & & \\
 & & . & . & . & & & \\
 & & & . & . & . & & \\
 & & & & . & . & . & \\
 & & & & & . & 2 & -1 \\
 & & & & & & -1 & 2
\end{pmatrix}
\end{array}
$$

Dimension: $dim(\mathfrak{g}) = n(n+2)$.

Coxeter number: $h = \frac{2|\Phi^+|}{n} = n+1$.

Weyl group: $W = S_{n+1}$, $|W| = (n+1)!$.

Longest element of W: $s_n(s_{n-1}s_n)\cdots(s_2\cdots s_n)(s_1\cdots s_n)$.

The root system: $\Phi = \{\epsilon_i - \epsilon_j : i,j = 1,2,\cdots,n+1,\ i \neq j\}$, where $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_{n+1}\}$ is an orthonormal basis. $|\Phi| = n(n+1)$

Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq n$.

Highest long root: $\alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Fundamental weights: $\varpi_i = \frac{1}{n+1}[(n-i+1)\alpha_1 + 2(n-i+1)\alpha_2 + \cdots + (i-1)(n-i+1)\alpha_{i-1} + i(n-i+1)\alpha_i + i(n-i)\alpha_{i+1} + \cdots + i\alpha_n]$ $i = 1, 2, \cdots, n$.

Minuscule fundamental weights Every fundamental weight is minuscule.

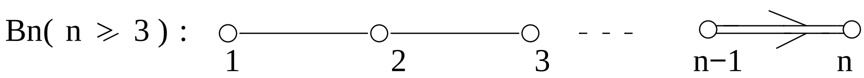
Fundamental group: $\mathbb{Z}_{n+1}$.

Group of diagram automorphisms: $\Gamma = \mathbb{Z}_2$.

Basic polynomial invariants of W: $f_k = \sum_{i=1}^{n} x_i^k$ $(1 \leq k \leq n)$.

Type B_n**:**

Dynkin diagram:



Cartan matrix:

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ n-2 \\ n-1 \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
1 & 2 & 3 & . & . & . & . & n-2 & n-1 & n
\end{array} \\
\begin{pmatrix}
2 & -1 & & & & & & & & \\
-1 & 2 & -1 & & & & & & & \\
 & -1 & 2 & . & & & & & & \\
 & & . & . & . & & & & & \\
 & & & . & . & . & & & & \\
 & & & & . & . & . & & & \\
 & & & & & & . & 2 & -1 & \\
 & & & & & & . & -1 & 2 & -1 \\
 & & & & & & & & -2 & 2
\end{pmatrix}
\end{array}
$$

Dimension: $dim(\mathfrak{g}) = n(2n+1)$.

Coxeter number: $h = 2n$.

Weyl group: $W = (\mathbb{Z}_2)^n \rtimes S_n$, $|W| = 2^n n!$.

Longest element of W**:** $w_0 = s_1(s_2s_1)(s_3s_2s_1)\cdots(s_ns_{n-1}\cdots s_2s_1)(s_ns_{n-1}\cdots s_2)(s_ns_{n-1}\cdots s_3)\cdots(s_ns_{n-1})s_n = -id$.

The root system: $\Phi = \Phi_1 \cup \Phi_2$: $\Phi_1 = \{\pm\epsilon_i \pm \epsilon_j : i, j = 1, 2, \cdots, n,\ i \neq j\}$, $\Phi_2 = \{\pm\epsilon_i : i = 1, 2, \cdots, n\}$ where $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ is an orthonormal basis. $|\Phi| = 2n^2$.

Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq n-1$, $\alpha_n = \epsilon_n$.

Highest long root: $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$.

Highest short root: $\alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Fundamental weights: $\varpi_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_n)$ $i = 1, 2, \cdots, n-1$.

$\varpi_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n)$.

Minuscule fundamental weights ϖ_n.

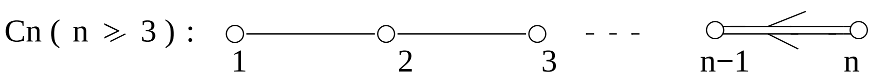
Fundamental group: $\mathbb{Z}_2$.

Group of diagram automorphisms: $\Gamma = 1$.

Basic polynomial invariants of W: $f_k = \sum_{i=1}^{n} x_i^{2k}$ $(1 \leq k \leq n)$.

Type C_n**:**

Dynkin diagram:



Cartan matrix:

$$
\begin{array}{c c}
 & \begin{array}{ccccccccc} 1 & 2 & 3 & . & . & . & n-2 & n-1 & n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ n-2 \\ n-1 \\ n \end{array} &
\begin{pmatrix}
2 & -1 & & & & & & & \\
-1 & 2 & -1 & & & & & & \\
 & -1 & 2 & . & & & & & \\
 & & . & . & . & & & & \\
 & & & . & . & . & & & \\
 & & & & . & . & . & & \\
 & & & & & . & 2 & -1 & \\
 & & & & & . & -1 & 2 & -2 \\
 & & & & & & & -1 & 2
\end{pmatrix}
\end{array}
$$

Dimension: $dim(\mathfrak{g}) = n(2n+1)$.

Coxeter number: $h = 2n$.

Weyl group: $W = (\mathbb{Z}_2)^n \rtimes S_n$, $|W| = 2^n n!$.

Longest element of W**:** $w_0 = s_1(s_2s_1)(s_3s_2s_1)\cdots(s_ns_{n-1}\cdots s_2s_1)(s_ns_{n-1}\cdots s_2)(s_ns_{n-1}\cdots s_3)\cdots(s_ns_{n-1})s_n = -id$.

The root system: $\Phi = \Phi_1 \cup \Phi_2$: $\Phi_1 = \{\pm\epsilon_i \pm \epsilon_j : i, j = 1, 2, \cdots, n,\ i \neq j\}$, $\Phi_2 = \{\pm 2\epsilon_i : i = 1, 2, \cdots, n\}$ where $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ is an orthonormal basis. $|\Phi| = 2n^2$.

Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq n-1$, $\alpha_n = 2\epsilon_n$.

Highest long root: $2\alpha_1 + 2\alpha_2 + \cdots + \alpha_n$.

Highest short root: $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$.

Fundamental weights: $\varpi_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n)$ $i = 1, 2, \cdots, n$.

Minuscule fundamental weights ϖ_1.

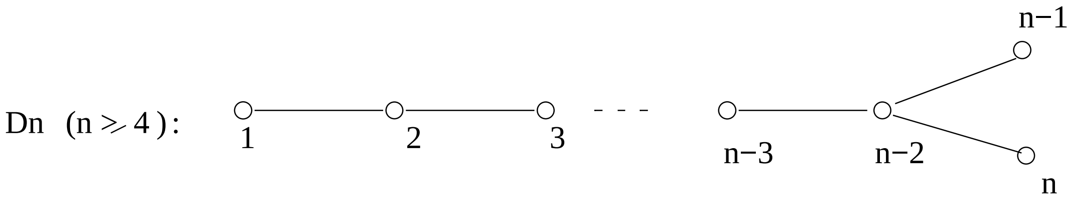
Fundamental group: $\mathbb{Z}_2$.

Group of diagram automorphisms: $\Gamma = 1$.

Basic polynomial invariants of W: $f_k = \sum_{i=1}^{n} x_i^{2k}$ $(1 \leq k \leq n)$.

Type D_n**:**

Dynkin diagram:



Cartan matrix:

$$
\begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & . & . & . & . & n-2 & n-1 & n \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ n-2 \\ n-1 \\ n \end{matrix} &
\begin{pmatrix}
2 & -1 & & & & & & & & \\
-1 & 2 & -1 & & & & & & & \\
 & -1 & 2 & . & & & & & & \\
 & & . & . & . & & & & & \\
 & & & . & . & . & & & & \\
 & & & & . & . & . & & & \\
 & & & & & & . & 2 & -1 & -1 \\
 & & & & & & . & -1 & 2 & \\
 & & & & & & & -1 & & 2
\end{pmatrix}
\end{array}
$$

Dimension: $dim(\mathfrak{g}) = n(2n-1)$.

Coxeter number: $h = 2n-2$.

Weyl group: $W = (\mathbb{Z}_2)^{n-1} \rtimes S_n$, $|W| = 2^{n-1}n!$.

Longest element of W**:** If n is odd: $w_0 = s_1(s_2s_1)(s_3s_2s_1)\cdots(s_{n-1}s_{n-2}\cdots s_2s_1)(s_ns_{n-2}\cdots s_2s_1)$ $(s_{n-1}s_{n-2}\cdots s_2)(s_ns_{n-2}\cdots s_3)(s_{n-1}s_{n-2}\cdots s_4)(s_ns_{n-2}\cdots s_5)\cdots(s_ns_{n-2})s_{n-1}$.

If n is even: $w_0 = s_1(s_2s_1)(s_3s_2s_1)\cdots(s_{n-1}s_{n-2}\cdots s_2s_1)(s_ns_{n-2}\cdots s_2s_1)(s_{n-1}s_{n-2}\cdots s_2)$ $(s_ns_{n-2}\cdots s_3)(s_{n-1}s_{n-2}\cdots s_4)(s_ns_{n-2}\cdots s_5)\cdots(s_{n-1}s_{n-2})s_n = -id$.

The root system: $\Phi = \{\pm\epsilon_i \pm \epsilon_j : i,j = 1,2,\cdots,n,\ i \neq j\}$, where $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ is an orthonormal basis. $|\Phi| = 2n(n-1)$.

Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq n-1$, $\alpha_n = \epsilon_{n-1} + \epsilon_n$.

Highest long root: $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$.

Fundamental weights: $\varpi_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2}) + \frac{1}{2}i(\alpha_{n-1} + \alpha_n)$ $i = 1,2,\cdots,n-2$.

$\varpi_{n-1} = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n)$.

$\varpi_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}n\alpha_n)$.

Minuscule fundamental weights $\varpi_1, \varpi_{n-1}, \varpi_n$.

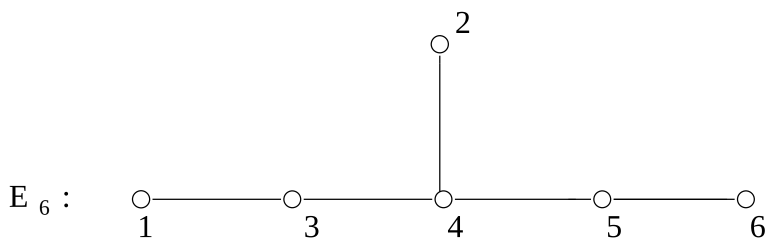
Fundamental group: $\mathbb{Z}_4$ if l is odd and $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even.

Group of diagram automorphisms: $\Gamma = \begin{cases} S_3 & \text{if } n = 4; \\ \mathbb{Z}_2 & \text{if } n > 4. \end{cases}$

Basic polynomial invariants of W: $f_k = \sum_{i=1}^{n} x_i^{2k}$ $(1 \leq k \leq n-1)$ and $f_n = x_1x_2\ldots x_n$.

Type E_6**:**

Dynkin diagram:



Cartan matrix:

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \end{array}$$

Dimension: $dim(\mathfrak{g}) = 78$.

Coxeter number: $h = 12$.

Order of the Weyl group: $|W| = 2^7 3^4 5$.

Longest element of W: $w_0 = s_1(s_2s_3s_1)(s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)$ $(s_6s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)(s_6s_5s_4s_3s_1)$.

The root system: $\Phi = \{\pm\epsilon_i \pm \epsilon_j \,:\, i, j = 1, 2, \cdots, 5,\ i \neq j\} \cup \{\frac{1}{2}\sum_{i=1}^{8} c_i\epsilon_i \,:\, c_i = \pm 1,\ \prod_{i=1}^{8} c_i = 1,\ c_6 = c_7 = c_8\}$, where $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_8\}$ is an orthonormal basis of $\mathbb{R}^8$. $|\Phi| = 72$.

Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq 4$, $\alpha_5 = \epsilon_4 + \epsilon_5$, $\alpha_6 = -\frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8)$.

Highest long root: $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$.

Fundamental weights: $\varpi_1 = \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6)$.

$\varpi_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$.

$\varpi_3 = \frac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6)$.

$\varpi_4 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6$.

$\varpi_5 = \frac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6)$.

$\varpi_6 = \frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6)$.

Minuscule fundamental weights: ϖ_1, ϖ_6.

Fundamental group: $\mathbb{Z}_3$.

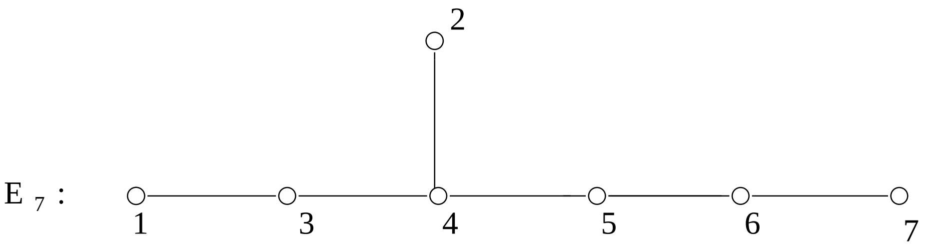
Group of diagram automorphisms: $\mathbb{Z}_2$.

Basic polynomial invariants of W: $f_m = \sum_{i=1}^{6}(y_i+y)^m + \sum_{i=1}^{6}(y_i-y)^m - \sum_{i<j}(y_i+y_j)^m$, $m = 2, 5, 6, 8, 9, 12$, where
$y_1 = 5x_1+4x_2+3x_3+2x_4+x_5$, $y_2 = -x_1+4x_2+3x_3+2x_4+x_5$, $y_3 = -x_1-2x_2+3x_3+2x_4+x_5$,

$y_4 = -x_1 - 2x_2 - 3x_3 + 2x_4 + x_5$, $y_5 = -x_1 - 2x_2 - 3x_3 - 42x_4 + x_5$, $y_6 = -\sum_{i=1}^{5} y_i$ and $y = -3(x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 + 2x_6)$.

Type E_7:

Dynkin diagram:



Cartan matrix:

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\end{array}
$$

Dimension: $dim(\mathfrak{g}) = 133$.

Coxeter number: $h = 18$.

Order of the Weyl group: $|W| = 2^{10}.3^4.5.7$.

Longest element of W: $w_0 = s_1(s_2s_3s_1)(s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)$ $(s_6s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)(s_6s_5s_4s_3s_1)(s_7s_6s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)(s_6s_5s_4s_3s_1)(s_7s_6s_5s_4s_2)$ $(s_3s_4s_5s_6s_7) = -id$.

The root system: $\Phi = \{\pm\epsilon_i \pm \epsilon_j : i, j = 1, 2, \cdots, 6,\ i \neq j\} \cup \{\pm(\epsilon_7 + \epsilon_8)\} \cup \{\frac{1}{2}\sum_{i=1}^{8} c_i\epsilon_i : c_i = \pm 1,\ \prod_{i=1}^{8} c_i = 1,\ c_7 = c_8\}$, where $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_8\}$ is an orthonormal basis of $\mathbb{R}^8$. $|\Phi| = 126$.

Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq 5$, $\alpha_6 = \epsilon_5 + \epsilon_6$, $\alpha_7 = -\frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8)$.

Highest long root: $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$.

Fundamental weights: $\varpi_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$.

$\varpi_2 = \frac{1}{2}(4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7)$.

$\varpi_3 = 3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7$.

$\varpi_4 = 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7$.

$\varpi_5 = \frac{1}{2}(6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7)$.

$\varpi_6 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7$.

$\varpi_7 = \frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)$.

Minuscule fundamental weights: ϖ_7.

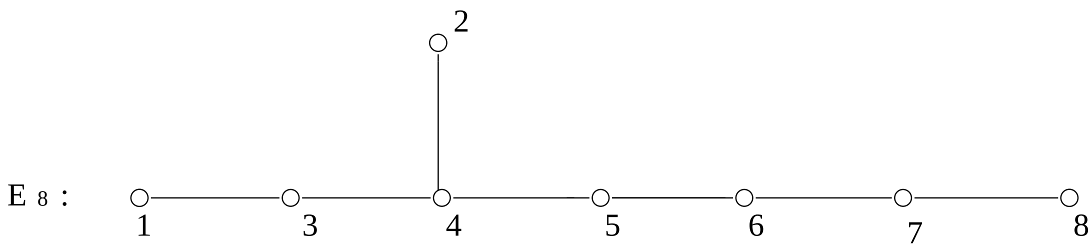
Fundamental group: $\mathbb{Z}_2$.

Group of diagram automorphisms: 1.

Basic polynomial invariants of W: $f_m = \sum_{i<j}(y_i + y_j)^m$, $i, j \in \{1, 2, \cdots, 8\}$, $m = 2, 6, 8, 10, 12, 14, 18$, where $y_1 = 3x_1 + 2x_2 + x_3 - x_7$, $y_2 = -x_1 + 2x_2 + x_3 - x_7$, $y_3 = -x_1 - 2x_2 + x_3 - x_7$, $y_4 = -x_1 - 2x_2 - 3x_3 - x_7$, $y_5 = -x_1 - 2x_2 - 3x_3 - 4x_4 - x_7$, $y_6 = -x_1 - 2x_2 - 3x_3 - 4x_4 - 4x_5 - x_7$, $y_7 = -x_1 - 2x_2 - 3x_3 - 4x_4 - 4x_5 - 4x_6 - x_7$, $y_8 = -\sum_{i=1}^{7} y_i$.

Type E_8:

Dynkin diagram:



Cartan matrix:

$$\begin{array}{c c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\end{array}$$

Dimension: $dim(\mathfrak{g}) = 248$.

Coxeter number: $h = 30$.

Order of the Weyl group: $|W| = 2^{14}.3^5.5^2.7$.

Longest element of W**:** $w_0 = s_1(s_2s_3s_1)(s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)$ $(s_6s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)(s_6s_5s_4s_3s_1)(s_7s_6s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)(s_6s_5s_4s_3s_1)(s_7s_6s_5s_4)$ $(s_2s_3s_4s_5s_6s_7)(s_8s_7s_6s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)(s_6s_5s_4s_3s_1)(s_7s_6s_5s_4s_2)(s_3s_4s_5s_6s_7)$ $(s_8s_7s_6s_5s_4s_2s_3s_1)(s_4s_3)(s_5s_4s_2)(s_6s_5s_4s_3s_1)(s_7s_6s_5s_4s_2)(s_3s_4s_5s_6s_7s_8) = -id$.

The root system: $\Phi = \{\pm\epsilon_i \pm \epsilon_j : i, j = 1, 2, \cdots, 8, \ i \neq j\} \cup \{\frac{1}{2}\sum_{i=1}^{8} c_i\epsilon_i : c_i = \pm1, \ \prod_{i=1}^{8} c_i = 1\}$, where $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_8\}$ is an orthonormal basis of $\mathbb{R}^8$. $|\Phi| = 240$.

Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq 6$, $\alpha_7 = \epsilon_6 + \epsilon_7$, $\alpha_7 = -\frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8)$.

Highest long root: $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

Fundamental weights: $\varpi_1 = 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8$.

$\varpi_2 = 5\alpha_1 + 8\alpha_2 + 10\alpha_3 + 15\alpha_4 + 12\alpha_5 + 9\alpha_6 + 6\alpha_7 + 3\alpha_8$.

$\varpi_3 = 7\alpha_1 + 10\alpha_2 + 14\alpha_3 + 20\alpha_4 + 16\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8$.

$\varpi_4 = 10\alpha_1 + 15\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8$.

$\varpi_5 = 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 24\alpha_4 + 20\alpha_5 + 15\alpha_6 + 10\alpha_7 + 5\alpha_8$.

$\varpi_6 = 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8$.

$\varpi_7 = 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 8\alpha_6 + 6\alpha_7 + 3\alpha_8$.

$\varpi_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

Minuscule fundamental weights: No miniscule fundamental weights.

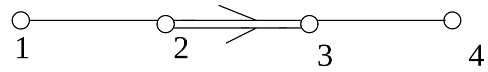
Fundamental group: 1.

Group of diagram automorphisms: 1.

Basic polynomial invariants of W: $f_m = 2\sum_{i<j}[(y_i+y_j)^m + (y_i-y_j)^m] + \sum_{\prod\epsilon_i=-1, \epsilon_i=\pm1} (\sum\epsilon_i x_i)^m$, $i, j \in \{1, 2, \cdots, 8\}$, $m = 2, 8, 12, 14, 18, 20, 24, 30$, where $y_1 = 2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_8$, $y_2 = 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_8$, $y_3 = 2x_3 + 2x_4 + 2x_5 + x_6 + x_8$, $y_4 = 2x_4 + 2x_5 + x_6 + x_8$, $y_5 = 2x_5 + x_6 + x_8$, $y_6 = x_6 + x_8$, $y_7 = x_6 + x_8$, $y_8 = 2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 + 7x_6 + 4x_7 + 5x_8$.

Type F_4**:**

Dynkin diagram:

F_4 :


Cartan matrix:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \end{array}$$

Dimension: $dim(\mathfrak{g}) = 52$.

Coxeter number: $h = 12$.

Weyl group: $W = (\mathbb{Z}_2^3 \rtimes S_4) \rtimes S_3$, $|W| = 2^7.3^2$.

Longest element of W: $w_0 = s_1s_2s_1s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_2s_3s_4 = -id$.

The root system: $\Phi = \{\pm\epsilon_i \pm \epsilon_j : i, j = 1, 2, 3, 4,\ i \neq j\} \cup \{\pm\epsilon_i : i = 1, 2, 3, 4\} \cup \{\frac{1}{2}\sum_{i=1}^4 c_i\epsilon_i : c_i = \pm 1,\ \prod_{i=1}^4 c_i = 1\}$, where $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ is an orthonormal basis. $|\Phi| = 48$.

Simple roots: $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$ $\alpha_3 = \epsilon_3$, $\alpha_4 = -frac12(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)$.

Highest long root: $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$.

Highest short root: $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$.

Fundamental weights: $\varpi_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$.

$\varpi_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4$.

$\varpi_3 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4$.

$\varpi_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$.

Minuscule fundamental weights: No miniscule fundamental weights.

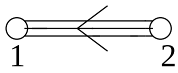
Fundamental group: 1.

Group of diagram automorphisms: 1.

Basic polynomial invariants of W: $f_{2k} = \sum_{1\leq i\leq j\leq 4}((x_i + x_j)^{2k} + (x_i - x_j)^{2k})$, $k = 1, 3, 4, 6$.

Type G_2:

Dynkin diagram:

G_2 : 

Cartan matrix:

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \end{array}$$

Dimension: $dim(\mathfrak{g}) = 14$.

Coxeter number: $h = 6$.

Weyl group: $W = D_6$: the dihedral group. $|W| = 12$.

Longest element of W: $w_0 = (s_1 s_2)(s_1 s_2)(s_1 s_2) = -id$.

The root system: $\Phi = \{\pm(-2\epsilon_1 + \epsilon_2 + \epsilon_3), \pm(\epsilon_1 - 2\epsilon_2 + \epsilon_3) + \pm(\epsilon_1 + \epsilon_2 - 2\epsilon_3), \pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_2 - \epsilon_3), \pm(\epsilon_1 - \epsilon_3)\}$, where $\epsilon_1, \epsilon_2, \epsilon_3$ is an orthonormal basis of $\mathbb{R}^3$. $|\Phi| = 12$.

Simple roots: $\alpha_1 = -2\epsilon_1 + \epsilon_2 + \epsilon_3$, $\alpha_2 = \epsilon_1 - \epsilon_2$.

Highest long root: $3\alpha_1 + 2\alpha_2$.

Highest short root: $2\alpha_1 + \alpha_2$.

Longest element of W: -id.

Fundamental weights: $\varpi_1 = 2\alpha_1 + \alpha_2$.

$\varpi_2 = 3\alpha_1 + 2\alpha_2$.

Minuscule fundamental weights: No miniscule fundamental weights.

Fundamental group: 1.

Group of diagram automorphisms: 1.

Basic polynomial invariants of W: $f_1 = \sum_{i<j} y_i y_j,\ \ f_2 = (y_1 y_2 y_3)^2$, $i, j = 1, 2, 3$, where $y_1 = 3x_1 + x_2$, $y_2 = x_2$ and $y_3 = -y_1 - y_2$.

Bibliography

[1] O. M. Adamovich, E. O. Golovina, *Simple Linear Lie Groups having a Free algebra of Invarisnts*, Questions of Group Theory and Homological Algebra Vyp 2, Yaroslav. Gros. Unir. Yaroslavl' 3-41 (1980) English Translation in Sel. Math. Sov. 3, 183-220 (1983/84).

[2] A. Borel, *Linear algebraic groups*, Second edition, Graduate Texts in Mathematics, 126. Springer Verlag, New York, 1991.

[3] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 4-6, Elements of Mathematics (Berlin), Springer-Verlag, 2002.

[4] A. Broer, *Remarks on invariant theory of finite groups*, preprint, Universite de Montreal, Montreal, 1997.

[5] A. Broer, *On Chevalley-Shephard-Todd's theorem in positive characteristic*, Symmetry and spaces, 21-34, Progr. Math., 278, Birkhauser Boston, Inc., Boston, MA, 2010.

[6] W. Bruns, J. Gubeladze, *Polytopes, Rings, and K-Theory*, Springer Monographs in Mathematics. Springer, Dordrecht, 2009.

[7] R. Bryant, G. Kemper, *Global degree bounds and the transfer principle for invariants*, J. Algebra 284 (2005), no. 1, 80-90.

[8] H. E. A. Campbell, A. V. Geramita, I. P. Hughes, J. R. Shank, D. L. Wehlau, *Non-Cohen-Macaulay vector invariants and a Noether bound for a Gorenstein ring of invariants*, Canad. Math. Bull. 42 (1999), no. 2, 155-161.

[9] H. E. A. Campbell, I. P. Hughes, G. Kemper, R. J. Shank and D. L. Wehlau, *Depth of modular invariant rings, Transformation Groups* 5 (2000) no. 1, 21-34.

[10] H. E. A. Campbell, D. L. Wehlau, *Modular Invariant Theory*, Encyclopaedia of Mathematical Sciences, 2011, Volume 139, Invariant Theory and Algebraic Transformation Groups, Springer-Verlag.

[11] E. Cartan, *"Les tenseurs irreductibles et les groupes lineaires simples et semi-simples"* Bull. Sci. Math., 49 (1925) pp. 130-152.

[12] R. W. Carter, *Finite Groups of Lie type*, John Wiley, New York, 1993.

[13] R. W. Carter, *Lie Algebras of Finite and Affine Type*, Cambridge studies in advanced mathematics 96 (2005).

[14] C. Chevalley, *Theorie des groupes de Lie. II*, Hermann, Paris, 1951.

[15] C. Chevalley, *Invariants of Finite Groups Generated by Reflections*, Amer. J. Math. 77(1955), 778-782.

[16] C. Chevalley, *Classification des groupes algebriques semi-simples*, Springer-Verlag, Berlin, 2005. Collected works. Vol. 3, Edited and with a preface by P. Cartier, With the collaboration of Cartier, A. Grothendieck and M. Lazard.

[17] H. Chu, S. J. Hu, M. C. Kang, *A note on Projective normality*, Proc. Amer. Math. Soc. 139 (2011), no. 8, pp. 863-867.

[18] D. Daigle, G. Freudenburg, *A counter example to Hilbert's fourteenth problem in dimension 5*. J. Algebra, 221(2), 528-535, 1999.

[19] M. Demazure, *Une demonstration algebrique d'un theoreme de Bott* (French), Invent. Math. 5 (1968) 349-356.

[20] M. Demazure, *A very simple proof of Bott's theorem*, Invent. Math. 33 (1976), no. 3, 271-272.

[21] H. Derksen, *Computation of invariants for reductive groups*, Adv. Math. 141, no. 2, 366-384. 1998.

[22] H. Derksen, G. Kemper, *Computational Invariant Theory*, Encyclopaedia of Mathematical Sciences 130, Springer-Verlag, Berlin, Heidelberg, New York 2002.

[23] L. E. Dickson, *Binary Modular Groups and their Invariants*, Amer. J. of Math. 33 (1911), 175-192.

[24] I. V. Dolgachev and Y. Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Atudes Sci. Publ. Math. (1998), no. 87, 5-56.

[25] I. V. Dolgachev, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series, 296.Cambridge University Press, Cambridge, 2003.

[26] M. Domokos, P. Hegedüs, *Noether's bound for polynomial invariants of finite groups*, Arch. Math. (Basel) 74 (2000), no. 3, 161-167.

[27] M. Domokos, *Vector invariants of a class of pseudo-reflection groups and multisymmetric syzygies*, J. Lie Theory 19 (2009), no. 3, 507-525.

[28] A. Dress, *On Finite Groups Generated by Pseudo-refiections*, J. of Algebra 11 (1969), 1-5.

[29] J. M. Drezet, M.S. Narasimhan, *Groupe de Picard des varieties de modules de fibrers semi-stables sur les courbes algebriques*, Invent. Math. 97, 53-94 (1989)

[30] G. Ellingsrud and T. Skjelbred, *Profondeur d'Anneaux d'Invariants en Caracteristique p*, Compos. Math.. 41, (1980), pp 233-244.

[31] P. Erdös, A. Ginzburg, A. Ziv, *A theorem in additive number theory*, Bull. Res. Council, Israel, 10 F(1961) 41-43.

[32] P. Fleischmann, *A new degree bound for the vector invariants of symmetric groups*, Trans. Amer. Math. Soc. 350 (1998), 1703-1712.

[33] P. Fleischmann, *The Noether bound in invariant theory of finite groups*, Adv. Math. 152 (2000), pp. 23-32.

[34] P. Fleischmann, G. Kemper and R.J. Shank, *Depth and Cohomological Connectivity in Modular Invariant Theory*, Transactions of the American Mathematical Society 357 (2005) no. 9, 3605-3621.

[35] P. Fleischmann, M. Sezer, R. J. Shank, C. F. Woodcock, *The Noether numbers for cyclic groups of prime order*, Adv. Math. 207 (2006), no. 1, 149-155.

[36] P. Fleischmann, C. F. Woodcock, *Non-linear Group Actions with Polynomial Invariant Rings and a Structure Theorem for Modular Galois Extensions*, arXiv:1011.5149v1.

[37] J. Fogarty, *On Noether's bound for polynomial invariants of a finite group*, Electron. Res. Announc. Amer. Math. Soc., 7, 5-7, 2001.

[38] R. Goodman, N. Wallach, *Symmetry, Representations, and Invariants*, Springer, 2009.

[39] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math., 52, Springer-Verlag, New York-Heidelberg, 197.

[40] D. Hilbert, *Über die Theorie der algebraischen Formen*, Mathematische Annalen 36, 473-534.

[41] M. Hochster and J. A. Eagon, *Cohen-Macaulay Rings, Invariant Theory, and the Generic Perfection of Determinantal Loci*, Amer. J. of Math. 93 (1971), 1020-1058.

[42] M. Hochster, J. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen Macaulay*, Adv.Math. 13 (1974) 115-175.

[43] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briancon-Skoda theorem*. J. Amer. Math. Soc. 3 (1990), pp. 31-116.

[44] J. E. Humphreys, *Modular representations of classical Lie algebras and semi-simple groups*, J. Algebra 19 (1971), 51-79.

[45] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, Berlin Heidelberg, 1972.

[46] J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, 1975.

[47] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge studies in advanced mathematics, 29 (1990).

[48] M. Hunziker, *Classical invariant theory for finite reflection groups*, Transform. Groups 2 (1997) (2), pp. 147-163.

[49] J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, New York (1987).

[50] V. G. Kac, V.L. Popov, E.B. Vinberg, *Sur les groupes lineages algebriques dont l'agebre des invaxiantes est libre*, C.R. Acad. Sci. Paris Ser. I Math. 283. 865-878 (1976).

[51] R. Kane, *Reflection Groups and Invariant theory*, CMS Books in Mathematics, Springer-Verlag, 2001.

[52] S. S. Kannan, *Torus quotients of homogeneous spaces*, Proc. Indian Acad. Sci.(Math. Sci), 108(1998), no 1, 1-12.

[53] S. S. Kannan, *Torus quotients of homogeneous spaces-II*, Proc. Indian Acad. Sci.(Math. Sci), 109(1999), no 1, 23-39.

[54] S. S. Kannan, *Cohomology of line bundles on Schubert varieties in the Kac-Moody setting*, J. Algebra, 310(2007) 88-107.

[55] S. S. Kannan, S.K. Pattanayak, Pranab Sardar, *Projective normality of finite groups quotients*, Proc. Amer. Math. Soc. 137 (2009), no. 3, pp. 863-867.

[56] S. S. Kannan, Pranab Sardar, *Torus quotients of homogeneous spaces of the general linear group and the standard representation of certain symmetric groups*, Proc. Indian Acad. Sci.(Math. Sci), 119(2009), no 1, 81-100.

[57] S. S. Kannan, S.K. Pattanayak, *Torus quotients of homogeneous spaces* - *minimal dimensional Schubert varieties admitting semi-stable points*, Proc. Indian Acad. Sci.(Math. Sci), 119(2009), no 4, 469-485.

[58] S. S. Kannan, S.K.Pattanayak, *Projective normality of Weyl group quotients*, Proc. Indian Acad. Sci.(Math. Sci), 121 (2011), no.1, pp. 1-8.

[59] S. S. Kannan, S.K. Pattanayak, *Normality, Projective normality and EGZ theorem*, will appear in Integers.

[60] G. Kemper, *Calculating invariant rings of finite groups over arbitrary fields*, J. Symbolic Comput. 21 (1996), 351-366.

[61] G. Kemper and G. Malle, *The finite irreducible linear groups with polynomial ring of invariants*, Transform. Groups 2 (1997), no. 1, 57-89.

[62] G. Kemper, *Lower degree bounds for modular invariants and a question of I. Hughes*, Transform. Groups 3 (1998), no. 2, 135-144.

[63] G. Kemper, *On the Cohen Macaulay property of modular invariant rings*, J. Algebra 215 (1999) 330-351.

[64] G. Kemper, *Die Cohen-Macaulay-Eigenschaft in der modularen Invariantentheorie*. Habilitationsschrift, Universität Heidelberg, 1999.

[65] G. Kemper, *A characterization of linearly reductive groups by their invariants*, Transform. Groups 5 (1) (2000) 85-92.

[66] G. Kemper, *The depth of invariant rings and cohomology*, with an appendix by K. Magaard. J. Algebra. 245 no. 2, 463-531 (2001).

[67] F. Knop, *Der kanonische Modul eines Invariantenrings*, J. Algebra 127 (1989), pp. 40-54.

[68] M. Kohls, *On the depth of invariant rings of infinite groups*, J. Algebra 322 (2009), 210-218.

[69] H. Kraft, P. Slodowy, T. A. Springer, *Algebraic Transformation Groups and Invariant Theory*, Birkhauser, 1989.

[70] H. P. Kraft and C. Procesi, *Classical invariant theory: A primer* lecture notes (1996), http://www.math.unibas.ch/ kraft/Papers/KP-Primer.pdf.

[71] D. Krause, *Die Noethersche Gradgrenze in der Invariantentheorie*, Diplomarbeit, University of G ttingen, 1999.

[72] V. Lakshmibai and K. N. Raghavan, *Standard Monomial Theory*, Encyclopaedia of Mathematical Sciences, vol. 137, Springer, 2008.

[73] C. Y. Lee, *Invariant polynomials of Weyl groups and applications to the centers of universal enveloping algebras*, Can. J. Math., XXVI (1974), 583-592.

[74] P. Littelmann, *Koregulare und aqnidimensionale Darstellungen*. J. Algebra 123, No. 1 193-222 (1989).

[75] Y. Matsushima, *Espaces homogenes de Stein des groupes de Lie complexes*, Nagoya Math. J. 16 (1960), 205-218.

[76] M. L. Mehta, *Basic set of invariant polynomials for finite reflection groups*, Com. in Alg., 16. (5), (1988), 1083-1098.

[77] T. Molien, *Über die Invarianten der linearen Substitutionsgruppen*, Sitzungsber. König. Preuss. Akad. Wiss. (1897), 1152-1156.

[78] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant theory*, Springer-Verlag, 1994.

[79] M. Nagata, *On the 14th problem of Hilbert*, Amer. J. Math., 8l (1959), 766-772.

[80] M. Nagata, *Complete reducibility of rational representations of a matric group*, J. Math. Kyoto Univ. 1 (1961) 87-99.

[81] M. Nagata, *"Invariants of a group in an affine ring"*, J. Math. Kyoto Univ., 3 (1964) pp. 369-377.

[82] H. Nakajima, *Invariants of Finite Groups Generated by Pseudoreflections in Positive Characteristic*, Tsukuba J. of Math. 3 (1979), 109-122.

[83] H. Nakajima, *Relative invariants of finite groups*, J. Algebra, 79 (1982) 218-234.

[84] H. Nakajima, *Modular Representations of Abelian Groups with Regular Rings of Invariants*, Nagoya Math. J. 86 (1982), 229-248.

[85] H. Nakajima, *Regular rings of invariants of unipotent groups*, J. Algebra 85 (1983), pp. 253-286.

[86] M. D. Neusel and L. Smith, *Invariant Theory of Finite Groups*, Math. Surveys and. Monographs Vol. 94, AMS, Providence RI 2002.

[87] P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, TIFR Lecture Notes, 1978.

[88] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen,* Math. Ann. 77, (1916), 89-92.

[89] E. Noether, *Der Endlichkeitssatz der Invarianten endlich linearer Gruppen der Charakteristik p,* Nachr. Akad. Wiss. Gottingen (1926), 28-35.

[90] V. L. Popov, *On Hilbert's theorem on invariants* (in Russian), Dokl. Akad. Nauk SSSR 249 (1979), 551–555.; English translation in Soviet Math. Dokl. 20 (1979), 1318-1322.

[91] V. Popov, *Constructive invariant theory*, Asterisque, 87-88 (1981), 303-334.

[92] V. L. Popov, *The Constructive Theory of Invariants*, Math. USSR Izvest. 10 (1982), 359-376.

[93] D. R. Richman, *Invariants of finite groups over fields of characteristic p*. Adv. Math. 124 (1996), no. 1, 25-48.

[94] P. Roberts, *An infinitely generated symbolic blow-up and a new counterexample to Hilbert's fourteenth problem*, J. of Algebra 132, pp. 461-473 (1990).

[95] D. J. S. Robinson, *A Course in the Theory of Groups*, 2nd ed., Springer-Verlag, 1996.

[96] B. J. Schmid, *Finite groups and invariant theory*, Topics in invariant theory (Paris, 1989-1990), pp. 35-66, Lecture Notes in Math. 1478, Springer, 1991.

[97] G. W. Schwarz, *Representations of Simple Lie Groups with Regular Rings on Invariants*, Inv. Math. 49 167-191 (1978).

[98] J. P. Serre, *Groupes finis d'automorphisms d'anneaux locaux reguliers*, Colloq. d'Alg. Ecole Norm. de Jeunes Filles, Paris (1967), 1-11.

[99] J. P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, vol. 42, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[100] C. S. Seshadri, *Mumford's conjecture for GL (2) and applications*, International Colloquim on Algebraic Geometry, Bombay, 16-23 January (1968).

[101] C. S. Seshadri, *Quotient spaces modulo reductive algebraic groups*, Ann. Math. 95(1972) 511-556.

[102] C. S. *Seshadri, Introduction to the theory of standard monomials*, Brandeis Lecture Notes 4, June 1985.

[103] M. Sezer, *Sharpening the generalized Noether bound in the invariant theory of finite groups*, J. Algebra 254 (2002) 252-263.

[104] R. J. Shank, *S.A.G.B.I. bases for rings of formal modular seminvariants*, Commentarii Mathematici Helvetici 73 (1998) no. 4, 548-565.

[105] R. J. Shank and D.L. Wehlau, *Computing modular invariants of p-groups*, Journal of Symbolic Computation 34 (2002) no. 5, 307-327.

[106] G. C. Shephard, J.A.Todd, *Finite Unitary Reflection Groups*, Canadian J. Math. 6, (1954), 274-304.

[107] A. N. Skorobogatov, *Automorphisms and forms of torus quotients of homogeneous spaces*, (Russian) Mat. Sb. 200 (2009), no. 10, 107-122.

[108] L. Smith and R. E. Stong, *Invariants of Finite Groups*, unpublished correspondence, 1979-1993.

[109] L. Smith, *Polynomial invariants of finite groups*, A. K. Peters, Wellesley, Mass. 1995.

[110] L. Smith, *E. Noether's bound in the invariant theory of finite groups*, Arch. Math.(Basel) 66 (1996), no. 2, 89-92.

[111] L. Smith, *Putting the squeeze on the Noether gap-the case of the alternating groups* A_n, Math. Ann. 315 (1999), no. 3, 503-510.

[112] L. Smith, *Noether's bound in the invariant theory of finite groups and vector invariants of iterated wreath products of symmetric groups*, Quartely. J. Math. 51 (2000), no. 1, 93-105.

[113] D. M. Snow, *The nef value and defect of homogeneous line bundles*, Trans. Amer. Math. Soc. 340 1 (1993), pp. 227-241.

[114] T. A. Springer, *Invariant theory*, Lecture Notes in Math., vol. 585, Springer-Verlag, Berlin and New York, 1977.

[115] T. A. Springer, *Linear algebraic groups*, Progress in Mathematics, vol. 9, Birkhauser, Boston; Basel, Stuttgart, 1981, 304 pp.

[116] E. Strickland, *Quotients of flag varieties by a maximal torus*. Math. Z. 234(2000, no.1, 1-7; translation in Sb. Math. 200 (2009), no. 9-10, 1521-1536.

[117] P. Symonds, *On the Castelnuovo-Mumford regularity of rings of polynomial invariants*, preprint, available at http://www.maths.manchester.ac.uk/ pas/preprints/ (2009).

[118] P. Tauvel, R. W. T. Yu, *Lie algebras and algebraic groups*, Springer Monographs in Mathematics, Springer Verlag, Berlin, 2005.

[119] N. R. Wallach, *Invariant differential operators on a reductive Lie algebra and Weyl group representations*, J. Amer. Math. Soc. 6 (1993) (4), pp. 779-816.

[120] D. Wehlau, *Constructive invariant theory for tori*, Ann. Inst. Fourier, Grenoble 43 (1993), pp. 1055-1066.

[121] D. Wehlau, *When is a ring of torus invariants a polynomial ring*, Manuscripta Math. 82, 161-170 (1994).

[122] R. Weitzenböck, *Über die Invarianten von Linearen Gruppen*, Acta Math., 58 (1932), p. 231-293.

[123] H. Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton Univ. Press (1946).

[124] Y. Wu, *Invariants of Modular Two-Row Groups*, Ph.D thesis, Queen's University, 2009.

[125] O. Zariski, *Interpretations algebrico-geometrique de quatorzieme probleme de Hilbert*, Bull. Sci. Math. 78 (1954), 155-168.

[126] V. S. Zhgun, *Variation of the Mumford quotient for torus actions on complete flag manifolds. I*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 71 (2007), no. 6, 29-46; translation in Izv. Math. 71 (2007), no. 6, 1105-1122.

[127] V. S. Zhgun, *Variation of the Mumford quotient for torus actions on complete flag manifolds. II*, (Russian) Mat. Sb. 199 (2008), no. 3, 25-44; translation in Sb. Math. 199 (2008), no. 3-4, 341-359.

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