# Tutorial 12 - Poincare Sphere

July 1, 2019

## Introduction

Today we will go back from chaotic flows and maps to flows on a plane. In previous lectures we obtained a comprehensive picture of the dynamics on a plane, both local and asymptotic (with respect to time). Results regarding local dynamics include linearization and center manifold reduction. The main result regarding asymptotic dynamics is the Poincare-Bendixson theorem. Recall that the theorem classifies the asymptotic dynamics possible on a plane, and states that all $\omega$-limit sets on the plane are either equilibria, periodic orbits or separatrix cycles. Essentially, this means that there is no chaos in 2D flows.

However, there is a remaining territory to cover: study of the asymptotic dynamics of unbounded orbits, as they tend towards infinity. Studying the character of "equilibria" at infinity can add to our understanding of the dynamics on the plane. In order to understand the behavior on the unbounded plane $\mathbb{R}^2$, we present the Poincare sphere, a compactification of the plane to a half-sphere on which $\infty$ becomes the edge and the asymptotic dynamics can be tracked.

## Warm-up: Poincare circle - Compactifying dynamics on the line $\mathbb{R}$

The behavior of an ODE $\dot{x} = f(x)$ near $\pm\infty$ on $\mathbb{R}$ is simple - assuming all of the fixed points of $f(x)$, i.e. all zeros of the function, are contained in some finite region, then after some large enough $|x|$, orbits move monotonically towards/away from $\pm\infty$, depending on the sign of $f(x)$ at the limit. Nevertheless we will show the compactification of the line as a warm-up to the related analysis on the plane.

Construction: Embed the phase line $\mathbb{R}$ into the plane, then project the line onto the

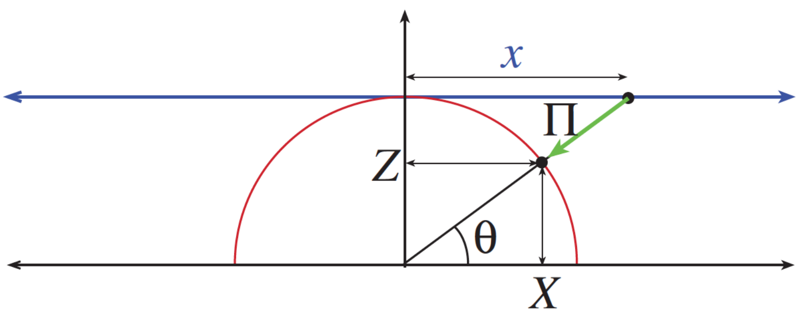


Figure 1: Coordinates for the Poincare circle.

half-circle with radius 1:

$$x \to (X, Z) = (\sin\theta, \cos\theta)\ ;\ \ x = \cot\theta.$$

With this projection, $+\infty \mapsto \theta = 0$ and $-\infty \mapsto \theta = 2\pi$.

Thus, given a 1D ODE $\dot{x} = f(x)$, the dynamics on the Poincare sphere are given by:

$$\dot{\theta} = \frac{\partial\theta}{\partial x}\dot{x} = -\frac{1}{x^2+1}\dot{x} = -\sin^2\theta f(\cot\theta) \equiv g(\theta).$$

How does this help us understand the behavior at infinity?

**Example:** power law asymptotic behavior

Assume $f(x)$ has a power law behavior near $\infty$,

$$f \sim ax^m + \mathcal{O}(x^{m-1}), a > 0.$$

What is $g(\theta)$ in this case?

$$g(\theta) \sim -a\sin^2\theta\cot^m\theta$$

When $x \to \infty$, then $\theta \to 0$ and the function asymptotically approaches the 1D dynamical system

$$\dot{\theta} = -a\theta^{2-m}$$

If $m < 2$, then $\theta \underset{t\to\infty}{\longrightarrow} 0$.

But if $m > 2$, then $\theta$ will approach 0 at a finite time, implying that the flow is not complete.

**Restoring completeness:** We rescale time to find a topologically equivalent system:

$$d\tau = \theta^{1-m}dt.$$

Thus,

$$\theta'(\tau) = \theta^{m-1}(-a\theta^{2-m}) = -a\theta(\tau)$$

and the system is no longer singular. This technique of extending the dynamics to $\infty$ by defining a topologically equivalent dynamics is called ”blowing up” the singularity.

## Poincare sphere - Compactifying $\mathbb{R}^2$

Consider the 2D dynamical system

$$\dot{x} = P(x, y)\ ,\ \ \dot{y} = Q(x, y).$$

Given a point on the plane $(x, y)$, we want to map this point uniquely to a point on the half sphere $S^{2+} = \{X, Y, Z : X^2 + Y^2 + Z^2 = 1, Z \geq 0\}$.

**Construction:** Center the half-sphere on the origin, place the plane on top of the ”north pole” of the half-sphere, and connect each point $(x, y, z = 1)$ on the plane to the origin. The intersection of this line with the sphere is the map. Thus, the coordinates of the new point on the sphere are:

$$X = \frac{x}{\sqrt{1+x^2+y^2}}\ ,\ \ Y = \frac{y}{\sqrt{1+x^2+y^2}}\ ,\ \ Z = \frac{1}{\sqrt{1+x^2+y^2}}.$$

The dynamics can be calculated:

$$\dot{X} = \frac{\dot{x}}{\sqrt{1+x^2+y^2}} - \frac{x(x\dot{x}+y\dot{y})}{(1+x^2+y^2)^{3/2}} = Z((1-X)P - XYQ),$$

and similarly:

$$\dot{Y} = Z(-XYP + (1-Y^2)Q)\ ,\ \dot{Z} = -Z^2(XP+YQ),$$

where $P = P(X/Z, Y/Z), Q = Q(X/Z, Y/Z)$. Note that we started out with 2 ODEs and now we have 3: that is because we haven't taken into accout that all of the dynamics occur on a sphere, giving us the extra constraint ODE:

$$\frac{d}{dt}(X^2+Y^2+Z^2) = 0.$$

Thus, the topological properties of the original planar flow near $\infty$ correspond to those of the flow on the Poincare sphere near the rim of the half-sphere $Z = 0$.

**Example: Power-law asymptotics**

Assume $(P,Q)$ have a power-law behavior near $\infty$ with a max degree of $m$. Then, near $Z = 0$,

$$P(X/Z, Y/Z) \sim Z^{-m},$$

and same for $Q$.

Therefore, near $Z = 0$, $X \sim Y \sim 1$ and the asymptotic dynamics are $\dot{X} \sim Z^{1-m}(1 - X - XY) \sim Z^{1-m}$. For $m > 1$, this is an asymptotic singularity. As in the 1D case, we rescale time, $d\tau = Z^{1-m}dt$, and further we define:

$$P^*(X,Y,Z) = Z^m P(X/Z, Y/Z)\ ,\ Q^*(X,Y,Z) = Z^m Q(X/Z, Y/Z),$$

and thus the dynamics are rewritten as

$$\begin{aligned} X'(\tau) &= (Y^2 - Z^2)P^* - XYQ^*, \\ Y'(\tau) &= -XYP^* + (X^2+Z^2)Q^*, \\ Z'(\tau) &= -Z(XP^* + YQ^*). \end{aligned}$$

As the dynamics approach $Z = 0$, only the highest order (i.e. degree $m$) term of $P^*$ and $Q^*$ will be retained. Define $P_m(X,Y)$, $Q_m(X,Y)$ as the degree-$m$ terms in the original $P$, $Q$, then $P^* \underset{Z\to 0}{\longrightarrow} P_m$ and same for $Q^*$. This allows us to study the (generally non-trivial) dynamics at infinity:

$$X' = -Y(XQ_m - YP_m)\ ,\ Y' = X(XQ_m - YP_m).$$

Finally we see what an equilibrium at infinity looks like:

**there will be an equilibrium at $\infty$ only when $XQ_m - YP_m = 0$.**

Note that since this equation is homogeneous of degree $m+1$, $(X,Y)$ is an equilibrium iff $(-X,-Y)$ is an equilibrium. Note that if $m$ is even then $XQ_m - YP_m = 0$ switches signs between opposing equilibria and each will have an opposite stability type, while if $m$ is odd they will have the same.

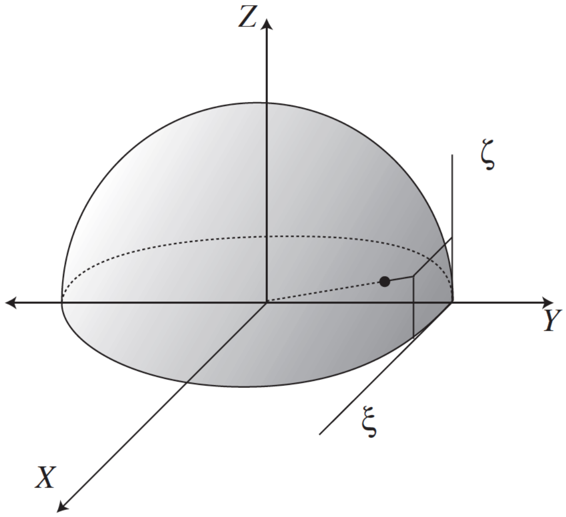


Figure 2: Second projection - understanding equilibria at infinity.

## Understanding the behavior around an equilibrium at ∞

Assume we found an equilibrium at $Z=0, X>0$. We would like to study its properties and understand behavior of trajectories around it, not just on the ∞ line. To do this, we perform another projection of coordinates onto a plane that is perpendicular to the half-sphere at $Z=0$:

Consider an equilibrium at $Z=0$ with $Y\neq 0$. Thus, a point $(X,Y,Z)$ is mapped to $\xi = X/Y$, $\zeta = Z/Y$. The differential equations governing the motion on the new plane are:

$$\dot{\xi} = Y^{m-1}\zeta^m(P-\xi Q)$$
$$\dot{\zeta} = -Y^{m-1}\zeta^{m+1}Q.$$

Rescaling time once more, $d\tau = Y^{1-m}dt$, obtain

$$\dot{\xi} = \zeta^m(P-\xi Q)$$

$$\dot{\zeta} = -\zeta^{m+1}Q,$$

where $P,Q$ are evaluated at $\xi/\zeta, 1/\zeta$.

**Example: Linear system**

Consider the linear system:

$$\dot{x} = x+y,$$

$$\dot{y} = 2x+y.$$

The equilibria at ∞ are determined by the equation $YP_m - XQ_m = 0$, where recall $P_m, Q_m$ have the rescaled time factors, so the equation becomes:

$$YZ^m(X/Z+Y/Z) - XZ^m(2X/Z+Y/Z) = -2X^2+Y^2 \stackrel{!}{=} 0,$$

and recalling $X^2+Y^2+Z^2=1$ we obtain that the system has equilibria at $(\pm 1/\sqrt{3}, \pm\sqrt{2/3}, 0)$.

Next, to analyze the behavior of the system around these fixed points, we perform the second projection:

$$\xi = X/Y\ ,\ \ \zeta = Z/Y.$$

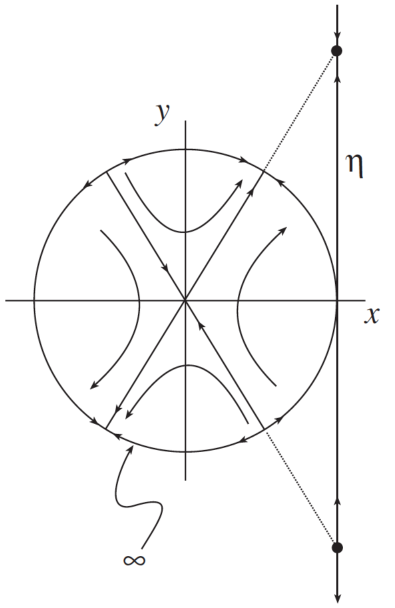


Figure 3: Phase space dynamics for the linear system example

The corresponding equations of motion are:

$$\dot{\xi} = \zeta(P - \xi Q) = \xi + 1 - 2\xi^2 - \xi = 1 - 2\xi^2$$

$$\dot{\zeta} = -\zeta^2 Q = -\zeta(2\xi + 1).$$

Equilibrium is obtained at:

$$\xi = \pm 1/\sqrt{2}\,,\ \ \zeta = 0,$$

corresponding (as it should) with the equilibria found before the second projection. In these new coordinates, linear stability analysis is an easy task, revealing that $+1/\sqrt{2}$ is a stable node while $-1/\sqrt{2}$ is unstable. The phase space picture of the original full plane $\mathbb{R}^2$ can thus be mapped onto the half-sphere, and viewed from above a qualitative picture of the dynamics is obtained and shown in Fig. 3.